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Abundancy "Outlaws" of the Form $\frac{\sigma(N)+t}{N}$

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Abstract

The abundancy index of a positive integer n is defined to be the rational number $I(n) = \sigma(n)/n$, where σ is the sum of divisors function $\sigma(n) = \sum_{d|n} d$. An abundancy outlaw is a rational number greater than 1 that fails to be in the image of of the map I. In this paper, we consider rational numbers of the form $(\sigma(N) + t)/N$ and prove that under certain conditions such rationals are abundancy outlaws.

1 Introduction

The abundancy index of a positive integer n is defined to be the rational number $I(n) = \sigma(n)/n$, where σ is the sum of divisors function, $\sigma(n) = \sum_{d|n} d$. Positive integers having integer-valued abundancy indices are said to be *multiperfect* numbers, and if $I(n) = 2$ in particular, then n is *perfect*. More generally, the abundancy index of a number n can be thought of as a measure of its perfection; if $I(n) < 2$ then n is said to be *deficient*, and if $I(n) > 2$ then n is *abundant*. In this way, the abundancy index is a useful tool in gaining a better understanding of perfect numbers. In fact, the following theorem provides conditions equivalent to the existence of an odd perfect number [2].

Theorem 1.1. *There exists an odd perfect number if and only if there exist positive integers* $p, n,$ *and* α *such that* $p \equiv \alpha \equiv 1 \mod 4$*, where* p *is a prime not dividing* n*, and*

$$I(n) = \frac{2p^\alpha(p-1)}{p^{\alpha+1}-1}.$$

So, for example, if one could find an integer n having abundancy index equal to $5/3$, then one would be able to produce an odd perfect number. Hence, it is useful to try to characterize those rational numbers in $(1,\infty)$ that do not appear as the abundancy index of some positive integer. We will call such numbers *abundancy outlaws.*

Definition 1.2. A rational number r/s greater than 1 is said to be an *abundancy outlaw* if $I(x)=r/s$ has no solution among the positive integers.

In this paper, we consider the sequence of rational numbers in $(1,\infty)$. For each numerator $a>1$, list the rationals a/b, with $\gcd(a,b)=1$, so that denominators $1\le b<a$ appear in ascending order:

$$\frac{2}{1},\frac{3}{1},\frac{3}{2},\frac{4}{1},\frac{4}{3},\frac{5}{1},\frac{5}{2},\frac{5}{3},\frac{5}{4},\frac{6}{1},\frac{6}{5},\frac{7}{1},\frac{7}{2},\frac{7}{3},\frac{7}{4},\frac{7}{5},\frac{7}{6},\dots$$

While each term in the sequence is either an abundancy index or an abundancy outlaw, it is generally difficult to determine the status of a given rational. The sequence can be partitioned into three sets: those rationals that are known to be abundancy indices, those that are known to be outlaws, and those with abundancy index/outlaw status unknown. Our goal is to capture outlaws from the third category, increasing the size of the second category. Since rationals of the form $(\sigma(N)-t)/N$, with $t\ge 1$, are known to be outlaws (see Property 2.3 below), we consider rational numbers of the form $(\sigma(N)+t)/N$. We prove that under certain conditions $(\sigma(N)+t)/N$ is an abundancy outlaw. It is worth noting that our original interest in such rationals stemmed from our interest in the fraction $5/3=(\sigma(3)+1)/3$. Unfortunately, our results do not allow us to say anything about rationals of the form $(\sigma(p)+1)/p$ where p is a prime. Such elusive rationals remain in category three. Nonetheless, we do prove that $(\sigma(2p)+1)/(2p)$ is an abundancy outlaw for all primes $p>3$. (Since $I(6)=(\sigma(2^2)+1)/2^2$ and $I(18)=(\sigma(6)+1)/6$, this provides a complete characterization of fractions of this form.)

2 Preliminaries

It is useful to think of the abundancy index I as a function mapping the natural numbers $n\ge 2$ into the set of rational numbers in $(1,\infty)$. Defining D to be the image of I:

$$D=\{I(n):n\in\mathbb{N},n\ge 2\},$$

we can ask many questions about D. For instance, how are the abundancy indices distributed among the set $(1,\infty)$? Certainly, we can find elements of D arbitrarily close to 1 because $I(p)=(p+1)/p$ for all primes p. Moreover, it is not hard to show that $I(n!)\ge\sum_i^n 1/i$, and therefore D is unbounded. In fact, D is dense in $(1,\infty)$ [3]. Even more interesting, P. Weiner proved that the set of abundancy outlaws is *also* dense in $(1,\infty)$ [5]! Hence it seems that the situation is both complex and interesting. For our purposes, the following properties will be helpful.

Property 2.1 $I(kN)\ge I(N)$ for all natural numbers k and $N\ge 2$. (See [3], page 84.)

Property 2.2 If $I(n) = k/m$ with $\gcd(k, m) = 1$, then $m|n$. This follows directly from setting $\sigma(n)/n = k/m$. Clearly, $m|(nk)$ and since k and m are relatively prime, it must be that $m|n$.

Property 2.3 If $m < k < \sigma(m)$ and k is relatively prime to m, then k/m is an abundancy outlaw. Hence if r/s is an abundancy index with $\gcd(r, s) = 1$, then $r \geq \sigma(s)$. (See [5], page 309. The property also appears in [1].)

Property 2.3 reveals a class of abundancy outlaws. Indeed, it was using this property that Weiner was able to prove that the set of outlaws is dense in $(1, \infty)$. It also worth noting that Property 2.3 implies that $(k+1)/k$ is an abundancy index if and only if k is prime. Similarly, $(k+2)/k$ is an abundancy outlaw whenever k is an odd composite number. If p is a prime greater than 2, then it is unknown whether $(p+2)/p$ is an outlaw. (See [4], pages 512-513 for more discussion about this.)

