Journal of Integer Sequences, Vol. 10 (2007),
Article 07.2.7

The Equation $(j+k+1)^2 - 4k = Qn^2$ and Related Dispersions

Clark Kimberling
Department of Mathematics
University of Evansville
1800 Lincoln Avenue
Evansville, IN 47722
USA
ck6@evansville.edu

Abstract

Suppose Q is a positive nonsquare integer congruent to 0 or 1 mod 4. Then for every positive integer n, there exists a unique pair (j, k) of positive integers such that $(j + k + 1)^2 - 4k = Qn^2$. This representation is used to generate the fixed-j array for Q and the fixed-k array for Q. These arrays are proved to be dispersions; i.e., each array contains every positive integer exactly once and has certain compositional and row-interspersion properties.

1 Introduction

The Pell-like equation $m^2 - 4k = Qn^2$, where Q is a positive nonsquare integer congruent to 0 or 1 mod 4, can be written in the form

$$(j + k + 1)^2 - 4k = Qn^2, \tag{1}$$

or equivalently

$$(j + k - 1)^2 + 4j = Qn^2. \tag{2}$$

When so written, there is, for each n, a unique solution (j, k); here and throughout this work, the symbols n, j, k represent positive integers. In section 2, the existence and uniqueness of such a solution is proved. In section 3, the definition of dispersion is recalled, and in section 4, it is proved that certain arrays associated with solutions of (2) are dispersions. In section 5, another class of arrays are proved to be dispersions. In sections 4 and 5, Q is restricted to 0-congruence mod 4, and in section 6, conjectures are given for $Q \equiv 1$ mod

4. In section 7, various numerical sequences associated with the dispersions in preceding sections are discussed.

Throughout, we abbreviate $\sqrt{Q}/2$ as Q_1.

2 Unique Representations

Theorem 1. *Suppose that* $Q \equiv 0$ *mod* 4. *Then the unique solution of (1) is given by*

$$j = (Q_1 n)^2 - \lfloor Q_1 n \rfloor^2, \tag{3}$$
$$k = (1 + \lfloor Q_1 n \rfloor)^2 - (Q_1 n)^2. \tag{4}$$

Proof: The method of proof is to assume that (1) has a solution, to find it, and then to observe that it is unique. Let $m = j + k + 1$. Clearly, in order for (1) to hold, m must be even, so we write $m = 2h$. The equation $m^2 - 4k = Qn^2$ then yields

$$k = h^2 - Qn^2/4.$$

Equation (2) can be written as $(m-2)^2 + 4j = Qn^2$, from which

$$j = Qn^2/4 - (h-1)^2,$$

so that the requirement that $j > 0$ is equivalent to

$$h < 1 + Q_1 n.$$

Also, the requirement that $k > 0$ is equivalent to

$$h > Q_1 n.$$

Clearly there is exactly one such integer:

$$h = 1 + \lfloor Q_1 n \rfloor,$$

from which (3) and (4) follow. ■

Theorem 2. *Suppose that* $Q \equiv 1$ *mod* 4. *Then the unique solution of (1) is given by*

$$j = \begin{cases} (Q_1 n)^2 - (\lfloor 1/2 + Q_1 n \rfloor - 1/2)^2 & \text{if } n \text{ is odd,} \\ (Q_1 n)^2 - \lfloor Q_1 n \rfloor^2 & \text{if } n \text{ is even,} \end{cases} \tag{5}$$

$$k = \begin{cases} (\lfloor 1/2 + Q_1 n \rfloor + 1/2)^2 - (Q_1 n)^2 & \text{if } n \text{ is odd,} \\ (1 + \lfloor Q_1 n \rfloor)^2 - (Q_1 n)^2 & \text{if } n \text{ is even.} \end{cases} \tag{6}$$

Proof: The method for even n is the same as for Theorem 1. If n is odd, then m must be odd. Put $m = 2h + 1$, and find the asserted result using essentially the same method as for Theorem 1. ■

Theorem 3. *Suppose that* $Q \equiv 2$ *mod* 4 *or* $Q \equiv 3$ *mod* 4 *and that* n *is even. Then the unique solution of (1) is given by (3) and (4). If* n *is odd, then (1) has no solution.*

Proof: The method for even n is the same as for Theorem 1. Now suppose n is odd. Write $Q = r + 4s$, where $r = 2$ or $r = 3$, and write $n = 2h + 1$. Then $m^2 = 4k + (r + 4s)(2h + 1)^2 \equiv r \bmod 4$. But this is contrary to the fact that the residues mod 4 of the squares are all 0 or 1. ∎

Corollary. *Suppose $Q \equiv 2 \bmod 4$ or $Q \equiv 3 \bmod 4$ and n is a positive integer. Then the unique solution of the equation*

$$(j + k + 1)^2 - 4k = 4Qn^2 \tag{7}$$

is given by (3) and (4)

Proof: The number $2n$ is even, so that Theorem 3 applies – or, as a second proof, apply Theorem 1, as $4Q \equiv 0 \bmod 4$. ∎

3 Dispersions

A dispersion is an array of positive integers in which each occurs exactly once, and certain conditions ([1], [4]) hold: specifically, an array $D = \{d(g, h)\}$ is *the dispersion of a strictly increasing sequence* $s(k)$ if the first column of D is the complement of $s(k)$ in increasing order, and

$$\begin{aligned} d(1, 2) &= s(1) \geq 2; \\ d(1, h) &= s(d(1, h - 1)) \text{ for all } h \geq 3; \\ d(g, h) &= s(d(g, h - 1)) \text{ for all } h \geq 2, \text{ for all } g \geq 2. \end{aligned}$$

In this section we introduce simple dispersions $D(r)$ and $E(r)$. In later sections, related dispersions will be generated in connection with equations (1) and (2).

