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A Parametrization of Equilateral Triangles Having Integer Coordinates

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Abstract

We study the existence of equilateral triangles of given side lengths and with integer coordinates in dimension three. We show that such a triangle exists if and only if their side lengths are of the form $\sqrt{2(m^2-mn+n^2)}$ for some integers m, n. We also show a similar characterization for the sides of a regular tetrahedron in $\mathbb{Z}^3$: such a tetrahedron exists if and only if the sides are of the form $k\sqrt{2}$, for some $k \in \mathbb{N}$. The classification of all the equilateral triangles in $\mathbb{Z}^3$ contained in a given plane is studied and the beginning analysis for small side lengths is included. A more general parametrization is proven under special assumptions. Some related questions about the exceptional situation are formulated in the end.

1 Introduction

It is known that there is no equilateral triangle whose vertices have integer coordinates in the plane. One can easily see this by calculating the area of such a triangle of side length l using the formula Area $= \frac{l^2\sqrt{3}}{4}$ and by using Pick's theorem for the area of a polygon with vertices of integer coordinates: Area $= \frac{\sharp b}{2} + \sharp i - 1$ where $\sharp b$ is the number of points of integer coordinates on the boundary of the polygon and $\sharp i$ is the number of such points in the interior of the polygon. Since Pick's theorem implies that this area is a rational number of square units, the formula Area $= \frac{l^2\sqrt{3}}{4}$ says that this is a rational multiple of $\sqrt{3}$ since l^2

$O(0,0,0)$	$A_1(9,9,0)$	$B_1(9,0,9)$	$C_1(0,9,9)$
$O(0,0,0)$	$A_2(-9,9,0)$	$B_2(-4,5,-11)$	$C_2(3,12,-3)$
$O(0,0,0)$	$A_3(12,3,-3)$	$B_3(7,-8,-7)$	$C_3(3,3,-12)$

Table 1: Coordinates of the three regular tetrahedra in Figure 1

must be a positive integer by the Pythagorean theorem. This contradiction implies that no such triangle exists.

The analog of this fact in three dimensions is not true since one can form a regular tetrahedron by taking as vertices the points $O(0,0,0)$, $A(1,1,0)$, $B(1,0,1)$ and $C(0,1,1)$. It turns out that the sides of such regular tetrahedra have to be of the form $k\sqrt{2}$, $k \in \mathbb{N}$. Moreover and as a curiosity, one can use the facts derived in this note to show that there are only three regular tetrahedra in $\mathbb{Z}^3$ having the origin as one of their vertices and of side lengths $9\sqrt{2}$, where the counting has been done up to symmetries of the cube. In the figure below, that we generated with Maple, we show three regular tetrahedra that together with all their cube symmetries fill out the class just described.

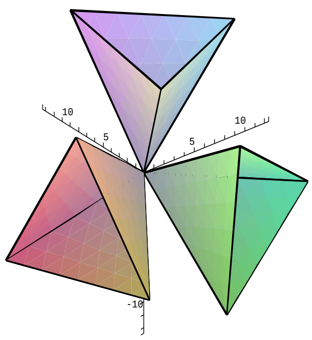

Figure 1: Regular tetrahedra of side lengths $9\sqrt{2}$

Along these lines, we mention the following related result of Schoenberg, [3], who proved that a regular n-simplex exists in $\mathbb{Z}^n$ in the following cases and no others:

(i) n is even and $n+1$ is a square;

(ii) $n \equiv 3 \pmod 4$;

(iii) $n \equiv 1 \pmod 4$ and $n+1$ is the sum of two squares.

In conjunction to the results of this note, Schoenberg's characterization opens more questions such as: what are the corresponding parameterizations in all these cases when regular n-simplexes do exist? Or, what is the similar characterization for the regular n-simplex side lengths?

n	1	2	3	4	5	6	7	8	9	10
$\mathcal{ET}(n)$	8	80	368	1264	3448	7792	16176	30696	54216	90104

Table 2: Sequence A 102698

Equilateral triangles with vertices of integer coordinates in the three-dimensional space are numerous as one could imagine from the situation just described. One less obvious example is the triangle CDO with $C(31, 19, 76)$ and $D(44, 71, 11)$ having side lengths equal to $13\sqrt{42}$. Generating all such triangles is a natural problem and we may start with one such triangle and then apply the group of affine transformations $T_{\alpha,O,\overrightarrow{y}}(\overrightarrow{x}) = \alpha O(\overrightarrow{x}) + \overrightarrow{y}$ where O is an orthogonal matrix with rational coefficients, $\alpha \in \mathbb{Z}$, and $\overrightarrow{y}$ a vector in $\mathbb{Z}^3$. Naturally, we may obtain triangles that have certain length sides but not all desired triangles can be obtained this way. Indeed, such a transformation multiplies the side lengths with the factor α and so for instance the triangle OAB with side lengths $\sqrt{2}$ cannot be transformed this way into the triangle CDO. We are interested in parametrizations that encompasses all equilateral triangles in $\mathbb{Z}^3$ that are contained in the same plane. This requires further restrictions on the type of transformations $T_{\alpha,O,\overrightarrow{y}}$.

In fact, in the next section we show that the side lengths which appear from such equilateral triangles are of the form $\sqrt{2N(\zeta)}$ where ζ is an Eisenstein-Jacobi integer and $N(\zeta)$ is its norm. The Eisenstein-Jacobi integers are defined as $\mathbb{Z}[\omega]$ where ω is a primitive cubic root of unity, i.e., the complex numbers of the form $\zeta = m - n\omega$ with $m, n \in \mathbb{Z}$ with their norm given by $N(\zeta) = m^2 - mn + n^2$.

What makes the existence of such triangles work in space that does not work in two dimensions? We show in Proposition 2 that the plane containing such a triangle must have a normal vector $\overrightarrow{n} = a\overrightarrow{i} + b\overrightarrow{j} + c\overrightarrow{k}$ where a, b, c are integers that satisfy the Diophantine equation

$$a^2 + b^2 + c^2 = 3d^2, \ d \in \mathbb{Z}. \tag{1}$$

This equation has no non-trivial solutions when $abc = 0$, according to Gauss' characterization for the numbers that can be written as sums of two perfect squares.

Our study of the existence of such triangles started with an American Mathematics Competition problem in the beginning of 2005. The problem was stated as follows:

Problem 1. *Determine the number of equilateral triangles whose vertices have coordinates in the set* $\{0, 1, 2\}$.