Finally, recent progress has been made by R. Ryan in finding abundancy outlaws. In 2002, Ryan produced an example of an abundancy outlaw not captured by Property 2.3. (See Theorem B.6 in [4].) Because the conditions describing Ryan's outlaw are quite technical, we will not restate them here. Nonetheless, we want to mention that the search for outlaws employed in the next section was inspired by Ryan's work.

3 A search for abundancy outlaws

As Property 2.1 implies, multiplying any number $N = \prod_{i=1}^{n} p_i^{k_i}$ by one of its prime divisors, p_j, will serve to increase its abundancy. The following lemma measures this increase.

Lemma 3.1. *Let $N = \prod_{i=1}^{n} p_i^{k_i}$ for primes $p_1, p_2, ..., p_n$. Then*

$$\frac{\sigma(p_j^{k_j+1})}{\sigma(p_j^{k_j+1}) - 1} = \frac{\sigma(p_j N)}{p_j \sigma(N)}$$

for all $1 \leq j \leq n$.

Proof. The result follows from the fact that

$$\begin{aligned} p_j\sigma(N) &= p_j\sigma(p_j^{k_j})\sigma(N/p_j^{k_j}) \\ &= p_j(\textstyle\sum_{i=1}^{k_j} p_j^i)\sigma(N/p_j^{k_j}) \\ &= (\sigma(p_j^{k_j+1}) - 1)\sigma(N/p_j^{k_j}). \end{aligned}$$

Therefore,

$$\frac{\sigma(p_j N)}{p_j\sigma(N)} = \frac{\sigma(p_j^{k_j+1})\sigma(N/p_j^{k_j})}{(\sigma(p_j^{k_j+1}) - 1)\sigma(N/p_j^{k_j})} \tag{3.1}$$

$$= \frac{\sigma(p_j^{k_j+1})}{\sigma(p_j^{k_j+1}) - 1}. \tag{3.2}$$

$\square$

Next we present criteria that can be used in the search for abundancy outlaws. As the assumptions given in Theorem 3.2 below indicate, our search focuses on those fractions r/s (in reduced form) that satisfy $I(N) < r/s < I(p_iN)$ for some prime divisor p_i of a positive integer N. To be more specific, keeping Properties 2.1 and 2.2 in mind, we look for those values of s having divisors that lead to abundancy values exceeding $I(p_iN)$.

Theorem 3.2. *Let $r/s > 1$ be a fraction in lowest terms such that there exists a divisor $N = \prod_{i=1}^{n} p_i^{k_i}$ of s satisfying the following two conditions:*

1. *$r/s < I(p_iN)$ for all $i \leq n$*
2. *The product $\sigma(N)(s/N)$ has a divisor M such that $(M, r) = 1$ and $I(M) \geq \frac{\sigma(p_j^{k_j+1})}{\sigma(p_j^{k_j+1})-1}$ for some positive integer $j \leq n$.*

Then r/s is an abundancy outlaw.

Proof. Let $r/s > 1$ be a fraction in lowest terms satisfying the above hypotheses. Suppose that $I(x) = r/s$ for some natural number x. Since r/s is in lowest terms, Property 2.2 ensures that $s|x$. Then, because $N|s$, $N|x$, and therefore, $x = dN$ for some positive integer d. However, since $r/s < I(p_iN)$ for $1 \leq i \leq n$, the first assumption requires that $p_i^{k_i+1} \nmid x$ for all $1 \leq i \leq n$, and thus, d is relatively prime to N.

Consequently, we can write

$$I(x) = I(dN) = I(d)I(N) = \frac{r}{s}.$$

Hence,

$$\sigma(d)\sigma(N)(s/N) = rd.$$

Now, by the second assumption, we know that there exists a positive integer M such that

$$M|\sigma(N)(s/N)$$

and M is relatively prime to r. Therefore, $M|d$ and we can say that

$$I(x) = I(d)I(N) \geq I(M)I(N)$$

by Property 2.1. Then, by assumption 2) and Lemma 3.1,

$$I(M) \geq \frac{\sigma(p_j^{k_j+1})}{\sigma(p_j^{k_j+1}) - 1} = \frac{\sigma(p_jN)}{p_j\sigma(N)}$$

for some $1 \leq j \leq n$, so we know that

$$I(x) \geq I(M)I(N) \geq \frac{\sigma(p_jN)}{p_j\sigma(N)}\frac{\sigma(N)}{N} = \frac{\sigma(p_jN)}{p_jN} = I(p_jN)$$

Therefore, $I(x) \geq I(p_jN)$, which contradicts our assumption that $I(x) = r/s < I(p_iN)$ for all $1 \leq i \leq n$. We conclude, then, that r/s is an abundancy outlaw. $\square$

Example 3.3. Theorem 3.2 can be used to show that $37/22$ is an abundancy outlaw. Certainly $37/22 < I(2^2) = 7/4$. Thus, assumption 1) is satisfied for $N = 2$. Next, note that $M = 3$ divides $\sigma(2) \cdot (22/2)$, and because $gcd(3, 37) = 1$, with $I(3) > \frac{\sigma(4)}{\sigma(4)-1} = 7/6$, assumption 2) is satisfied as well. A computer search reveals many more examples. (See Table 3.4)