Suppose $r > 2$ is an irrational number, and let $D(r)$ denote the dispersion of the sequence $\{\lfloor rn \rfloor\}$. In order to construct $D(r)$, let r' be the number given by

$$1/r + 1/r' = 1,$$

so that the Beatty sequences $\{\lfloor rn \rfloor\}$ and $\{\lfloor r'n \rfloor\}$ partition the set of positive integers. Writing $d(g, h)$ for the general term of $D(r)$, we have

$$\begin{aligned} d(g, 1) &= \lfloor r'g \rfloor \text{ for all } g \geq 1, \\ d(g, h) &= \lfloor rd(g, h - 1) \rfloor \text{ for all } g \geq 1 \text{ and } h \geq 2. \end{aligned}$$

Thus, all the terms of the sequence $\{\lfloor rn \rfloor\}$ are dispersed in the columns of $D(r)$ excluding the first.

Example 1. The dispersion $D(r)$ for $r = 3 + 2\sqrt{2}$ is represented here and as A120858 in [3]. Column 1 is the sequence $d(g, 1) = \lfloor r'g \rfloor$ for $r' = (1 + \sqrt{2})/2$.

1	5	29	169	985	$\cdots$
2	11	64	373	2174	
3	17	99	577	3363	
4	23	134	781	4552	
6	34	198	1154	6726	
7	40	233	1358	7915	
8	46	268	1562	9104	
9	52	303	1766	10293	
10	58	338	1970	11482	
12	69	402	2343	13656	
13	75	437	2547	14845	
$\vdots$					

Now suppose $r > 1$ is an irrational number, and define

$$E(1, 1) = 1, \qquad E(1, h) = \lfloor rE(1, h - 1) \rfloor + 1 \ \text{ for } h \geq 2,$$

and inductively define, for $g \geq 2$,

$$\begin{aligned} E(g, 1) &= \text{least positive integer not among } E(i, h) \text{ for } 1 \leq i \leq g - 1,\ h \geq 1; \\ E(g, h) &= \lfloor rE(1, h - 1) \rfloor + 1 \ \text{ for } h \geq 2. \end{aligned}$$

Then $E = \{E(g, h)\}$ is clearly the dispersion of the sequence $s(k)$ given by

$$s(k) = \lfloor rk \rfloor + 1,$$

and the complement $\{s'(k)\}$ of $\{s(k)\}$ is given by

$$s'(k) = \lfloor r'(k - 1) \rfloor + 1, \quad \text{where } 1/r + 1/r' = 1.$$

Example 2. The dispersion $E(3 + 2\sqrt{2})$ is represented here and as A120859 in [3]:

1	6	35	204	1189	$\cdots$
2	12	70	408	2378	
3	18	105	612	4756	
4	24	140	816	4348	
5	30	175	1020	5945	
7	41	239	1393	8119	
8	47	274	1597	9308	
9	53	309	1801	10497	
10	59	344	2005	11686	
11	65	379	2209	12875	
13	76	443	2582	15049	
$\vdots$					

Next, suppose that r is a quadratic irrational number: $r = a + b\sqrt{Q}$, where Q is not a square. By Theorem 4 in [1], the rows of $D(r)$ and also the rows of $E(r)$ satisfy the recurrence

$$x_n = 2ax_{n-1} + (b^2Q - a^2)x_{n-2}.$$

For example, the row recurrence for the dispersions in Examples 1 and 2 is

$$x_n = 6x_{n-1} - x_{n-2}.$$

4 Fixed-j Dispersions for $Q \equiv 0$ mod 4

Suppose that $Q \equiv 0$ mod 4 and that (2) holds for a triple (j, k, n). Keeping j fixed, we ask what other pairs (k_1, n_1) there are for which (j, k_1, n_1) is another solution of (1). This question is answered by Theorem 4, which, with Theorems 5 and 6, shows that for given j, the set of such n_1 forms a row, or in some cases, several rows, of a dispersion.