It turns out that the answer to this question is 80. Let us introduce the notation $\mathcal{ET}(n)$ for the number of equilateral triangles whose vertices have coordinates in the set $\{0, 1, 2, \ldots, n\}$ for $n \in \mathbb{N}$. Some of the values of $\mathcal{ET}(n)$ are tabulated in Table 2.

This sequence was entered in the on-line Encyclopedia of Integer Sequences [5] by Joshua Zucker on February 4th, 2005. The first 34 terms in this sequence were calculated by Hugo Pfoertner using a program in Fortran (private communication). Our hope is that the results obtained here may be used in designing a program that could calculate $\mathcal{ET}(n)$ for significantly more values of n.

2 Planes containing equilateral triangles in $\mathbb{Z}^3$

Let us denote the side lengths of an equilateral triangle $\triangle OPQ$ by l. We are going to discard translations, so we may assume that one of the vertices of such a triangle is $O(0,0,0)$. If the other two points, P and Q, have coordinates (x,y,z) and (u,v,w) respectively, then as we have seen before the area of $\triangle OPQ$ is given by

$$\text{Area} = \frac{l^2\sqrt{3}}{4} = \frac{1}{2}|\overrightarrow{OP} \times \overrightarrow{OQ}| = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ u & v & w \end{vmatrix}. \tag{2}$$

This implies the following simple proposition but essential in our discussion:

Proposition 2. *Assume the triangle $\triangle OPQ$ is equilateral and its vertices have integer coordinates with O the origin and $l = ||\overrightarrow{OP}||$. Then the points P and Q are contained in a plane of equation $a\alpha + b\beta + c\gamma = 0$, where $a, b, c,$ and d are integers which satisfy (1) and $l^2 = 2d$.*

Proof. Assume the coordinates of P and Q are denoted as before. Let us observe that $l^2 = ||\overrightarrow{OP} - \overrightarrow{OQ}||^2 = (x-u)^2 + (y-v)^2 + (z-w)^2 = 2l^2 - 2(xu + yv + zw)$ which implies $xu + yv + zw = \frac{l^2}{2} = d \in \mathbb{Z}$. Then using the fact that $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ are contained in the plane orthogonal on the vector $\overrightarrow{OP} \times \overrightarrow{OQ} = a\vec{i} + b\vec{j} + c\vec{k}$ with $a = yw - vz$, $b = zu - xw$ and $c = xv - yu$ the statement follows from (2). $\square$

The equation (1) has infinitely many integer solutions besides the obvious ones $a = \pm d$, $b = \pm d$, $c = \pm d$. For instance we can take $a = -19$, $b = 11$, $c = 5$ and $d = 13$ and the triangle OCD given in the Introduction has C and D in the plane $\{(\alpha,\beta,\gamma) \in \mathbb{R}^3 | -19\alpha + 11\beta + 5\gamma = 0\}$. For the purpose of computing $\mathcal{ET}(n)$ one has to consider all the planes of form below although we are going to concentrate only on those that contain the origin.

Definition 3. *Let us consider the set $\mathcal{P}$ of all planes $\{(\alpha,\beta,\gamma) \in \mathbb{R}^3 |\ a\alpha + b\beta + c\gamma = e\}$, such that $a^2 + b^2 + c^2 = 3d^2$ for some $a, b, c, d, e \in \mathbb{Z}$ and $\gcd(a,b,c) = 1$.*

So, if one starts with a plane π in $\mathcal{P}$, picks two points of integer coordinates that belong to π, which amounts to solving a simple linear Diophantine equation, the natural question is weather or not there exists a third point of integer coordinates, contained in π, that completes the picture to an equilateral triangle (see Figure 2, where O and P are the chosen points and the third point is denoted here by Q_+ or Q_- since there are two possible such candidates). In order for the third point to exist one needs to take the first two points in a certain way. But if one requires only that the new point have rational coordinates it turns out that this is always possible and the next theorem gives a way to find the coordinates of the third point in terms of the given data.

Theorem 4. *Assume that $P(u,v,w)$ $(u,v,w \in \mathbb{Q})$ is an arbitrary point of a plane $\pi \in \mathcal{P}$ of normal vector (a,b,c) and passing through the origin O. Then the coordinates of a point $Q(x,y,z)$ situated in π and such that the triangle $\triangle OPQ$ is equilateral are all rational numbers given by:*

$$\begin{cases} x = \dfrac{u}{2} \pm \dfrac{cv-bw}{2d} \\ y = \dfrac{v}{2} \pm \dfrac{aw-cu}{2d} \\ z = \dfrac{w}{2} \pm \dfrac{bu-av}{2d}. \end{cases} \tag{3}$$

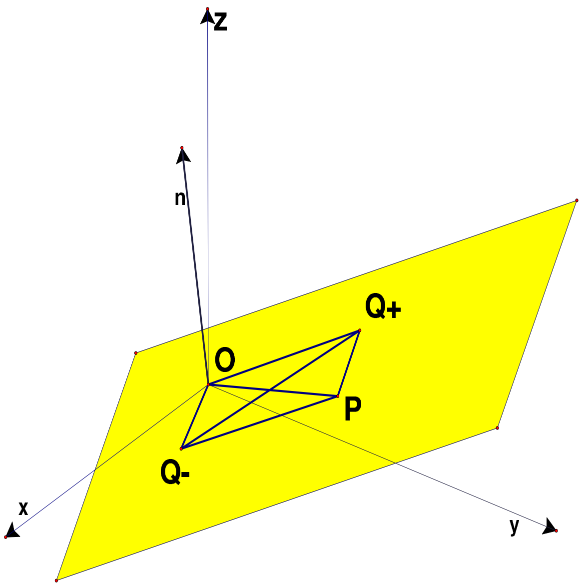

Figure 2: Plane of normal (a,b,c)

Proof. From the geometric interpretation of the problem we see that a point $Q \in \pi$ such that $\triangle OQP$ becomes equilateral is at the intersection of plane π and two spheres of radius OP and centers O and P. There are exactly two points with this property lying on a segment perpendicular to OP and passing through its midpoint. We want to show that one point is given by taking the plus sign in all equalities in (3) and the other point corresponds to the minus sign in all equalities in (3). We are going to set $\overrightarrow{n} = \dfrac{1}{d\sqrt{3}}(a,b,c)$ which is one of the two unit vectors normal to the plane π and let $\overrightarrow{r} = \overrightarrow{OP} = (u,v,w)$. Then the cross product $\overrightarrow{r} \times \overrightarrow{n}$ is given by

$$\overrightarrow{r} \times \overrightarrow{n} = \frac{1}{d\sqrt{3}}(cv-bw, aw-cu, bu-av).$$