Abundancy outlaw r/s	s	$\sigma(s)$
29/12	$2^2 \cdot 3$	28
37/22	$2 \cdot 11$	36
43/20	$2^2 \cdot 5$	42
43/26	$2 \cdot 13$	42
55/34	$2 \cdot 17$	54
59/34	$2 \cdot 17$	54
61/24	$2^3 \cdot 3$	60
61/38	$2 \cdot 19$	60
65/38	$2 \cdot 19$	60
73/30	$2 \cdot 3 \cdot 5$	72
73/46	$2 \cdot 23$	72
73/51	$3 \cdot 17$	72
77/46	$2 \cdot 23$	72
79/45	$3^2 \cdot 5$	78
79/46	$2 \cdot 23$	72
91/40	$2^3 \cdot 5$	90
91/58	$2 \cdot 29$	90
95/58	$2 \cdot 29$	90
97/42	$2 \cdot 3 \cdot 7$	96
97/58	$2 \cdot 29$	90
97/62	$2 \cdot 31$	96
97/69	$3 \cdot 23$	96
101/58	$2 \cdot 29$	90
101/62	$2 \cdot 31$	96
103/62	$2 \cdot 31$	96
107/62	$2 \cdot 31$	96
115/74	$2 \cdot 37$	114
119/74	$2 \cdot 37$	114
121/56	$2^3 \cdot 7$	120
121/74	$2 \cdot 37$	114
121/87	$3 \cdot 29$	120
125/48	$2^4 \cdot 3$	124

Abundancy outlaw r/s	s	$\sigma(s)$
125/74	$2 \cdot 37$	114
125/87	$3 \cdot 29$	120
127/68	$2^2 \cdot 17$	126
127/74	$2 \cdot 37$	114
127/82	$2 \cdot 41$	126
131/82	$2 \cdot 41$	126
131/93	$3 \cdot 31$	128
133/82	$2 \cdot 41$	126
133/86	$2 \cdot 43$	132
133/93	$3 \cdot 31$	128
137/82	$2 \cdot 41$	126
137/86	$2 \cdot 43$	132
139/82	$2 \cdot 41$	126
139/86	$2 \cdot 43$	132
141/76	$2^2 \cdot 19$	140
143/82	$2 \cdot 41$	126
143/86	$2 \cdot 43$	132
145/66	$2 \cdot 3 \cdot 11$	144
145/86	$2 \cdot 43$	132
145/94	$2 \cdot 47$	144
149/86	$2 \cdot 43$	132
149/94	$2 \cdot 47$	144
151/94	$2 \cdot 47$	144
155/94	$2 \cdot 47$	144
155/111	$3 \cdot 37$	152
157/94	$2 \cdot 47$	144
157/99	$3^2 \cdot 11$	156
157/111	$3 \cdot 37$	152
161/94	$2 \cdot 47$	144
163/94	$2 \cdot 47$	144
163/106	$2 \cdot 53$	162
167/106	$2 \cdot 53$	162

Table 3.4: A list of abundancy outlaws found using Theorem 3.2. These outlaws are captured by Theorem 3.2, but not by Property 2.3.

4 The main results

Table 3.4 reveals some recognizable patterns. Most obvious is the indication that, for small odd values of t (and p prime), reduced rational numbers of the form

$$\frac{r}{s} = \frac{\sigma(2^m p^n) + t}{2^m p^n}$$

often lead to abundancy outlaws. In fact, the results of this section will show that $\frac{\sigma(2^m p^{2n+1})+1}{2^m p^{2n+1}}$ is an abundancy outlaw whenever $\gcd(p, \sigma(2^m)) = 1$. In particular, if $p > 3$, then $(\sigma(2p) + 1)/(2p)$ is *always* an abundancy outlaw. Our main result, however, will address the more general situation where r/s is a reduced rational number of the form $(\sigma(N) + t)/N$. First we will need a lemma.

Lemma 4.1. *Let $N = \prod_{i=1}^{n} p_i^{k_i}$, where p_i is a prime for all $1 \le i \le n$. Then, for a given $1 \le j \le n$ and a positive integer t,*

$$p_j < \frac{1}{t}\sigma\left(\frac{N}{p_j^{k_j}}\right)$$

if and only if

$$\frac{\sigma(N) + t}{N} < I(p_j N).$$

Proof. Assume that

$$\frac{\sigma(N) + t}{N} < I(p_j N)$$

for a given natural number $j \le n$. This is equivalent to

$$p_j \sigma(N) + p_j t < \sigma(p_j N).$$

Then, since $p_j \sigma(p_j^{k_j}) = \sigma(p_j^{k_j+1}) - 1$, we find that this inequality is equivalent to

$$(\sigma(p_j^{k_j+1}) - 1)\sigma\left(\frac{N}{p_j^{k_j}}\right) + p_j t \quad < \quad \sigma(p_j N),$$

or

$$\sigma(p_j N) - \sigma\left(\frac{N}{p_j^{k_j}}\right) + p_j t \quad < \quad \sigma(p_j N).$$

Therefore,

$$p_j < \frac{1}{t}\sigma\left(\frac{N}{p_j^{k_j}}\right).$$

□

We are now ready to present the main result.

Theorem 4.2. *For a positive integer t, let $\frac{\sigma(N)+t}{N}$ be a fraction in lowest terms, and let $N = \prod_{i=1}^{n} p_i^{k_i}$ for primes $p_1, p_2, ..., p_n$. If there exists a positive integer $j \leq n$ such that $p_j < \frac{1}{t}\sigma(N/p_j^{k_j})$ and $\sigma(p_j^{k_j})$ has a divisor $D > 1$ such that at least one of the following is true:*

1. $I(p_j^{k_j})I(D) > \frac{\sigma(N)+t}{N}$ *and* $gcd(D,t) = 1$
2. $gcd(D, Nt) = 1$

then $\frac{\sigma(N)+t}{N}$ is an abundancy outlaw.