Theorem 4. *Suppose that $Q \equiv 0$ mod 4 and that n, j, k satisfy (2). Let u be the least positive integer x such that*

$$x^2 - 1 = Qy^2/4$$

for some positive integer y, and let

$$n_1 = un + y\lfloor Q_1 n\rfloor, \tag{8}$$
$$k_1 = (1 + \lfloor Q_1 n_1\rfloor)^2 - (Q_1 n_1)^2. \tag{9}$$

Then

$$(j + k_1 - 1)^2 + 4j = Qn_1^2. \tag{2A}$$

Proof: In view of (2) and (2A), it suffices to prove that

$$(j + k - 1)^2 - Qn^2 = (j + k_1 - 1)^2 - Qn_1^2. \tag{10}$$

Using (3) and (4), we have

$$j + k - 1 = 2\lfloor Q_1 n\rfloor,$$

which, along with (3) and (8), leads to the following equivalent of (10):

$$\begin{aligned}(Q_1^2 n^2 - \lfloor Q_1 n\rfloor^2 + k_1 - 1)^2 = {} & 4\lfloor Q_1 n\rfloor^2 - Qn^2 + Qu^2 n^2 \\ & + 2Quny\lfloor Q_1 n\rfloor + Qy^2\lfloor Q_1 n\rfloor^2.\end{aligned} \tag{11}$$

Next using

$$u^2 - 1 = Q_1^2 y^2 \tag{12}$$

and $Q_1^2 = Q/4$, we find, after simplifications, that (11) is equivalent to

$$(Q_1^2 n^2 - \lfloor Q_1 n\rfloor^2 + k_1 - 1)^2 = (2u\lfloor Q_1 n\rfloor + 2nyQ_1^2)^2. \tag{13}$$

Taking square roots leads to

$$k_1 = \lfloor Q_1 n \rfloor^2 - Q_1^2 n^2 + 2u \lfloor Q_1 n \rfloor + 2ynQ_1^2 + 1 \tag{14}$$

as an equivalent of (13). Thus, using (9), what we must prove is that

$$\begin{aligned}(1 + \lfloor Q_1 n_1 \rfloor)^2 = {} & \lfloor Q_1 n \rfloor^2 - Q_1^2 n^2 + 2u \lfloor Q_1 n \rfloor + 2ynQ_1^2 + 1 \\ & + Q_1^2 (un + y \lfloor Q_1 n \rfloor)^2. \end{aligned} \tag{15}$$

Expanding the right side of (15) and again using (12), we find that the right-hand side of (15) is a square, and taking square roots leaves the following equivalent of (10):

$$1 + \lfloor Q_1 n_1 \rfloor = u \lfloor Q_1 n \rfloor + ynQ_1^2 + 1,$$

so that, using (8), what we must prove has now been reduced to

$$\lfloor Q_1 (un + y \lfloor Q_1 n \rfloor) \rfloor = u \lfloor Q_1 n \rfloor + ynQ_1^2. \tag{16}$$

To prove (16), we begin by noting from (12) that

$$(u - yQ_1)(u + yQ_1) = 1, \tag{17}$$

showing that

$$u - yQ_1 > 0.$$

Let $\epsilon = Q_1 n - \lfloor Q_1 n \rfloor$, the fractional part of $Q_1 n$. The fact that $(u - yQ_1)\epsilon > 0$ readily renders

$$u \lfloor Q_1 n \rfloor + ynQ_1^2 < Q_1 (un + y \lfloor Q_1 n \rfloor). \tag{18}$$

Also, since $u + yQ_1 > 1$, we have from (17)

$$u - yQ_1 < 1, \tag{19}$$

so that $(u - yQ_1)\epsilon < 1$, and consequently,

$$Q_1 (un + y \lfloor Q_1 n \rfloor) < u \lfloor Q_1 n \rfloor + ynQ_1^2 + 1.$$

This and (18) imply (16) and hence (10). ■

Regarding u and y in Theorem 4 as sequences with terms $u(n)$ and $y(n)$, where $n = Q/4$, we note that u and y are registered in [3] as A033313 and A033317. Several initial terms are shown here:

Q	8	12	20	24	28	32	40	44	48
u	3	2	9	5	8	3	19	10	7
y	2	1	4	2	3	1	6	3	4

Theorem 5. *Suppose that Q, j, k, u, y, n, and n_1 are as in Theorem 4, and let*

$$n_2 = un_1 + y \lfloor Q_1 n_1 \rfloor. \tag{20}$$

Then

$$n_2 = 2un_1 - n. \tag{21}$$

Proof: It suffices to prove that the right-hand sides of (21) and (22) are equal, or equivalently, that

$$n = un_1 - y \lfloor Q_1 n_1 \rfloor . \tag{22}$$

Using (8) to substitute for n shows that (22) is equivalent to

$$\lfloor unQ_1 + yQ_1 \lfloor Q_1 n_1 \rfloor \rfloor = nyQ_1^2 + u \lfloor Q_1 n \rfloor . \tag{23}$$

Now since $\lfloor nQ_1 \rfloor < nQ_1$, we have

$$nyQ_1^2 + u \lfloor Q_1 n \rfloor < unQ_1 + yQ_1 \lfloor Q_1 n_1 \rfloor . \tag{24}$$