So we observe that the solution (x,y,z) is, in fact, if written in vector notation, $\overrightarrow{OQ_{\pm}} = \frac{1}{2}\overrightarrow{r} \pm \frac{\sqrt{3}}{2}\overrightarrow{r} \times \overrightarrow{n}$. It is easy now to check that $\overrightarrow{r}$ and $\overrightarrow{OQ_{\pm}}$ have the same norm and make a $60°$ angle in between. $\square$

3 Solutions of the Diophantine equation $a^2+b^2+c^2 = 3d^2$

We have enough evidence to believe that for each (a, b, c) satisfying (1) for some $d \in \mathbb{Z}$, there are infinitely many equilateral triangles in $\mathbb{Z}^3$ that belong to a plane of normal vector (a, b, c) and the purpose of this paper is to determine a way to generate all these triangles. But how many planes do we have in $\mathcal{P}$? Let us observe that if d is even, then not all of a, b, c can be odd integers, so at least one of them must be even. Then the sum of the other two is a number divisible by 4 which is possible only if they are also even. Therefore we may reduce all numbers by a factor of two in this case. So, if we assume without loss of generality that $\gcd(a, b, c) = 1$, then such solutions of (1) must have d an odd integer and then this forces that a, b and c must be all odd integers too. If in addition, disregarding the signs, we have $a, b, c \in \mathbb{N}$ and $a \le b \le c$, then such a solution will be referred to as a *primitive* solution of (1). One can find lots of solutions of (1) in the following way.

Proposition 5. *(i) The following formulae give a three integer parameter solution of (1):*

$$\begin{cases} a = -x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 - 2x_1x_3 \\ b = x_1^2 - x_2^2 + x_3^2 - 2x_xx_1 - 2x_2x_3 \\ c = x_1^2 + x_2^2 - x_3^2 - 2x_3x_1 - 2x_3x_2 \\ d = x_1^2 + x_2^2 + x_3^2 \end{cases}, \quad x_1, x_2, x_3 \in \mathbb{Z}. \tag{4}$$

(ii) Every nontrivial primitive solution of (1) is of the form (4) with $x_1, x_2, x_3 \in \mathbb{Z}[\frac{1}{\sqrt{k}}]$ with $k = (3d - a - b - c)/2 \in \mathbb{N}$.

Proof. For the first part of this proposition, one can check that a, b, c and d given by (4) satisfy (1). For the second, we assume that (a, b, c, d) is a primitive solution of (1) not equal to $(1, 1, 1, 1)$. Then we can introduce $x_1 = (d - a)t$, $x_2 = (d - b)t$, $x_3 = (d - c)t$ where $t \in \mathbb{R}$ such that $x_1^2 + x_2^2 + x_3^2 = d$. This gives

$$t^2 = \frac{d}{(d-a)^2 + (d-b)^2 + (d-c)^2} = \frac{d}{6d^2 - 2d(a+b+c)} =$$

$$\frac{1}{2(3d - a - b - c)} = \frac{1}{2}\frac{t}{x_1 + x_2 + x_3},$$

and so $t = \frac{1}{2s}$ with $s = x_1 + x_2 + x_3 = \sqrt{(3d - a - b - c)/2}$. Every primitive solution of (1) must have a, b, c, and d all odd numbers as we have observed. This makes $k = (3d-a-b-c)/2$ an integer. Since $3d > a+b+c$ is equivalent to $3(a^2+b^2+c^2)-(a+b+c)^2 = (a-b)^2+(b-c)^2 + (a-c)^2 > 0$, it follows that $k \in \mathbb{N}$. Then, for instance, $x_1 = \frac{d-a}{2}\frac{1}{\sqrt{k}} \in \mathbb{Z}[\frac{1}{\sqrt{k}}]$ because $d - a$ is even. Therefore, in general, every nontrivial primitive solution of (1) is of the form (4) with $x_1, x_2, x_3 \in \mathbb{Z}[\frac{1}{\sqrt{k}}]$. $\square$

Example: The following example shows that (4) does not cover all solutions of (1). Suppose $a = 5$, $b = 11$, $c = 19$ and $d = 13$. As we have seen in the proof of Proposition 5, x_1, x_2, x_3 in (4) are uniquely determined: $x_1 = \frac{4}{\sqrt{2}}$, $x_2 = \frac{1}{\sqrt{2}}$ and $x_3 = -\frac{3}{\sqrt{2}}$.

d	(a,b,c)	d	(a,b,c)
1	$\{(1,1,1)\}$	9	$\{(1,11,11),(5,7,13)\}$
3	$\{(1,1,5)\}$	11	$\{(1,1,19),(5,7,17),(5,13,13)\}$
5	$\{(1,5,7)\}$	13	$\{(5,11,19),(7,13,17)\}$
7	$\{(1,5,11)\}$	15	$\{(1,7,25),(5,11,25),(5,17,19)\}$

Table 3: Primitive solutions of (1)

Although formulae (4) provide infinitely many solutions of (1), the following proposition brings additional information about its integer solutions.

Proposition 6. *The equation (1) has non-trivial solutions for every odd integer $d \geq 3$.*

Proof. If $d = 2p + 1$ for some $p \in \mathbb{Z}$, $p \geq 1$, then $3d^2 = 3[4p(p + 1) + 1] = 8l + 3$ which shows that $3q^2 \equiv 3 \pmod 8$. From Gauss's Theorem about the number of representations of a number as a sum of three squares ([2, Thm. 2, p. 51]), we see that the number of representations of $3d^2$ as a sum of three squares is at least 24. Here, the change of signs is counted so each solution actually generates eight solutions by just changing the signs. Also, the six permutations of a, b and c get into the counting process. So, there must be at least one solution which is nontrivial since the solution $3d^2 = d^2 + d^2 + d^2$ generates only eight solutions by using all possible change of signs. $\square$

Remark: One can generate an infinite family of solutions of (1) by reducing it two separate equations, say for example $146 = 3d^2 - c^2$ and $a^2 + b^2 = 146$. The second equation admits as solution, for instance, $a = 11$ and $b = 5$. The first equation has a particular solution $d = 7$ and $c = 1$. Then one can use the recurrence formulae to obtain infinitely many solutions of $146 = 3d^2 - c^2$:

$$d_{n+1} = 2d_n + c_n, \quad c_{n+1} = 3d_n + 2c_n \ for\ n \in \mathbb{N}$$

and $d_1 = 7$, $c_1 = 1$. A simple calculation shows that $3d_{n+1}^2 - c_{n+1}^2 = 3d_n^2 - c_n^2$ so, by induction, (d_n, c_n) is a solution of the equation $3d^2 - c^2 = 146$ for all $n \in \mathbb{N}$. It is easy to see that q_n and c_n are increasing sequences so this procedure generates infinitely many solutions of (1).