Proof. Case 1: Let N and t be natural numbers such that $\frac{\sigma(N)+t}{N}$ is a fraction in lowest terms, and let $j \leq n$ be a natural number satisfying $p_j < \frac{1}{t}\sigma(N/p_j^{k_j})$ and D a divisor of $\sigma(p_j^{k_j})$ satisfying hypothesis (1). Suppose further that $I(x) = \frac{\sigma(N)+t}{N}$ for some natural number x. Since $(\sigma(N)+t, N) = 1$, $N|x$ by Property 2.2. Say $x = dN$, where d is a positive integer. Since p_j satisfies $p_j < \frac{1}{t}\sigma(N/p_j^{k_j})$, Lemma 4.1 implies that $I(x) = \frac{\sigma(N)+t}{N} < I(p_jN)$. Hence, $gcd(p_j^{k_j}, dN/p_j^{k_j}) = 1$ and

$$I(x) = I(p_j^{k_j})I(dN/p_j^{k_j}) = \frac{\sigma(N)+t}{N}.$$

Equivalently,

$$\sigma(p_j^{k_j})\sigma(dN/p_j^{k_j}) = (\sigma(N)+t)d.$$

Since $\sigma(p_j^{k_j})|\sigma(N)$ and $gcd(D,t) = 1$, the divisor D of $\sigma(p_j^{k_j})$ satisfying hypothesis (1) also divides d. Hence D divides $dN/p_j^{k_j}$, and by Property 2.1, $I(D) \leq I(dN/p_j^{k_j})$. Thus

$$I(p_j^{k_j})I(D) \leq I(p_j^{k_j})I(dN/p_j^{k_j}) = I(x).$$

Given $I(x) = \frac{\sigma(N)+t}{N}$, we conclude then that

$$I(p_j^{k_j})I(D) \leq \frac{\sigma(N)+t}{N}.$$

This contradicts the assumption that $I(p_j^{k_j})I(D) > \frac{\sigma(N)+t}{N}$. Therefore, $\frac{\sigma(N)+t}{N}$ is an abundancy outlaw.

Case 2: Now let $N = \prod_{i=1}^{n} p_i^{k_i}$ and t be natural numbers such that $\sigma(p_j)^{k_j}$ has a divisor D satisfying hypothesis (2) for some $1 \leq j \leq n$, and assume that $I(x) = \frac{\sigma(N)+t}{N}$ for some natural number x. Since $\frac{\sigma(N)+t}{N}$ is in lowest terms, $N|x$, so that $x = sN$ for some natural number s. By assumption, there exists a natural number $j \leq n$ so that $p_j < \frac{1}{t}\sigma(N/p_j^{k_j})$. By Lemma 4.1, $I(x) = \frac{\sigma(N)+t}{N} < I(p_jN)$. Hence, $p_j \nmid s$, so that

$$\begin{aligned} I(x) &= I\left(p_j^{k_j} s \frac{N}{p_j^{k_j}}\right) \\ &= I(p_j^{k_j}) I\left(s \frac{N}{p_j^{k_j}}\right). \end{aligned}$$

Next, we factor out the part of s, $\prod_{i=1}^n p_i^{\gamma_i}$, that has divisors in common with $N/p_j^{k_j}$, so that

$$\overline{s} = \frac{s}{\prod_{i=1}^n p_i^{\gamma_i}},$$

where γ_i is a non-negative integer for all $i \leq n$.

Then, we can rewrite $I(x)$ once more in the following form:

$$I(x) = I(p_j^{k_j})I(\overline{s})I\left(\frac{N}{p_j^{k_j}}\prod_{i=1}^n p_i^{\gamma_i}\right)$$

Because $I(x) = \frac{\sigma(N)+t}{N}$, we can see, then, that

$$I(p_r^{k_r})I(\overline{s})I\left(\tfrac{N}{p_r^{k_r}}\textstyle\prod_{i=1}^n p_i^{\gamma_i}\right) \quad = \quad \tfrac{\sigma(N)+t}{N},$$

or equivalently,

$$\sigma(p_r^{k_r})\sigma(\overline{s})\sigma\left(\tfrac{N}{p_r^{k_r}}\textstyle\prod_{i=1}^n p_i^{\gamma_i}\right) \quad = \quad (\sigma(N)+t)\overline{s}\textstyle\prod_{i=1}^n p_i^{\gamma_i}.$$

Now consider the divisor D of $\sigma(p_j^{k_j})$ satisfying hypothesis (2). Since $\sigma(p_j^{k_j})|\sigma(N)$ and $gcd(D, Nt) = 1$, D divides $\overline{s}$ (so $\overline{s} > 1$). This, then, means that $I(\overline{s}) \geq I(D)$. Then, since $p_j < \frac{1}{t}\sigma(N/p_j^{k_j})$,

$$p_j\sigma(p_j^{k_j}) < \frac{1}{t}\sigma(N),$$

which implies that the following is true:

$$\frac{1}{p_j\sigma(p_j^{k_j})} > \frac{t}{\sigma(N)}.$$

Thus, since $D|\sigma(p_j^{k_j})$, $D < p_j\sigma(p_j^{k_j})$, so that $1/D > 1/p_j\sigma(p_j^{k_j})$.

Hence, the following are true:

$$\begin{aligned}
I(\overline{s}) &\geq I(D) \\
&\geq 1 + \frac{1}{D} \\
&> 1 + \frac{1}{p_j\sigma(p_j^{k_j})} \\
&> 1 + \frac{t}{\sigma(N)} \\
&= \frac{\sigma(N)+t}{\sigma(N)} \\
&= \frac{\sigma(N)+t}{I(p_j^{k_j})I(N/p_j^{k_j})N} \\
&\geq \frac{\sigma(N)+t}{I(p_j^{k_j})I((N/p_j^{k_j})\prod_{i=1}^n p_i^{\gamma_i})N}.
\end{aligned}$$

Hence

$$I(p_j^{k_j})I(\overline{s})I\left(\frac{N}{p_j^{k_j}}\prod_{i=1}^{n}p_i^{\gamma_i}\right) = I(x) > \frac{\sigma(N)+t}{N},$$

which contradicts our original assumption that $I(x) = \frac{\sigma(N)+t}{N}$. Thus, $\frac{\sigma(N)+t}{N}$ is an abundancy outlaw.