Also, in connection with (17),

$$Q_1 n - \lfloor Q_1 n \rfloor < 1/(u - yQ_1),$$

which implies

$$unQ_1 + yQ_1 \lfloor Q_1 n_1 \rfloor < nyQ_1^2 + u \lfloor Q_1 n \rfloor + 1,$$

which with (24) establishes (23) and hence (20). ■

We turn now to the construction of the *fixed-j array of Q*. By Theorem 1, every n has a unique representation of a certain form depending on a pair (j, k) which we shall now write as (j_n, k_n). By Theorem 4, for each j_n, there are infinitely many pairs (n', k') such that (1) and (2) hold; that is,

$$(j_n + k_{n'} - 1)^2 + 4j_n = Q(n')^2. \tag{25}$$

Let S_1 be the set of all n' for which (1A) holds for $n = 1$ and some $k_{n'}$. By Theorem 4, $n_1 = u + y \lfloor Q_1 \rfloor \in S_1$. (Note that in (25),

$$j_1 = Q/4 - \left\lfloor \sqrt{Q}/2 \right\rfloor ,$$

this being the "fixed j" used to define S_1). By Theorem 5, the numbers given by the recurrence relation

$$x_i = 2ux_{i-1} - x_{i-1},$$

with initial values $x_1 = 1$ and $x_2 = n_1$, all lie in S_1. Possibly they comprise all of S_1, but if not, let w be the least number in S_1 that is not an x_i. By Theorems 4 and 5, we obtain another recurrence sequence $\{w_i\}$ lying in S_1. If S_1 contains a number not in $\{x_i\}$ and not in $\{w_i\}$, the production of recurrence sequences with all terms in S_1 can be continued inductively, so that S_1 is partitioned into a set of such sequences.

Next, let v be the least positive integer not in S_1, and let S_2 be the set of all v' for which (25) holds for $n = v$ and some $k_{v'}$. Again Theorems 4 and 5 apply, so that S_2 is partitioned into a set of recurrence sequences.

Continuing in this manner, the set of all positive integers is partitioned into sets S_i, which are themselves partitioned into sets of recurrence sequences. Next, arrange these sequences so that their first terms form an increasing sequence, thus producing an array of all the positive integers, with increasing first column, and whose rows all satisfy the recurrence relation $x_i = 2ux_{i-1} - x_{i-1}$.

Example 3. The fixed-j array for $Q = 8$. (This is A120860 in [3].)

1	5	29	169	985	$\cdots$
2	10	58	338	1970	
3	17	99	577	3363	
4	22	128	746	4348	
6	34	198	1154	6726	
7	39	227	1323	7711	
8	46	268	1562	9104	
9	51	297	1731	10089	
11	63	367	2139	12467	
12	68	396	2308	13452	
13	75	437	2547	14845	
$\vdots$					

We prove below that the array in Example 3 is the dispersion of the ordered sequence s of numbers not in its first column; this sequence appears to be A098021, described in [3] as "Irrational rotation of the square root of 2 as an implicit sequence with an uneven Cantor cartoon." (A Cantor cartoon is a kind of geometric fractal, so that A098021 indicates a connection between geometric fractals and fractal integer sequences as defined just before Example 15. The sequence s and A098021 agree for 30 terms and no later discrepancies were detected.)

In order to prove that the fixed-j array of Q is actually a dispersion, we begin with a lemma which applies to a wider class of arrays all of whose row sequences satisfy a common second-order recurrence.

Lemma. *Suppose that p and q are integers, that $p > 0$, and that $A = \{a(g, h)\}$ is an array consisting of all the positive integers, each exactly once. Suppose further that the first column of A is increasing and that every row sequence satisfies the recurrence*

$$x_n = px_{n-1} + qx_{n-2}$$

for $n \geq 3$. Finally, suppose for arbitrary indices g, g_1, g_2, and h that

$$\text{if } a(g_1, h) < a(g, 1) < a(g_1, h+1), \text{ then } a(g_1, h+1) < a(g, 2) < a(g_1, h+2). \tag{26}$$

Then A is a dispersion.

Proof: An array is a dispersion if and only if it is an interspersion. In [1], the four defining properties of an interspersion are given, and the first three are assumed in our present hypothesis. It remains to prove the fourth, which is as follows:

$$\text{if } (\sigma_i) \text{ and } (\tau_i) \text{ are rows of } A \text{ and } \sigma_g < \tau_h < \sigma_{g+1}, \text{ then } \sigma_{g+1} < \tau_{h+1} < \sigma_{g+2}. \tag{27}$$

(Property (27) essentially means that, beginning at the first term of any row having greater initial term than that of another row, all the following terms individually separate the individual terms of the other row; in this sense, *every* pair of rows are mutually interspersed, as in Examples 1-4.)

Now suppose (σ_i) and (τ_i) are distinct rows, that $g \geq 1$ and $h \geq 1$, and that

$$\sigma_g < \tau_h < \sigma_{g+1}, \tag{28}$$

$$\sigma_{g+1} < \tau_{h+1} < \sigma_{g+2}. \tag{29}$$

Then since

$$\sigma_{g+2} = p\sigma_{g+1} + q\sigma_g \quad \text{and} \quad \tau_{g+2} = p\tau_{g+1} + q\tau_g,$$

we have $\sigma_{g+2} < \tau_{h+2} < \sigma_{g+3}$, and, inductively,

$$\sigma_{g'} < \tau_{h'} < \sigma_{g'+1}$$

for all $g' \geq g$ and $h' \geq h$. This is to say, if (28) and (29) hold, then the rows (σ_i) and (τ_i) are interspersed beginning at term τ_h. The hypothesis (26) implies (28) and (29) for $h = 1$, so that (27) holds. ■

Theorem 6. *Suppose that $Q \equiv 0$ mod 4. Then the fixed-j array for Q is a dispersion.*