The primitive solutions of (1) for small values of d are included in the Table 3.

As a curiosity the number of primitive representations as in (1) corresponding to $d = 2007$ is 333.

4 The first six parametrizations

The simplest solution of (1) is $a = b = c = d = 1$. We are going to introduce some more notation here before we give the parametrization for this case.

Definition 7. *For every (a, b, c), a primitive solution of (1), denote by $\mathcal{T}_{a,b,c}$ the set of all equilateral triangles with integer coordinates having the origin as one of the vertices and the other two lie in the plane $\{(\alpha, \beta, \gamma) \in \mathbb{R}^3 | a\alpha + b\beta + c\gamma = 0\}$.*

It is clear now that in light of Proposition 2, every equilateral triangle having integer coordinates after a translation, interchange of coordinates, or maybe a change of signs of some of the coordinates, belongs to one of the classes $\mathcal{T}_{a,b,c}$. Let us introduce also the notation $\mathcal{T}$ for all the equilateral triangles in $\mathbb{Z}^3$. If we have different values for a, b and c, how many different planes can one obtain by permuting a, b and c in between and changing their signs? That will be 6 permutations and essentially 4 change of signs (note that $ax+by+cz = 0$ is the same plane as $(-a)x+(-b)y+(-c)z = 0$) which gives a total of 24 such transformations. We are going to denote the group of symmetries of the space determined by these transformations and leave the origin fixed, by $\mathcal{S}_{cube}$ (it is actually the group of symmetries of the cube). Hence we have

$$\mathcal{T} = \bigcup_{\substack{s \in \mathcal{S}_{cube}, s(O) = O, \\ a^2+b^2+c^2 = 3d^2 \\ 0 < a \leq b \leq c, \gcd(a,b,c) = 1 \\ a,b,c,d \in \mathbb{Z}, v \in \mathbb{Z}^3}} s\left(\mathcal{T}_{a,b,c}\right) + v. \tag{5}$$

Theorem 8. *Every triangle $OAB \in \mathcal{T}_{1,1,1}$ is of the form $\{A, B, O\} = \{(m, -n, n-m), (m-n, -m, n), (0, 0, 0)\}$ for some $m, n \in \mathbb{Z}$. The side lengths of $\triangle OAB_{m,n}$ are given by*

$$l = \sqrt{2(m^2 - mn + n^2)}.$$

Proof. Let us assume A has coordinates (u, v, w) with $u + v + w = 0$ and $B(x, y, z)$ with $x + y + z = 0$. From (3) we get that $x = \frac{u}{2} + \frac{v-w}{2}$, $y = \frac{v}{2} + \frac{w-u}{2}$, $z = \frac{v}{2} + \frac{u-v}{2}$ if we choose the plus signs. This choice is without loss of generality since we can interchange the roles of A and B if necessary. This implies $x = -w$, $y = -u$ and $z = -v$. So, if we denote $u = m$, $v = -n$ then $w = n - m$ and so $x = m + n$, $y = -m$, $z = -n$. □

We are introducing the notation $N(m, n) = 2(m^2 - mn + n^2)$ for $m, n \in \mathbb{Z}$. For the next cases, $d \in \{3, 5, 7\}$, as we have recorded in the Table 3, $3d^2$ has also a unique primitive representation. One can use basically the same technique as in the proof of Theorem 8 to derive the corresponding parameterizations for the vertices in $\mathcal{T}_{a,b,c}$ ($d \in \{3, 5, 7\}$) and the corresponding side lengths but for each individual set of formulae, that are given below, we had something specific to work out in order to get rid of denominators that naturally arise from applying (3):

$$d = 3,\ l = 3\sqrt{N(m,n)}$$
$$\mathcal{T}_{1,1,5} = \{[O, (4m-3n, m+3n, -m), (3m+n, -3m+4n, -n)] : m, n \in \mathbb{Z}, l \neq 0\},$$

$$d = 5,\ l = 5\sqrt{N(m,n)}$$
$$\mathcal{T}_{1,5,7} = \{[O, (7m-4n, 5n, -m-3n), (3m-7n, 5m, -4m+n)] : m, n \in \mathbb{Z}, l \neq 0\},$$

$$d = 7,\ l = 7\sqrt{N(m,n)},\ \mathcal{T}_{1,5,11} = \{[O, (8m-9n, 5m+4n, -3m-n), (-m-8n, 9m-5n, -4m+3n)] : m, n \in \mathbb{Z}, l \neq 0\}.$$

Remark: Every triangle in one particular family, $s(\mathcal{T}_{a,b,c}) + v$, is different of all the other triangles in other families since they live in different planes. So if we take in (5) only the symmetries $s \in \mathcal{S}_{cube}$, $s(O) = O$, that give different normal vectors (two of the numbers a, b, c may be equal) then (5) is a partition of $\mathcal{T}$.

The case $d = 9$ is the first in which there are two essentially different primitive representations of $3d^2$: $3(9)^2 = 1^2 + 11^2 + 11^2 + 1 = 5^2 + 7^2 + 13^2$. The corresponding parameterizations are included below:

$$d = 9,\ l = 9\sqrt{N(m,n)}, \mathcal{T}_{1,11,11} = \{[O, (11m-11n, 4m+5n, -5m-4n), (-11n, 9m-4n, -9m+5n)] : m, n \in \mathbb{Z}, l \neq 0\},$$

$$\mathcal{T}_{5,7,13} = \{[O, (7m+5n, 8m-11n, -7m+4n), (12m-7n, -3m-8n, -3m+7n)] : m, n \in \mathbb{Z}, l \neq 0\}.$$

To give an idea of how we obtained these parametrizations we will include the proof of the case $d = 9$, $a = 5$, $b = 7$, $c = 13$, that gave us the last of the above formulae.