$\square$

If $t = 1$ we get the following corollary.

Corollary 4.3. *Let $\frac{\sigma(N)+1}{N}$ be a fraction in lowest terms, and let $N = \prod_{i=1}^{n} p_i^{k_i}$ for primes $p_1, p_2, ..., p_n$. If there exists a natural number $j \leq n$ such that $p_j < \sigma(N/p_j^{k_j})$ and $\sigma(p_j^{k_j})$ has a divisor D such that at least one of the following is true:*

1. $I(p_j^{k_j})I(D) > \frac{\sigma(N)+1}{N}$
2. $gcd(D, N) = 1$

then $\frac{\sigma(N)+1}{N}$ is an abundancy outlaw.

5 Constructing Sequences of Abundancy Outlaws

Using the results in the previous section, we can find and construct sequences of abundancy outlaws. The following lemma will be helpful.

Lemma 5.1. *Let $N = \prod_{i=1}^{n} p_i^{k_i}$ for primes $p_1, p_2, ..., p_n$. Then N is relatively prime to $\sigma(N)+1$ if and only if p_i is relatively prime to $\sigma(N/p_i^{k_i})+1$ for all $1 \leq i \leq n$.*

Proof. Since $\sigma(p_i^{k_i}) \equiv 1 \ (mod\ p_i)$,

$$\begin{aligned} \sigma(N)+1 &= \sigma(p_i^{k_i})\sigma(N/p_i^{k_i})+1 \\ &\equiv \sigma(N/p_i^{k_i})+1 \ (mod\ p_i). \end{aligned}$$

Thus, any prime divisor p_i of N and $\sigma(N)+1$ must also be a prime divisor of $\sigma(N/p_i^{k_i})+1$, and conversely, if $p_i|(\sigma(N/p_i^{k_i})+1)$, then p_i divides both N and $\sigma(N)+1$. $\square$

Outlaws with even denominators

Corollary 5.2. *For all natural numbers m and nonnegative integers n, and for all odd primes p such that $gcd(p, \sigma(2^m)) = 1$, the rational number*

$$\frac{\sigma(2^m p^{2n+1})+1}{2^m p^{2n+1}}$$

is an abundancy outlaw.

Proof. To see that $(\sigma(2^m p^{2n+1})+1)/2^m p^{2n+1}$ is in lowest terms, apply Lemma 5.1. Since p and $2n+1$ are odd, $\sigma(p^{2n+1})+1$ is odd, so $gcd(2^m, \sigma(p^{2n+1})+1) = 1$. Also, $gcd(p^{2n+1}, \sigma(2^m)+1) = 1$, because $\sigma(2^m)+1 = 2^{m+1}$. Next, we apply Corollary 4.3. Since $2 < \sigma(p^{2n+1})$, we consider divisors D of $\sigma(2^m)$. Clearly, $gcd(\sigma(2^m), 2^m) = 1$, and because p does not divide $\sigma(2^m)$, $gcd(\sigma(2^m), 2^m p^{2n+1}) = 1$. Hence, $D = \sigma(2^m)$ is the divisor required to apply Corollary 4.3. Thus, $(\sigma(2^m p^{2n+1}))/2^m p^{2n+1}$ is an abundancy outlaw.

$\square$

Corollary 5.3. *For all primes $p > 3$,*

$$\frac{\sigma(2p)+1}{2p}$$

is an abundancy outlaw. If $p = 2$ or $p = 3$ then $\frac{\sigma(2p)+1}{2p}$ is an abundancy index.

Proof. This result follows directly from Corollary 5.2. To prove that $\frac{\sigma(2p)+1}{2p}$ is an abundancy index when $p = 2$ and $p = 3$, note that $I(6) = \frac{\sigma(2^2)+1}{2^2}$ and $I(18) = \frac{\sigma(6)+1}{6}$. $\square$

Remark 5.4. Corollary 5.3 actually has a very simple proof (and in fact, the results presented in section 4 represent our attempt to push this simple proof as far as possible). Clearly, $(\sigma(2p)+1)/(2p) = (3p+4)/(2p)$ is in lowest terms. Thus, if $I(N) = (\sigma(2p)+1)/2p$, $2p|N$. Because $p > 3$, it can be shown that $I(4p) > (\sigma(2p)+1)/2p$, so $4 \nmid N$. Therefore, $\sigma(2)|\sigma(N)$, and since $\sigma(2) = 3$ does not divide $\sigma(2p)+1$, 3 divides N. Hence, $I(N) > I(6p) > 2 > (\sigma(2p)+1)/2p$. This is a contradiction, so $(\sigma(2p)+1)/2p$ is an abundancy outlaw, and we have captured the following sequence of outlaws:

$$\frac{19}{10}, \frac{25}{14}, \frac{37}{22}, \frac{43}{26}, \frac{55}{34}, \frac{61}{38}, \frac{73}{46}, \frac{91}{58}, \frac{97}{62}, \frac{115}{74}, \frac{127}{82}, \frac{133}{86}, \frac{145}{94}, \frac{163}{106}, \frac{181}{118}, \frac{187}{122}, \ldots$$

Corollary 5.3 also captures another (potentially infinite) set of outlaws having even denominators...

Corollary 5.5. *If N is an even perfect number,*

$$\frac{\sigma(2N)+1}{2N}$$

is an abundancy outlaw.