Proof: Let $\{d(g, h\}$ be the fixed-j array for Q, and suppose that g, g_1, and h are indices such that

$$d(g_1, h) < d(g, 1) < d(g_1, h+1). \tag{30}$$

With reference to the lemma, we wish to prove that

$$d(g_1, h+1) < d(g, 2) < d(g_1, h+2). \tag{31}$$

Abbreviate (30) and (31) as

$$n < w < n' \tag{32}$$

and

$$n' < w' < n'', \tag{33}$$

where, by Theorem 5,

$$\begin{aligned} n' &= un + y\lfloor Q_1 n \rfloor, \\ w' &= uw + y\lfloor Q_1 w \rfloor, \\ n'' &= 2un' - n. \end{aligned}$$

The obvious inequality

$$0 < u(w - n) + y(\lfloor Q_1 w \rfloor - \lfloor Q_1 n \rfloor)$$

easily implies $n' < w'$. To prove that $w' < n''$, we appeal to (19) to see that

$$(u - yQ_1)(Q_1 n - \lfloor Q_1 n \rfloor) < 1, \tag{34}$$

which readily yields

$$unQ_1 + yQ_1 \lfloor Q_1 n \rfloor < yQ_1^2 + u \lfloor Q_1 n \rfloor + 1.$$

The hypothesis that

$$w < un + y \lfloor Q_1 n \rfloor \tag{35}$$

implies

$$Q_1 w < unQ_1 + yQ_1 \lfloor Q_1 n \rfloor ,$$

so that by (34),

$$Q_1 w < ynQ_1^2 + u \lfloor Q_1 n \rfloor + 1,$$

which implies

$$\lfloor Q_1 w \rfloor - u \lfloor Q_1 n \rfloor < ynQ_1^2.$$

Multiply through by y and apply (12) to obtain

$$n + y \lfloor Q_1 w \rfloor < u^2 n + uy \lfloor Q_1 n \rfloor ,$$

equivalent to

$$u^2 n + uy \lfloor Q_1 n \rfloor + y \lfloor Q_1 w \rfloor < 2u^2 n + 2uy \lfloor Q_1 n \rfloor - n. \tag{36}$$

Returning to the hypothesis, we have

$$uw < u^2 n + uy \lfloor Q_1 n \rfloor ,$$

so that

$$uw + y \lfloor Q_1 w \rfloor < u^2 n + uy \lfloor Q_1 n \rfloor + y \lfloor Q_1 w \rfloor .$$

This and (36) give

$$uw + y \lfloor Q_1 w \rfloor < 2u(un + y \lfloor Q_1 n \rfloor) - n,$$

which is $w' < n''$. To summarize, (30) implies (31). Therefore, the lemma applies, and $\{d(g, h)\}$ is a dispersion. ■

5 Fixed-k Dispersions for $Q \equiv 0$ mod 4

Loosely speaking, if we hold k fixed instead of j, then the methods of section 4 yield another kind of dispersion. The purpose of this section is to define these fixed-k arrays and to prove that they are dispersions. Theorems 7 and 8 are similar to Theorems 4 and 5, and the fixed-k array is then defined with reference to Theorems 7 and 8. Thereafter, Example 4 using $Q = 8$ is presented, and then lemma of section 4 is used to prove Theorem 9, similar to Theorem 6.

Theorem 7. *Suppose that $Q \equiv 0$ mod 4 and that n, j, k satisfy (1). Let u, y, and r be as in Theorem 4, and let*

$$n_1 = un + y \lfloor Q_1 n \rfloor + y, \tag{37}$$
$$j_1 = (Q_1 n_1)^2 - \lfloor Q_1 n_1 \rfloor^2 . \tag{38}$$

Then

$$(j_1 + k + 1)^2 - 4k = Qn_1^2. \tag{1A}$$

Proof: In view of (1) and (1A), it suffices to prove that

$$(j + k + 1)^2 - Qn^2 = (j + k_1 - 1)^2 - Qn_1^2. \tag{39}$$

Following the method of proof of Theorem 4, we find (39) equivalent to

$$\begin{aligned}(j_1 + (1 + \lfloor Q_1 n \rfloor)^2 - Q_1^2 n^2 + 1)^2 &= (2 + 2\lfloor Q_1 n \rfloor)^2 + Q_1 n_1^2 - Q_1 n^2 \\ &= 4(u + u\lfloor Q_1 n \rfloor + yQ_1^2 n)^2.\end{aligned}$$

Taking square roots leads to

$$j_1 = -\lfloor Q_1 n \rfloor^2 + Q_1^2 n^2 + 2(u-1)\lfloor Q_1 n \rfloor + 2ynQ_1^2 n + 2(u-1) \tag{40}$$

as an equivalent of (39). Thus, using (36), what we must prove is that

$$\begin{aligned}\lfloor Q_1 n_1 \rfloor^2 = Q_1^2(un + y\lfloor Q_1 n \rfloor + y)^2 + \lfloor Q_1 n \rfloor^2 - Q_1^2 n^2 \\ -2(u-1)\lfloor Q_1 n \rfloor - 2ynQ_1^2 n - 2(u-1).\end{aligned} \tag{41}$$