Proof. Assume that one of the points, P, has coordinates (u, v, w). If one solves the Diophantine equation $5u + 7v + 13w = 0$ finds that a general solution may be written as

$$\begin{cases} w = 5u + 7t, \\ v = -10u - 13t \ \textit{and}\ t, w \in \mathbb{Z}. \end{cases}$$

Using Theorem 4 we see that the coordinates of a point Q, say (x, y, z), such that $\triangle OPQ \in \mathcal{T}_{5,7,13}$ must be given by (3). Switching P with Q, if necessary, we may take all plus signs in (3). This gives

$$\begin{cases} x = -\dfrac{26u}{3} - \dfrac{109t}{9}, \\[2ex] y = -\dfrac{13u}{3} - \dfrac{41t}{9}, \\[2ex] z = \dfrac{17u}{3} + \dfrac{64t}{9}. \end{cases}$$

Since $x = -9u - 12t - \frac{t-3u}{9}$ must be an integer we need to have $t = 3u + 9g$ for some $g \in \mathbb{Z}$. Substituting we find that all other coordinates are automatically integers: $x = -45u - 109g$, $y = -18u - 41g$, $z = 27u + 64g$. Calculating $l^2 = u^2 + v^2 + w^2$ we get $l^2 = 3078u^2 + 14742ug + 17658g^2 = 2(9)^2(19u^2 + 91ug + 109g^2)$. Or $l^2 = 2(9)^2[(2u+5g)^2 + (2u+5g)(3u+7g) + (3u+7g)^2]$ which suggests that we can change the variables $2u + 5g = -m$ and $3u + 7g = n$ to obtain the statement from the above parametrization. If we solve this system for u and g it turns out that the solution preserves integers values since $u = 7m + 5n$ and $g = -3m - 2n$. $\square$

A natural question that we may ask at this point is whether or not every $\mathcal{T}_{a,b,c}$ admits such a parametrization. In the next section we prove that this is indeed the case under the assumption that $\min\{\gcd(a,d), \gcd(b,d), \gcd(c,d)\} = 1$. However this is not always the case. In the last section we address the existence of solutions to (1) for which $\min\{\gcd(a,d), \gcd(b,d), \gcd(c,d)\} > 1$.

5 Characterization of side lengths

We will begin with two preliminary results. The first we just need to recall it since it is a known fact that can be found in number theory books mostly as an exercise or as an implicit corollary of more general theorems about quadratic forms or Euler's $6k+1$ theorem (see [4], pp. 568 and [1], pp. 56).

Proposition 9. *An integer t can be written as $m^2 - mn + n^2$ for some $m, n \in \mathbb{Z}$ if and only if in the prime factorization of t, 2 and the primes of the form $6k - 1$ appear to an even exponent.*

The next lemma is probably also known in algebraic number theory but we do not have straight reference for it so we are going to include a proof of it.

Lemma 10. *An integer t which can be written as $t = 3x^2 - y^2$ with $x, y \in \mathbb{Z}$ is the sum of two squares if and only if t is of the form $t = 2(m^2 - mn + n^2)$ for some integers m and n.*

Proof. . For necessity, by Proposition 9, we have to show that t is even and $t/2$ does not contain in its prime factor decomposition any of the primes 2 or those of the form $6k - 1$ except to an even power. First, let us show that t must be even and the exponent of 2 in its prime factorization is odd. Since $t = 3x^2 - y^2 = a^2 + b^2$ implies $3x^2 = a^2 + b^2 + y^2$ we have observed that either all x, y, a, and b are even or all odd.

If x, y, a and b are all even we can factor out a 2 from all these numbers and reduce the problem to $t/4$ instead of t. Applying this arguments several times one can see that $t = 2^{2l+1}t'$ with t' odd and $l \in \mathbb{Z}$. Without loss of generality we may assume that $l = 0$. In this case t contains only one power of 2 in its prime decomposition and so x, y, a and b must be all odd.

Let us then suppose that $t/2$ is divisible by a prime $p = 6k - 1$ for some $k \in \mathbb{N}$. We need to show that the exponent of p in the prime factorization of t is even. Since p divides $t = 3x^2 - y^2$ we get that $3x^2 \equiv y^2 \pmod{p}$. If p divides x, then p divides y and so p^2 divides t which reduces the problem to t/p^2. Applying this argument several times we arrive to a point when p does not divide x/p^i. So, discarding an even number of p's from t, we may assume that $i=0$. This implies that x has an inverse modulo p and then $z^2 \equiv 3 \pmod{p}$, where $z=x^{-1}y$. Using the Legendre symbol this says that $(\frac{3}{p})=1$. By the Law of Quadratic Reciprocity ([4, Thm. 11.7]) we see that $(\frac{p}{3})=(-1)^{\frac{(p-1)}{2}\frac{3-1}{2}}=(-1)^{3k-1}$. But the equation in w, $w^2 \equiv p \pmod{3}$, is equivalent to $w^2 \equiv -1 \pmod{3}$ which obviously has no solution in w. This implies $(\frac{p}{3})=-1$ and so k has to be even. Therefore $p=12k'-1=4j+3$ for some $j\in\mathbb{Z}$. But by hypothesis, t is a sum of two squares and so, from Euler's characterization of those numbers, p must have an even exponent in the prime decomposition of t.

For sufficiency, let us assume that $t=3x^2-y^2=2(m^2-mn+n^2)$ for some $x,y,m,n\in\mathbb{Z}$. Using Euler's characterization we have to show that if $p=4k+3$ is a prime dividing t then the exponent in its prime decomposition is even. If $p=3$, then 3 divides y which implies $x^2-3y'^2=2(m'^2-m'n'+n'^2)$. This is true because of Proposition 9 which one has to use in both directions.

If $m'^2-m'n'+n'^2$ is not divisible by 3 then $m'^2-m'n'+n'^2\equiv 1 \pmod{3}$ since all the prime factors of the form $6k-1$ and 2 appear to even exponents. This implies $x^2-3y'^2\equiv 2(m'^2-m'n'+n'^2) \pmod{3}$ or $x^2\equiv 2 \pmod{3}$ which is a contradiction. So, $m'^2-m'n'+n'^2$ must contain another factor of 3 and so the problem could be then reduced to $t/9$ instead of t. Hence 3 must have an even exponent in the prime decomposition of t.