Proof. Since N is an even perfect number, $N = 2^{p-1}(2^p - 1)$, where p and $2^p - 1$ are both prime. Hence,

$$2N = 2^p(2^p - 1).$$

Applying Corollary 5.2, we need only show that $gcd(2^p - 1, \sigma(2^p)) = 1$. This follows from the fact that

$$2(2^p - 1) = (2^{p+1} - 1) - 1,$$

which is clearly relatively prime to $\sigma(2^p) = 2^{p+1} - 1$. Therefore, $(\sigma(2N)+1)/2N$ is an abundancy outlaw. $\square$

Outlaws with odd denominators

Corollary 5.6. *Let M be an odd natural number, and let p, α, and t be odd natural numbers such that $p \nmid M$ and $p < \frac{1}{t}\sigma(M)$. Then, if $(\sigma(p^\alpha M) + t)/p^\alpha M$ is in lowest terms,*

$$\frac{\sigma(p^\alpha M) + t}{p^\alpha M}$$

is an abundancy outlaw.

Proof. Since $(\sigma(p^\alpha M) + t)/p^\alpha M$ is in lowest terms, we apply Theorem 4.2 to show that it is an abundancy outlaw. Since p and α are both odd, $\sigma(p^\alpha)$ is even. Hence, $D = 2$ divides $\sigma(p^\alpha)$, and since p, M, and t are all odd, $gcd(D, pMt) = 1$. Thus, $(\sigma(p^\alpha M) + t)/p^\alpha M$ is an abundancy outlaw. $\square$

Corollary 5.7. *For primes p and q, with $3 < q$, $p < q$, and $gcd(p, q+2) = gcd(q, p+2) = 1$,*

$$\frac{\sigma(pq) + 1}{pq}$$

is an abundancy outlaw.

Proof. The case for $p = 2$ follows from Corollary 5.3. Now suppose that $q > p > 2$. In this case, neither p nor q divides $\sigma(pq)+1 = pq+p+q+2$, because $gcd(p, q+2) = gcd(q, p+2) = 1$. Thus, $\frac{\sigma(pq)+1}{pq}$ is in lowest terms. The result now follows from Corollary 5.6. $\square$

Corollary 5.7 produces outlaws with ease. To illustrate, let p and q be odd primes with $3 < p < q$, and assume $q \equiv 1 \ (mod\ p)$. Then $p \nmid q+2$ and $q \nmid p+2$. Since Dirichlet's theorem on arithmetic progressions of primes ensures the existence of an infinite sequence of primes q satisfying $q \equiv 1 \ (mod\ p)$, Corollary 5.7 reveals an infinite class of outlaws corresponding to each odd prime $p > 3$. The sequences corresponding to the primes 5, 7, and 11 follow.

$p = 5$ **:** $\frac{73}{55}, \frac{193}{155}, \frac{253}{205}, \frac{373}{305}, \frac{433}{355}, \frac{613}{505}, \frac{793}{655}, \frac{913}{755}, \frac{1093}{905}, \frac{1153}{955}, \frac{1273}{1055}, \frac{1513}{1255}, \frac{1633}{1355}, \frac{1693}{1405}, \frac{1873}{1555}, \frac{1993}{1655}, \frac{2413}{2005}, \frac{2533}{2105}, \ldots$

$p = 7$ **:** $\frac{241}{203}, \frac{353}{301}, \frac{577}{497}, \frac{913}{791}, \frac{1025}{889}, \frac{1585}{1379}, \frac{1697}{1477}, \frac{1921}{1673}, \frac{2257}{1967}, \frac{2705}{2359}, \frac{3041}{2653}, \frac{3377}{2947}, \frac{3601}{3143}, \frac{3713}{3241}, \frac{3937}{3437}, \frac{4385}{3829}, \frac{4945}{4319}, \ldots$

$p = 11$ **:** $\frac{289}{253}, \frac{817}{737}, \frac{1081}{979}, \frac{2401}{2189}, \frac{3985}{3641}, \frac{4249}{3883}, \frac{4777}{4367}, \frac{5041}{4609}, \frac{5569}{5093}, \frac{7417}{6787}, \frac{7945}{7271}, \frac{8209}{7513}, \frac{8737}{7997}, \frac{10321}{9449}, \frac{10585}{9691}, \frac{11377}{10417}, \ldots$

Remark 5.8. It is interesting to note that if p and $q = p + 2$ are twin primes then Corollary 5.7 does not apply, and in this case

$$\frac{\sigma(p(p+2)) + 1}{p(p+2)} = \frac{\sigma(p) + 1}{p} = \frac{p+2}{p}.$$

Once again, we are faced with that elusive ratio $\frac{p+2}{p}$!

6 Further explorations

R. Ryan considered the equation

$$I(x) = \frac{p+2}{p} = \frac{\sigma(p)+1}{p}, \tag{6.1}$$

where p is an odd prime [4]. Whether or not a solution x exists is still an open question. This problem is interesting for two reasons. First, $\frac{p+2}{p}$ is just barely out of reach of Weiner's 2000 result, since $p+2 = \sigma(p)+1$ for all primes p. Second, as we have already mentioned, if $\frac{5}{3} = \frac{3+2}{3}$ is an abundancy index then there exists an odd perfect number.

It seems to be just as difficult to find a solution to the above equation as it is to show that no such x exists. R. Ryan reports that $I(x) = \frac{p+2}{p}$ has no solution for $x < 10^{16}$ [4]. The investigations that led us to Theorems 3.2 and 4.2 were motivated in part by a desire to show that equation 5.1 has no solutions. However, $\frac{p+2}{p}$ has proven to be an elusive fraction. The techniques we have employed in this paper cannot be applied to it. The difficulty lies in the fact that

$$\frac{p+2}{p} > \frac{p}{p-1} > I(p^\alpha)$$

for all primes p and natural numbers α. Thus, one can never "trap" $\frac{p+2}{p}$ between two numbers:

$$I(p^\alpha N) < \frac{p+2}{p} < I(p^{\alpha+1}N).$$

Hence, Theorems 3.2 and 4.2 fail to capture this potential outlaw. In fact, proving that $\frac{p+2}{p}$ is an abundancy outlaw seems to require the discovery of an entirely new category of abundancy outlaws.