Expanding the right side of (41) and using (12) leads to factoring the right-hand side as a square, and then taking square roots leaves the following equivalent of (39):

$$\lfloor Q_1 n_1 \rfloor^2 = u\lfloor Q_1 n \rfloor + ynQ_1^2 + u - 1.$$

Thus, in view of (37), we need only prove that

$$\lfloor Q_1(un + y\lfloor Q_1 n \rfloor + y) \rfloor = u\lfloor Q_1 n \rfloor + ynQ_1^2 + u - 1;$$

but this now follows easily from

$$(u - yQ_1)(1 - \epsilon) > 0,$$

where $\epsilon = Q_1 n - \lfloor Q_1 n \rfloor$. ■

Theorem 8. *Suppose that Q, j, k, u, y, n, and n_1 are as in Theorem 5, and let*

$$n_2 = un_1 + y\lfloor Q_1 n_1 \rfloor + y. \tag{42}$$

Then

$$n_2 = 2un_1 - n. \tag{21}$$

A proof similar to that of Theorem 7 is omitted. We do note, however, that (42) and (21) yield

$$n = un_1 - y\lfloor Q_1 n_1 \rfloor - y,$$

an inversion formula for (37), as (22) is for (8).

We are now in a position to define, by construction, the fixed-k array for Q. Each n is uniquely represented as in (1) by a pair (j, k), which we write as (j_n, k_n). By Theorem 7, for each k_n, there are infinitely many pairs (n', j') such that (1) and (2) hold; that is,

$$(j_{n'} + k_n + 1)^2 - 4k_n = Q(n')^2. \tag{46}$$

Let S_1 be the set of all n' for which (1A) holds for $n = 1$ and some $j_{n'}$. By Theorem 7, $n_1 = u + y \lfloor Q_1 \rfloor \in S_1$. (Note that in (46),

$$k_1 = \left\lfloor 1 + \sqrt{Q}/2 \right\rfloor^2 - Q/4,$$

this being the "fixed k" used to define S_1). By Theorem 8, the numbers given by the recurrence relation

$$x_i = 2ux_{i-1} - x_{i-1},$$

with initial values $x_1 = 1$ and $x_2 = n_1$, all lie in S_1. Possibly they comprise all of S_1, but if not, let w be the least number in S_1 that is not an x_i. By Theorems 7 and 8, we obtain another recurrence sequence $\{w_i\}$ lying in S_1. If S_1 contains a number not in $\{x_i\}$ and not in $\{w_i\}$, the production of recurrence sequences with all terms in S_1 can be continued inductively, so that S_1 is partitioned into a set of such sequences. Next, let v be the least positive integer not in S_1, and let S_2 be the set of all v' for which (46) holds for $n = v$ and some $k_{v'}$. Again Theorems 7 and 8 apply, so that S_2 is partitioned into a set of recurrence sequences.

Continuing in this manner, the set of all positive integers is partitioned into sets S_i, which are themselves partitioned into sets of recurrence sequences. Next, arrange these sequences so that their first terms form an increasing sequence, thus producing an array of all the positive integers, with increasing first column, and whose rows all satisfying the recurrence relation $x_i = 2ux_{i-1} - x_{i-1}$. This array is the fixed-k array.

Example 4. The fixed-k array for $Q = 8$. (This is A120861 in [3].)

1	7	41	239	1393	$\cdots$
2	12	70	408	2378	
3	19	111	647	3771	
4	24	140	816	4756	
6	31	181	1055	6149	
7	36	210	1224	7134	
8	48	280	1632	9512	
9	53	309	1801	10497	
11	60	350	2040	11890	
12	65	379	2209	12875	
13	77	449	2617	15253	
$\vdots$					

Theorem 9. *Suppose that $Q \equiv 0$ mod 4. Then the fixed-k array for Q is a dispersion.*

A proof using

$$\begin{aligned} n' &= un + y\lfloor Q_1 n\rfloor + y, \\ w' &= uw + y\lfloor Q_1 w\rfloor + y, \\ n'' &= 2un' - n \end{aligned}$$

is very similar to that of Theorem 6 and is omitted.

6 Dispersions for $Q \equiv 1$ mod 4

If $Q \equiv 1$ mod 4, the parity of n leads to two cases, as in Theorem 2, and to more subtle results of the sort given in Theorems 4 and 5 (and Theorems 7 and 8). We offer the following conjectures:

Conjecture 1. *Suppose that $Q \equiv 1$ mod 4 and that n, j, k satisfy (2). Let u be the least positive integer x such that*

$$x^2 - 4 = Qy^2$$

for some positive integer y, let

$$\begin{aligned} r &= (u + y\sqrt{Q})/2, \\ n_1 &= \lfloor rn\rfloor - \lfloor yf(n)\rfloor, \end{aligned}$$

where $f(n)$ denotes the fractional part of $(1+\sqrt{Q})n/2$, and let

$$k_1 = \begin{cases} (\lfloor 1/2 + Q_1 n_1\rfloor + 1/2)^2 - (Q_1 n_1)^2 & \text{if } n_1 \text{ is odd,} \\ (1 + \lfloor Q_1 n_1\rfloor)^2 - (Q_1 n_1)^2 & \text{if } n_1 \text{ is even.} \end{cases}$$

Then

$$(j + k_1 - 1)^2 + 4j = Qn_1^2.$$

Moreover, if

$$n_2 = \lfloor rn_1\rfloor - \lfloor yf(n_1)\rfloor,$$

then the recurrence (21) holds.