Let us assume that $k=3j-1$ with $j\in\mathbb{Z}$. Then $p=12j-1=6(2j)-1$ and so these primes must appear to an even power in the decomposition of m^2-mn+n^2. The case $k=3j+1$ ($p=12k+7$) is not possible because that will contradict the Law of Quadratic Reciprocity: $(\frac{3}{p})(\frac{p}{3})=1\neq(-1)^{\frac{(p-1)}{2}\frac{3-1}{2}}$. ☐

Theorem 11. *An equilateral triangle of side lengths l and having integer coordinates in $\mathbb{R}^3$ exists, if and only if $l=\sqrt{2(m^2-mn+n^2)}$ for some integers m and n (not both zero) .*

Proof. The sufficiency part of the theorem is given by the triangles in $\mathcal{T}_{\infty,\infty,\infty}$ (Theorem 8). For necessity let us start with an arbitrary equilateral triangle having integer coordinates and non zero side lengths l. Without loss of generality we may assume that one of its vertices is the origin. Denote the triangle as before $\triangle OPQ$, with $P(u,v,w)$ and $Q(x,y,z)$. As we have shown in Proposition 2 we know that $au+bv+wc=0$ and $ax+by+cz=0$ for some a,b,c satisfying $a^2+b^2+c^2=3d^2$ and $\gcd(a,b,c)=1$. We noticed too that all a,b,c have to be odd integers and so, in particular, they are all non-zero numbers.

Then

$$l^2=u^2+v^2+w^2=\left(\frac{bv+cw}{a}\right)^2+v^2+w^2=\frac{(a^2+b^2)v^2+2bcvw+(a^2+c^2)w^2}{a^2}.$$

Completing the square we have

$$a^2l^2=(a^2+b^2)v^2+2bcvw+(a^2+c^2)w^2=(3d^2-c^2)\left(v+\frac{bcw}{3d^2-c^2}\right)^2+(3d^2-b^2)w^2-$$

$$\frac{b^2c^2w^2}{3d^2-c^2}=(3d^2-c^2)\left(v+\frac{bcw}{3d^2-c^2}\right)^2+3\frac{d^2a^2w^2}{3d^2-c^2},$$

or

$$a^2(3d^2-c^2)l^2 = [(3d^2-c^2)v + bcw]^2 + 3d^2a^2w^2.$$

This calculation shows that $a^2(3d^2-c^2)l^2 = m'^2 - m'n' + n'^2$ where $m' = (3d^2-c^2)v + bcw + daw$ and $n' = 2daw$. By Lemma 10 we can write $3d^2 - c^2 = a^2 + b^2 = 2(m''^2 - m''n'' + n''^2)$ for some $m'', n'' \in \mathbb{Z}$. Hence

$$l^2 = 2\frac{1}{(2a)^2}\frac{m'^2 - m'n' + n'^2}{m''^2 - m''n'' + n''^2}.$$

Because $l^2 \in \mathbb{Z}$ and Proposition 9 we see that $l^2 = 2(m^2 - mn + n^2)$. □

We include here a similar result and the last of this section which is only based on Proposition 2.

Proposition 12. *A regular tetrahedra of side lengths l and having integer coordinates in $\mathbb{R}^3$ exists, if and only if $l = m\sqrt{2}$ for some $m \in \mathbb{N}$.*

Proof. For sufficiency, we can take the tetrahedra $OPQR$ with $P(m, 0, m)$, $Q(m, m, 0)$ and $R(0, m, m)$.

For necessity, without loss of generality we assume the tetrahedra $OPQR$ is regular and has all its coordinates integers. As before, we assume $P(u, v, w)$ and $Q(x, y, z)$.

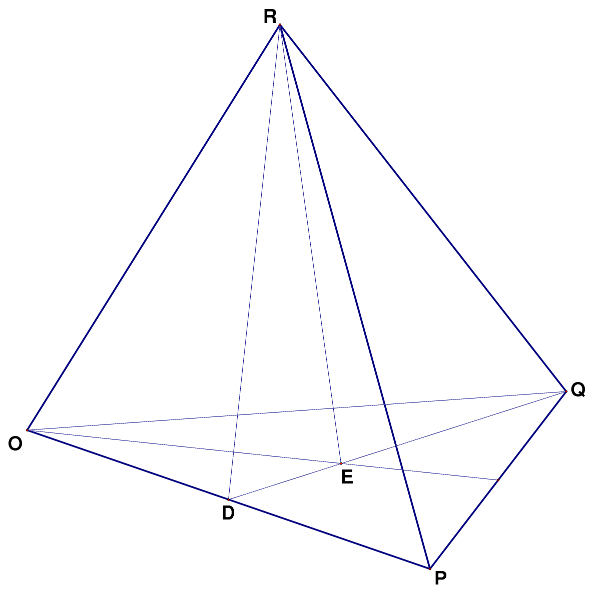

Figure 3: Regular tetrahedra

Let E be the center of the face $\triangle OPQ$. Then from Proposition 2 we know that $\frac{\overrightarrow{ER}}{|\overrightarrow{ER}|} = \frac{(a,b,c)}{\sqrt{a^2+b^2+c^2}} = \frac{1}{\sqrt{3d}}(a, b, c)$ for some $a, b, c, d \in \mathbb{Z}$, $l^2 = 2d$. The coordinates of E are $(\frac{u+x}{3}, \frac{y+v}{3}, \frac{z+w}{3})$.

From the Pythagorean theorem one can find easily that $RE = l\sqrt{\frac{2}{3}}$. Since $\overrightarrow{OR} = \overrightarrow{OE} + \overrightarrow{ER}$, the coordinates of R must be given by

$$\left(\frac{u+x}{3} \pm l\sqrt{\frac{2}{3}}\frac{1}{\sqrt{3}d}a, \frac{y+v}{3} \pm l\sqrt{\frac{2}{3}}\frac{1}{\sqrt{3}d}b, \frac{z+w}{3} \pm l\sqrt{\frac{2}{3}}\frac{1}{\sqrt{3}d}c\right)$$

or

$$\left(\frac{u+x}{3} \pm \frac{2\sqrt{2}}{3l}a, \frac{y+v}{3} \pm \frac{2\sqrt{2}}{3l}b, \frac{z+w}{3} \pm \frac{2\sqrt{2}}{3l}c\right).$$

Since these coordinates are assumed to be integers we see that $l = m\sqrt{2}$. □

Another natural question that one may ask at this point is weather or not every family $\mathcal{T}_{a,b,c}$ contains triangles which are faces of regular tetrahedra with integer coordinates. We believe that every triangle in $\mathcal{T}_{a,b,c}$ is a face of such a tetrahedra as long as its sides, in light of Theorem 11, are of the form $l = \sqrt{2N(m,n)}$ with $N(m,n)$ a perfect square. We leave this conjecture for further study. To find values of $m, n \in \mathbb{Z}$ such that $N(m,n) = m^2-mn+n^2$ is a perfect square, of course one can accomplish this in the trivial way, by taking $m = 0$ or $n = 0$ but there are also infinitely many non-trivial solutions as one can see from Proposition 9.