There are also many interesting questions to consider relating to the size of these sets of abundancy outlaws, in the sense of asymptotic density. Based on some preliminary computer experiments, the proportion of outlaws captured by Theorem 4.2 seems to approach 1.7 percent. See the appendix for empirical data. Our future work will involve a deeper exploration of the sizes of the sets of outlaws, indices, and rationals of unknown status.

7 Appendix

The following is a table containing empirical data relating to the asymptotic density of the set of outlaws captured by Theorem 3.4. The second and third columns give the number of outlaws with numerators less than or equal to n captured by Theorem 3.4 and Proposition 2.3, respectively. The fourth column gives the number of rationals in the image of the abundancy index of the first one million natural numbers with numerators less than or equal to n. The fifth column gives the total number of rationals with numerators less than or equal to n, and the sixth column gives the value of column 2 divided by column 5. In other words, the sixth column gives the proportion of outlaws captured by Theorem 3.4 in the set of rationals with numerator less than or equal to n.

n	Thm. 3.4	Prop. 2.3	Abundancy Index	Total rationals	Proportion
10	0	4	20	32	0
100	44	720	553	3044	0.0145
1000	5170	74927	7803	304192	0.01700
10000	518193	750174	62064	30397486	0.01705

Table 7.1: A table of empirical data on the asymptotic densities of the abundancy outlaws and the abundancy indices.

As a way to visualize the distribution of abundancy outlaws, we include a list of the rationals with numerators less than or equal to 100 (under the ordering described in the introduction). Each rational number q is colored according to its abundancy index/outlaw status:

- Blue: q is in the image of the abundancy index of the first one million natural numbers, or a natural number from 2 to 11 (the abundancy index of a known multiperfect number)
- Green: q is an outlaw captured by Property 2.3.
- Red: q is an outlaw captured by Theorem 4.2.
- Black: q is not in any of the other categories (the outlaw status of q is "unknown")

In the following list, 450 rationals are blue, 720 are green, 44 are red, and 1961 are black.