Conjecture 1 depends on the existence of the pairs (u, y). Regarding u and y as sequences with terms $u(n)$ and $y(n)$, where $n = (Q-1)/4$, we note that u and y are registered in [3] as A077428 and A078355. Several initial terms are shown here:

Q	5	13	17	21	29	33	37	41	45
u	3	11	66	5	27	46	146	4098	7
y	1	3	16	1	5	8	24	640	1

Conjecture 2. *Suppose that $Q \equiv 1$ mod 4 and that n, j, k satisfy (1). Let u, y, r be as in Conjecture 1, let*

$$n_1 = \lfloor rn\rfloor - \lfloor yf(n)\rfloor + y,$$

where $f(n)$ *denotes the fractional part of* $(1+\sqrt{Q})n/2$, *and let*

$$j_1 = \begin{cases} (Q_1 n_1)^2 - (\lfloor 1/2 + Q_1 n_1 \rfloor - 1/2)^2 & \text{if } n_1 \text{ is odd,} \\ (Q_1 n_1)^2 - \lfloor Q_1 n_1 \rfloor^2 & \text{if } n_1 \text{ is even.} \end{cases}$$

Then

$$(j_1 + k + 1)^2 - 4k = Qn_1^2.$$

Moreover, if

$$n_2 = \lfloor rn_1 \rfloor - \lfloor yf(n_1) \rfloor + y,$$

then the recurrence (21) holds.

Assuming the two conjectures valid, we construct fixed-j and fixed-k arrays exactly as in sections 4 and 5, and we conjecture that they are dispersions.

Example 5. The fixed-j array for $Q = 13$.

1	10	109	1189	$\cdots$
2	20	218	2378	
3	30	327	3567	
4	43	469	5116	
5	53	578	6305	
6	63	687	7494	
7	76	829	9043	
8	86	938	10232	
9	96	1047	11421	
11	119	1298	14159	
$\vdots$				

Example 6. The fixed-k array for $Q = 13$.

1	13	142	1549	$\cdots$
2	23	251	2738	
3	33	360	3927	
4	46	502	5476	
5	56	611	6665	
6	66	720	7854	
7	79	862	9403	
8	89	971	10592	
9	99	1080	11781	
10	112	1222	13330	
11	122	1331	14519	
$\vdots$				

7 Examples

This section consists of Examples 7-14 illustrating Theorems 1 and 2 for small values of Q, followed by fractal sequences associated with the fixed-j and fixed-k arrays for $Q = 5$.

Example 7. Taking $Q = 8$ in Theorem 1, for each n, there is a unique pair $(j, k) = (j(n), k(n))$ such that $(j + k + 1) - 4k = 8n^2$. The sequences j and k are given by

n	1	2	3	4	5	6	7	8	9	10	11	12	13	$\cdots$
j	1	4	2	7	1	8	17	7	18	4	17	32	14	$\cdots$
k	2	1	7	4	14	9	2	16	7	25	14	1	23	$\cdots$

The sequences j and k appear in [3] as A087056 and A087059, respectively.

Example 8. Taking $Q = 12$ in Theorem 1, for each n, there is a unique pair $(j, k) = (j(n), k(n))$ such that $(j + k + 1) - 4k = 12n^2$. The sequences j and k are given by

n	1	2	3	4	5	6	7	8	9	10	11	12	13	$\cdots$
j	2	3	2	2	11	8	3	23	18	11	2	32	23	$\cdots$
k	1	4	9	1	6	13	22	4	13	24	37	9	22	$\cdots$

The sequences j and k appear in [3] as A120864 and A120865, respectively.

Example 9. Taking $Q = 20$ in Theorem 1, for each n, there is a unique pair $(j, k) = (j(n), k(n))$ such that $(j + k + 1) - 4k = 20n^2$. The sequences j and k are given by

n	1	2	3	4	5	6	7	8	9	10	11	12	13	$\cdots$
j	1	4	9	16	4	11	20	31	5	16	29	44	4	$\cdots$
k	4	5	4	1	19	16	11	4	36	29	20	9	55	$\cdots$

The sequences j and k appear in [3] as A120866 and A120867, respectively.

Example 10. Taking $Q = 5$ in Theorem 2, for each n, there is a unique pair $(j, k) = (j(n), k(n))$ such that $(j + k + 1) - 4k = 5n^2$. The sequences j and k are given by

n	1	2	3	4	5	6	7	8	9	10	11	12	13	$\cdots$
j	1	1	5	4	1	9	5	16	11	4	19	11	1	$\cdots$
k	1	4	1	5	11	4	11	1	9	19	5	16	29	$\cdots$

The sequences j and k appear in [3] as A005752 and A120868, respectively.

Example 11. Taking $Q = 13$ in Theorem 2, for each n, there is a unique pair $(j, k) = (j(n), k(n))$ such that $(j + k + 1) - 4k = 13n^2$. The sequences j and k are given by

n	1	2	3	4	5	6	7	8	9	10	11	12	13	$\cdots$
j	1	4	9	3	9	17	3	12	23	1	13	27	43	$\cdots$
k	3	3	1	12	9	4	23	17	9	36	27	16	3	$\cdots$

The sequences j and k appear in [3] as A120869 and A120870, respectively.