6 A more general parametrization

Our construction depends on a particular solution, $(r,s) \in \mathbb{Z}^2$, of the equation:

$$2(a^2+b^2) = s^2 + 3r^2. \tag{6}$$

As before let us assume that a, b, c and q are integers satisfying $a^2+b^2+c^2 = 3d^2$ with d an odd positive integer and $\gcd(a,b,c) = 1$. By Lemma 10 we see that $3d^2 - c^2 = a^2 + b^2 = 2(f^2 - fg + g^2)$ for some $f, g \in \mathbb{Z}$ and so $2(a^2+b^2) = (2f-g)^2 + 3g^2$ which says that equation (6) has always an integer solution.

Theorem 13. *Let a, b, c, d be odd positive integers such that $a^2+b^2+c^2 = 3d^2$, $a \le b \le c$ and* $\gcd(d,c) = 1$. *Then $\mathcal{T}_{a,b,c} = \{\triangle OPQ |\ m,n \in \mathbb{Z}\}$ where the points $P(u,v,w)$ and $Q(x,y,z)$ are given by*

$$\begin{cases} u = m_u m - n_u n, \\ v = m_v m - n_v n, \\ w = m_w m - n_w n, \end{cases} \quad \text{and} \quad \begin{cases} x = m_x m - n_x n, \\ y = m_y m - n_y n, \\ z = m_z m - n_z n, \end{cases} \tag{7}$$

with

$$\begin{cases} m_x = -\frac{1}{2}[db(3r+s) + ac(r-s)]/q, & n_x = -(rac+dbs)/q \\ m_y = \frac{1}{2}[da(3r+s) - bc(r-s)]/q, & n_y = (das - bcr)/q \\ m_z = (r-s)/2, & n_z = r \end{cases}$$
$$\text{and} \tag{8}$$
$$\begin{cases} m_u = -(rac+dbs)/q, & n_u = -\frac{1}{2}[db(s-3r) + ac(r+s)]/q \\ m_v = (das - rbc)/q, & n_v = \frac{1}{2}[da(s-3r) - bc(r+s)]/q \\ m_w = r, & n_w = (r+s)/2 \end{cases}$$

where $q = a^2 + b^2$ and (r, s) is a suitable solution of (6).

In order to prove Proposition 13 we need the following lemma.

Lemma 14. *Suppose A and B are integers in such a way A^2+3B^2 is divisible by q where $2q$ can be written as $s'^2 + 3r'^2$ with $r', s' \in \mathbb{Z}$. Then there exist a representation $2q = s^2 + 3r^2$, $r, s \in \mathbb{Z}$, such that*

$$Ar + Bs \equiv 0 \pmod{2q},$$

and

$$As - 3Br \equiv 0 \pmod{2q}.$$

Proof. Let us observe that every number of the form $A^2 + 3B^2$ is an Eisenstein-Jacobi integer since $A^2 + 3B^2 = (A + B)^2 - (A + B)(2B) + (2B)^2$. Conversely if n is even then $m^2 - mn + n^2 = (m - n/2)^2 + 3(n/2)^2$ and since $m^2 - mn + n^2 = (n - m)^2 - (n - m)n + n^2$ we see that every Eisenstein-Jacobi integer is of the form $A^2 + 3B^2$ for some $A, B \in \mathbb{Z}$. Using Proposition 9 we can write

$$A^2 + 3B^2 = 2^{2\alpha}(\prod_{t\in T} p_t)^2 \prod_{j\in J} p_j, \quad \alpha \in \mathbb{N},$$

and

$$2q = 2^{2\beta}(\prod_{t\in I'} p_t)^2 \prod_{j\in J'} p_j, \quad 1 \le \beta \le \alpha,\ T' \subset T,\ J' \subset J,$$

with p_t prime of the form $6k - 1$ for $t \in T$ and p_j prime of the form $6k + 1$ or equal to 3 for all $j \in J$. From Euler's $6k + 1$ theorem, for each $j \in J$ we can write $p_j = (m_j + n_j\sqrt{3}i)(m_j - n_j\sqrt{3}i)$ and we make the choice of m_j and n_j in $\mathbb{Z}$ in such a way $A + B\sqrt{3}i = s\prod_{j\in J}(m_j + n_j\sqrt{3}i)$ using the prime factorization in $\mathbb{Z}[\sqrt{3}i]$ of $A + B\sqrt{3}i$ and $h = 2^\alpha \prod_{t\in T} p_t$.

Then we take r and s such that $s + r\sqrt{3}i = h' \prod_{j\in J'}(m_j - n_j\sqrt{3}i)$ with $h' = 2^\beta \prod_{t\in T'} p_t$. Notice that $2q = (s + r\sqrt{3}i)(s - r\sqrt{3}i) = s^2 + 3r^2$ and $(A + B\sqrt{3}i)(s + r\sqrt{3}i) = 2q(A' + B'\sqrt{3}i)$. Identifying the coefficients we get $As + 3Br = 2qA'$ and $Ar + Bs = 2qB'$ and the conclusion of our lemma follows from this. $\square$

To return to the proof of Theorem 13 we begin with the next proposition.

Proposition 15. *For some particular solution $(r, s) \in \mathbb{Z}^2$ of (6) all of the coefficients m_u, m_v, m_w, n_u, n_v, n_w, m_x, m_y, m_z, n_x,n_y, n_z in (8) are integers.*

Proof. One can check that

$$\begin{cases} m_x^2 + m_y^2 + m_z^2 = n_x^2 + n_y^2 + n_z^2 = m_u^2 + m_v^2 + m_w^2 = n_u^2 + n_v^2 + n_w^2 = 2d^2, \\ m_xn_x + m_yn_y + m_zn_z = m_un_u + m_vn_v + m_wn_w = d^2, \\ am_x + bm_y + cm_z = am_u + bm_v + cm_w = an_x + bn_y + cn_z = an_u + bn_v + cn_w = 0 \end{cases} \tag{9}$$

These identities insures that the points $P(u,v,z)$ and $Q(x,y,z)$ are in the plane of normal vector (a,b,c) and containing the origin, the $\triangle OPQ$ is equilateral for every values of m, n and its side lengths are $l=d\sqrt{2(m^2-mn+n^2)}$. These calculations are tedious and so we are not going to include them here. The only ingredients that are used in establishing all these identities are the two relations between a, b, c, d, r and s.

From (6) we see that r and s have to be of the same parity. Then, it is clear that m_z, n_z, m_w, and n_w are all integers. Because the equalities in (9) are satisfied it suffices to show that m_x, n_x, m_u, and n_u are integers for some choice of (r,s) solution of (6). To show that n_x is an integer we need to show that q divides $N=rac+dbs$.