2, 3, 3/2, 4, 4/3, 5, 5/2, 5/3, 5/4, 6, 6/5, 7, 7/2, 7/3, 7/4, 7/5, 7/6, 8, 8/3, 8/5, 8/7, 9, 9/2, 9/4, 9/5, 9/7, 9/8, 10, 10/3, 10/7, 10/9, 11, 11/2, 11/3, 11/4, 11/5, 11/6, 11/7, 11/8, 11/9, 11/10, 12, 12/5, 12/7, 12/11, 13, 13/2, 13/3, 13/4, 13/5, 13/6, 13/7, 13/8, 13/9, 13/10, 13/11, 13/12, 14, 14/3, 14/5, 14/9, 14/11, 14/13, 15, 15/2, 15/4, 15/7, 15/8, 15/11, 15/13, 15/14, 16, 16/3, 16/5, 16/7, 16/9, 16/11, 16/13, 16/15, 17, 17/2, 17/3, 17/4, 17/5, 17/6, 17/7, 17/8, 17/9, 17/10, 17/11, 17/12, 17/13, 17/14, 17/15, 17/16, 18, 18/5, 18/7, 18/11, 18/13, 18/17, 19, 19/2, 19/3, 19/4, 19/5, 19/6, 19/7, 19/8, 19/9, 19/10, 19/11, 19/12, 19/13, 19/14, 19/15, 19/16, 19/17, 19/18, 20, 20/3, 20/7, 20/9, 20/11, 20/13, 20/17, 20/19, 21, 21/2, 21/4, 21/5, 21/8, 21/10, 21/11, 21/13, 21/16, 21/17, 21/19, 21/20, 22, 22/3, 22/5, 22/7, 22/9, 22/13, 22/15, 22/17, 22/19, 22/21, 23, 23/2, 23/3, 23/4, 23/5, 23/6, 23/7, 23/8, 23/9, 23/10, 23/11, 23/12, 23/13, 23/14, 23/15, 23/16, 23/17, 23/18, 23/19, 23/20, 23/21, 23/22, 24, 24/5, 24/7, 24/11, 24/13, 24/17, 24/19, 24/23, 25, 25/2, 25/3, 25/4, 25/6, 25/7, 25/8, 25/9, 25/11, 25/12, 25/13, 25/14, 25/16, 25/17, 25/18, 25/19, 25/21, 25/22, 25/23, 25/24, 26, 26/3, 26/5, 26/7, 26/9, 26/11, 26/15, 26/17, 26/19, 26/21, 26/23, 26/25, 27, 27/2, 27/4, 27/5, 27/7, 27/8, 27/10, 27/11, 27/13, 27/14, 27/16, 27/17, 27/19, 27/20, 27/22, 27/23, 27/25, 27/26, 28, 28/3, 28/5, 28/9, 28/11, 28/13, 28/15, 28/17, 28/19, 28/23, 28/25, 28/27, 29, 29/2, 29/3, 29/4, 29/5, 29/6, 29/7, 29/8, 29/9, 29/10, 29/11, 29/12, 29/13, 29/14, 29/15, 29/16, 29/17, 29/18, 29/19, 29/20, 29/21, 29/22, 29/23, 29/24, 29/25, 29/26, 29/27, 29/28, 30, 30/7, 30/11, 30/13, 30/17, 30/19, 30/23, 30/29, 31, 31/2, 31/3, 31/4, 31/5, 31/6, 31/7, 31/8, 31/9, 31/10, 31/11, 31/12, 31/13, 31/14, 31/15, 31/16, 31/17, 31/18, 31/19, 31/20, 31/21, 31/22, 31/23, 31/24, 31/25, 31/26, 31/27, 31/28, 31/29, 31/30, 32, 32/3, 32/5, 32/7, 32/9, 32/11, 32/13, 32/15, 32/17, 32/19, 32/21, 32/23, 32/25, 32/27, 32/29, 32/31, 33, 33/2, 33/4, 33/5, 33/7, 33/8, 33/10, 33/13, 33/14, 33/16, 33/17, 33/19, 33/20, 33/23, 33/25, 33/26, 33/28, 33/29, 33/31, 33/32, 34, 34/3, 34/5, 34/7, 34/9, 34/11, 34/13, 34/15, 34/19, 34/21, 34/23, 34/25, 34/27, 34/29, 34/31, 34/33, 35, 35/2, 35/3, 35/4, 35/6, 35/8, 35/9, 35/11, 35/12, 35/13, 35/16, 35/17, 35/18, 35/19, 35/22, 35/23, 35/24, 35/26, 35/27, 35/29, 35/31, 35/32, 35/33, 35/34, 36, 36/5, 36/7, 36/11, 36/13, 36/17, 36/19, 36/23, 36/25, 36/29, 36/31, 36/35, 37, 37/2, 37/3, 37/4, 37/5, 37/6, 37/7, 37/8, 37/9, 37/10, 37/11, 37/12, 37/13, 37/14, 37/15, 37/16, 37/17, 37/18, 37/19, 37/20, 37/21, 37/22, 37/23, 37/24, 37/25, 37/26, 37/27, 37/28, 37/29, 37/30, 37/31, 37/32, 37/33, 37/34, 37/35, 37/36, 38, 38/3, 38/5, 38/7, 38/9, 38/11, 38/13, 38/15, 38/17, 38/21, 38/23, 38/25, 38/27, 38/29, 38/31, 38/33, 38/35, 38/37, 39, 39/2, 39/4, 39/5, 39/7, 39/8, 39/10, 39/11, 39/14, 39/16, 39/17, 39/19, 39/20, 39/22, 39/23, 39/25, 39/28, 39/29, 39/31, 39/32, 39/34, 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50/37, 50/39, 50/41, 50/43, 50/47, 50/49, 51, 51/2, 51/4, 51/5, 51/7, 51/8, 51/10, 51/11, 51/13, 51/14, 51/16, 51/19, 51/20, 51/22, 51/23, 51/25, 51/26, 51/28, 51/29, 51/31, 51/32, 51/35, 51/37, 51/38, 51/40, 51/41, 51/43, 51/44, 51/46, 51/47, 51/49, 51/50, 52, 52/3, 52/5, 52/7, 52/9, 52/11, 52/15, 52/17, 52/19, 52/21, 52/23, 52/25, 52/27, 52/29, 52/31, 52/33, 52/35, 52/37, 52/41, 52/43, 52/45, 52/47, 52/49, 52/51, 53, 53/2, 53/3, 53/4, 53/5, 53/6, 53/7, 53/8, 53/9, 53/10, 53/11, 53/12, 53/13, 53/14, 53/15, 53/16, 53/17, 53/18, 53/19, 53/20, 53/21, 53/22, 53/23, 53/24, 53/25, 53/26, 53/27, 53/28, 53/29, 53/30, 53/31, 53/32, 53/33, 53/34, 53/35, 53/36, 53/37, 53/38, 53/39, 53/40, 53/41, 53/42, 53/43, 53/44, 53/45, 53/46, 53/47, 53/48, 53/49, 53/50, 53/51, 53/52, 54, 54/5, 54/7, 54/11, 54/13, 54/17, 54/19, 54/23, 54/25, 54/29, 54/31, 54/35, 54/37, 54/41, 54/43, 54/47, 54/49, 54/53, 55, 55/2, 55/3, 55/4, 55/6, 55/7, 55/8, 55/9, 55/12, 55/13, 55/14, 55/16, 55/17, 55/18, 55/19, 55/21, 55/23, 55/24, 55/26, 55/27, 55/28, 55/29, 55/31, 55/32, 55/34, 55/36, 55/37, 55/38, 55/39, 55/41, 55/42, 55/43, 55/46, 55/47, 55/48, 55/49, 55/51, 55/52, 55/53, 55/54, 56, 56/3, 56/5, 56/9, 56/11, 56/13, 56/15, 56/17, 56/19, 56/23, 56/25, 56/27, 56/29, 56/31, 56/33, 56/37, 56/39, 56/41, 56/43, 56/45, 56/47, 56/51, 56/53, 56/55, 57, 57/2, 57/4, 57/5, 57/7, 57/8, 57/10, 57/11, 57/13, 57/14, 57/16, 57/17, 57/20, 57/22, 57/23, 57/25, 57/26, 57/28, 57/29, 57/31, 57/32, 57/34, 57/35, 57/37, 57/40, 57/41, 57/43, 57/44, 57/46, 57/47, 57/49, 57/50, 57/52, 57/53, 57/55, 57/56, 58, 58/3, 58/5, 58/7, 58/9, 58/11, 58/13, 58/15, 58/17, 58/19, 58/21, 58/23, 58/25, 58/27, 58/31, 58/33, 58/35, 58/37, 58/39, 58/41, 58/43, 58/45, 58/47, 58/49, 58/51, 58/53, 58/55, 58/57, 59, 59/2, 59/3, 59/4, 59/5, 59/6, 59/7, 59/8, 59/9, 59/10, 59/11, 59/12, 59/13, 59/14, 59/15, 59/16, 59/17, 59/18, 59/19, 59/20, 59/21, 59/22, 59/23, 59/24, 59/25, 59/26, 59/27, 59/28, 59/29, 59/30, 59/31, 59/32, 59/33, 59/34, 59/35, 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