Example 12. The table shows, in row g and column h, the unique pair (j,k) corresponding to the fixed-j array $\{d(g,h)\}$ for $Q = 8$; i.e., $(j+k+1)^2 - 4k = 8n^2$, where $n = d(g,h)$ is as in Example 3:

$$\begin{array}{lllll}
(1,2) & (1,14) & (1,82) & (1,478) & \cdots \\
(4,1) & (4,25) & (4,161) & (4,953) & \\
(2,7) & (2,47) & (2,279) & (2,1631) & \\
(7,4) & (7,56) & (7,356) & (7,2104) & \\
(9,6) & (9,89) & (9,553) & (9,3257) & \\
\vdots & & & &
\end{array}$$

The values of j, fixed for each row, form the sequence A120871:

$$(1,4,2,7,8,17,7,18,17,32,14,31,9,28,23,46,16,41,34,63,25,56,14,47,\ldots).$$

These numbers are terms of the j-sequence in Example 7, but without duplicates.

Example 13. The table shows, in row g and column h, the unique pair (j,k) corresponding to the fixed-k array $\{d(g,h)\}$ for $Q = 8$; i.e., $(j+k+1)^2 - 4k = 8n^2$, where $n = d(g,h)$ is as in Example 4:

$$\begin{array}{lllll}
(1,2) & (17,2) & (43,2) & (673,2) & \cdots \\
(4,1) & (32,1) & (196,1) & (1152,1) & \\
(2,7) & (46,7) & (306,7) & (1822,7) & \\
(7,4) & (63,4) & (391,4) & (2303,4) & \\
(1,14) & (72,14) & (497,14) & (2969,14) & \\
\vdots & & & &
\end{array}$$

The values of k, fixed for each column, form the sequence A120872:

$$(2,1,7,4,14,9,16,7,25,14,23,8,34,17,47,28,41,18,\ldots).$$

These numbers are terms of the k-sequence in Example 7, but without duplicates.

Example 14. The unique pair $(j,k) = (j(n),k(n))$ such that

$$(j+k+1)^2 - 4k = 5n^2$$

is given in Example 10. If the duplicates in $\{j(n)\}$ are expelled, the remaining sequence gives the j values for the fixed-j array for $Q = 5$ (cf., Example 12); this remaining sequence appears to be essentially A022344, described as "Allan Wechsler's 'J determinant' sequence". To compare the two sequences, check successive terms of A022344 and A005742.

We conclude with a few notes about the case $Q = 5$, the least Q covered by Theorem 2. The fixed-j array is the Wythoff difference array (A080164 in [3]) and the fixed-k array is the

Fraenkel array (A038150). These and their relationship to the equation $(j+k+1)^2-4k = 5n^2$ are discussed in [2].

As noted in [4], there is a fractal sequence associated with every dispersion. Specifically, if $d(g,h)$ is the general term of the dispersion, and we define $f(n)$ to be the value of g for which $n = d(g,h)$, then f is the associated fractal sequence. (A fractal sequence contains itself as a proper subsequence; e.g., if you delete the first occurrence of each positive integer in f, the remaining sequence is f; iterating this procedure shows that the sequence properly contains itself infinitely many times.)

Example 15. The fractal sequence associated with the Wythoff difference array $\{d(g,h)\}$ is A120873:

$$f = (1, 1, 2, 3, 1, 4, 2, 5, 6, 3, 7, 8, 1, 9, 4, 10, 11, 2, 12, 5, 13, 14, 6, \ldots).$$

Example 16. The fractal sequence associated with the Fraenkel array $\{d(g,h)\}$ is A120874:

$$f = (1, 2, 1, 3, 4, 2, 5, 1, 6, 7, 3, 8, 9, 4, 10, 2, 11, 12, 5, 13, 1, 14, 15, \ldots).$$

References

[1] Clark Kimberling, Interspersions and dispersions, *Proc. Amer. Math. Soc.* **117** (1993) 313–321.

[2] Clark Kimberling, The equation $m^2 - 4k = 5n^2$ and unique representations of positive integers, submitted.

[3] N. J. A. Sloane, editor, *The On-Line Encyclopedia of Integer Sequences,* `http://www.research.att.com/~njas/sequences/`

[4] N. J. A. Sloane, Classic sequences in the On-Line Encyclopedia of Integer Sequences, Part 1, The Wythoff Array and The Para-Fibonacci Sequence.

2000 *Mathematics Subject Classification*: Primary 11B37; Secondary 11D09, 11D85 .
Keywords: Beatty sequence, complementary equation, dispersion, interspersion, Pell equation, recurrences.

(Concerned with sequences A005752, A033313, A033317, A038150, A077428, A078355, A080164, A087076, A087079, A098021, A120858, A120859, A120860, A120861, A120862, A120863, A120864, A120865, A120866, A120867, A120868, A120869, A120870, A120871, A120872, A120873, and A120874.)

Received July 10 2006; revised version received February 20 2007. Published in *Journal of Integer Sequences*, March 13 2007.

Return to Journal of Integer Sequences home page.