Let us observe that $c^2\equiv 3d^2$ and $a^2\equiv -b^2 \pmod{q}$. Multiplying together these two congruences we obtain $(ac)^2+3(db)^2\equiv 0 \pmod{q}$. Using Lemma 14 we see that N is divisible by $2q$ for some choice of r and s as in (6). So, $n_x\in\mathbb{Z}$.

Next we want to show that m_x is an integer. First let us observe that if $M=3dbr-acs$ we can apply the second part of Lemma 14 to conclude that $2q$ divides M also. Hence $2q$ divides $M+N=db(3r+s)+ac(r-s)$ and so, m_x is an integer. Because $m_x+n_u=n_x$ and $m_u=n_x$ it follows that n_u and m_u are also integers. $\square$

Remark: Let us observe that the parametric formulae (7) and (8) exist under no extra assumption on a, b and c. The question is whether or not every triangle in $\mathcal{T}_{a,b,c}$ is given by these formulae. So, with these preparations we can return to prove that this is indeed the case under the assumption of Theorem 13.

Proof. We start with a triangle in $\mathcal{T}_{a,b,c}$ say $\triangle OPQ$ with the notation as before. We know that $P(u_0,v_0,w_0)$ and $Q(x_0,y_0,z_0)$ belong to the plane of equation $a\alpha+b\beta+c\gamma=0$ and by Theorem 4 we see that the coordinates of P and Q should satisfy (3). Hence, using the same notation, cv_0-bw_0, aw_0-cu_0 and bu_0-aw_0 are divisible by d. A relatively simple calculation shows that $m_vn_w-m_wn_v=ad$, $m_wn_u-m_un_w=bd$ and $m_un_v-m_vn_u=cd$. We would like to solve the following system in m and n:

$$\begin{cases} u_0=m_um-n_un, \\ v_0=m_vm-n_vn, \\ w_0=m_wm-n_wn. \end{cases} \tag{10}$$

The equalities (9) and the fact that (u_0,v_0,w_0) is in the plane $a\alpha+b\beta+c\gamma=0$ insures that (10) has a unique real solution in m and n. We want to show that this solution is in fact an integer solution. The value of n can be solved from each pair of these equations to get

$$n=\frac{v_0m_w-w_0m_v}{ad}=\frac{w_0m_u-u_0m_w}{bd}=\frac{u_0m_v-v_0m_u}{cd}.$$

Since $\gcd(a,b,c)=1$ we can find integers a', b', c' such that $aa'+bb'+cc'=1$. Hence the above sequence of equalities gives

$$n=\frac{a'(v_0m_w-w_0m_v)+b'(w_0m_u-u_0m_w)+c'(u_0m_v-v_0m_u)}{d}$$

So, it suffices to show that d divides $v_0m_w-w_0m_v$, $w_0m_u-u_0m_w$ and $u_0m_v-v_0m_u$.

Next, let us calculate for instance $v_0 m_w - w_0 m_v$ in more detail:

$$v_0 m_w - w_0 m_v = v_0 r - \frac{das - rbc}{q} w_0 = \frac{v_0 qr - (das - rbc) w_0}{q} =$$

$$\frac{-dasw_0 + v_0(3d^2 - c^2)r + rbcw_0}{q} = \frac{c(bw_0 - v_0 c)r + 3rv_0 d^2 - dasw_0}{q}.$$

From Theorem 2 we see that $bw_0 - v_0 c = \pm d(2x_0 - u_0)$. Hence,

$$v_0 m_w - w_0 m_v = \frac{d[\pm c(2x_0 - u_0)r + 3rv_0 d - asw_0]}{q}.$$

This shows that d divides $v_0 m_w - w_0 m_v$ provided that $\gcd(q, d) = 1$. This is true since $\gcd(d, c) = 1$ implies $\gcd(q, d) = 1$. Similar calculations show that d divides $w_0 m_u - u_0 m_w$ and $u_0 m_v - v_0 m_u$. Hence n must be an integer. Similarly one shows that m is an integer. By replacing P with Q if necessary all the equalities in (7) have to hold true by Theorem 2. $\square$

7 Acknowledgements and further investigations

It turns out that the Diophantine equation $a^2 + b^2 + c^2 = 3d^2$ has plenty of solutions which satisfy $\gcd(a, b, c) = 1$, $\gcd(a, d) > 1$, $\gcd(b, d) > 1$ and $\gcd(c, d) > 1$. Let us refer to these as *degenerate* solutions. We have searched for an example of such a degenerate solution but did not find one between all odd $d \leq 4095$. Thanks to Professor Florian Luca who pointed us in the right direction, we have found the following first concrete degenerate solution: $a = (17)(41)(79)$, $b = (23)(31)(3361)$, $c = (5)(13)(71)(241)(541)(2017)$ and $d = (3)(13)(41)(241)(3361)$. It will be interesting to find the smallest d for which a corresponding degenerate solution exists. One needs to investigate whether or not the proof of Theorem 13 can be adapted to include the case of degenerate solutions.

We would like to thank the referee of this paper who has helped us understand more about the existence of such solutions by providing a sketch for the proof of the following facts:

$$\#\{(a, b, c, d) : 1 \leq a \leq b \leq c \leq x, a^2 + b^2 + c^2 = 3d^2, \gcd(a, b, c) = 1\} \ll x^2 (\ln x)^2, \qquad (11)$$

and

$$\begin{aligned} &\#\{(a, b, c, d) : 1 \leq a \leq b \leq c \leq x, a^2 + b^2 + c^2 = 3d^2, \gcd(a, b, c) = 1, \\ &\min(\gcd(a, d), \gcd(b, d), \gcd(c, d)) > 1\} \gg \tfrac{x^2}{(\ln x)^\alpha}, \end{aligned} \qquad (12)$$

for some $\alpha > 0$ and large enough x.

Also, determining the density of the degenerate solutions in the set of all solutions becomes an interesting nontrivial problem in analytic number theory.

References

[1] R. Guy, *Unsolved Problems in Number Theory*, Springer-Verlag, 2004.

[2] E. Grosswald, *Representations of Integers as Sums of Squares*, Springer Verlag, 1985.

[3] I. J. Schoenberg, *Regular simplices and quadratic forms*, *J. London Math. Soc.* **12** (1937), 48–55.

[4] K. Rosen, *Elementary Number Theory*, Fifth Edition, Addison-Wesley, 2004.

[5] Neil J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, 2007, published electronically at `http://www.research.att.com/~njas/sequences/`.

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