Journal of Integer Sequences, Vol. 10 (2007),
Article 07.3.1

Direct and Elementary Approach to Enumerate Topologies on a Finite Set

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Abstract

Let $\mathbb{E}$ be a set with n elements, and let $\tau(n,k)$ be the set of all labelled topologies on $\mathbb{E}$, having k open sets, and $T(n,k) = |\tau(n,k)|$. In this paper, we use a direct approach to compute $T(n,k)$ for all $n \geq 4$ and $k \geq 6 \cdot 2^{n-4}$.

1 Introduction

Let $\mathbb{E}$ be a set with n elements. The problem of determining the total number of labelled topologies $T(n)$ one can define on $\mathbb{E}$ is still an open question. Sharp [3], and Stephen [6] had shown that every topology which is not discrete contains $k \leq 3 \cdot 2^{n-2}$ open sets, and that this bound is optimal. Stanley [5] computed all labelled topologies on $\mathbb{E}$, with $k \geq 7 \cdot 2^{n-4}$ open sets. In the opposite sense, Benoumhani [1] computed, for all n, the total number of labelled topologies with $k \leq 12$ open sets. In the other hand, Erné and Stege [2] computed the total number of topologies, for $n \leq 14$. In this paper, we use a direct approach to compute all labelled topologies on $\mathbb{E}$ having $k \geq 6 \cdot 2^{n-4}$ open sets. Furthermore, we confirm the results in [3, 5, 6]. This work is a continuation of the results of [1, 5]. Here is our approach. The set $\tau(n,k)$ is partitioned into two disjoint parts as follows:

$$\tau(n,k) = \tau_1(n,k) \cup \tau_2(n,k),$$

where

$$\begin{aligned}
\tau_1(n,k) &= \left\{ \tau = \{\varnothing, A_1, \ldots, A_{k-2},\ \mathbb{E}\} \in \tau(n,k), \quad \text{such that } \bigcap_{i=1}^{k-2} A_i \neq \varnothing \right\}, \\
\tau_2(n,k) &= \tau(n,k) - \tau_1(n,k).
\end{aligned}$$

In Theorem 2.1, we prove that the cardinal $T_1(n,k) = |\tau_1(n,k)|$ satisfies

$$T_1(n,k) = \sum_{l=1}^{n-1} \binom{n}{l} T(l,k-1), \qquad \forall\ n \geq 1.$$

This relation enables us to compute $T_1(n,k)$ for $k > 5 \cdot 2^{n-4}$. For the determination of the cardinal $T_2(n,k) = |\tau_2(n,k)|$, we introduce the notion of minimal open set (Definition 2.2), and we designate by $\tau_2(n,k,\alpha)$ the labelled topologies in $\tau_2(n,k)$ having $\alpha \geq 2$ minimal open sets. In Lemma 2.2, it is proved that if $k > 5 \cdot 2^{n-4}$ such that $k \neq 6 \cdot 2^{n-4}$, and $k \neq 2^{n-1}$, then all the minimal open sets of τ are necessarily singletons. So, we can compute the numbers $T_2(n,k,\alpha)$ for all $n \geq 4$, $k \geq 6 \cdot 2^{n-4}$, and $\alpha \geq 2$.

2 Basic Results

Theorem 2.1. *For every integer $n > 1$, and $2 \leq k \leq 2^n$, we have*

$$T_1(n,k) = \sum_{l=1}^{n-1} \binom{n}{l} T(l,k-1),$$

with the convention that $T(l,1) = 0$.

Proof. Let $A \subset \mathbb{E}$, with $|A| = l \leq n-1$, and let τ' be a topology on A, and having $k-1$ open sets. To this topology we associate the following one

$$\Phi_A(\tau') = \tau = \{O \cup A^c, \qquad O \in \tau'\} \cup \{\varnothing\}.$$

Obviously Φ_A is an injective mapping on $\tau(l,k-1)$ into $\tau_1(n,k)$. In the other hand, if $|A| = |B| = l \leq n-1$ and $A \neq B$, then

$$R(\Phi_A) \cap R(\Phi_B) = \varnothing,$$

where $R(\Phi_A)$ is the image of Φ_A. This shows that

$$T_1(n,k) \geq \sum_{l=1}^{n-1} \binom{n}{l} T(l,k-1).$$

Conversely, if $\tau = \{\varnothing,\ A_1, \ldots,\ A_{k-2},\ \mathbb{E}\} \in \tau_1(n,k)$, with $A_1 = \bigcap_{i=1}^{k-2} A_i$, then $\tau' = \{O - A_1, \quad O \in \tau\}$ is a topology on A_1^c, having $k-1$ open sets, and $\Phi_{A_1^c}(\tau') = \tau$. This shows the other inequality, and completes the proof. $\square$

The following definition will be needed in the sequel.

Definition 2.2. *Let $\tau = \{\varnothing,\ A_1, \dots,\ A_{k-2},\ \mathbb{E}\} \in \tau(n,k)$. The element A_i is called a minimal open set, if it satisfies:*

$$A_i \cap A_j = A_i \quad or \quad \varnothing, \qquad \forall\ j = 1, \dots, k-2.$$

Remark 2.3. i) A topology on $\mathbb{E}$ is a bounded lattice with ($1 = \mathbb{E}$, $0 = \varnothing$). A minimal open set is in fact an atom. Recall that an atom in a partially ordered set is an element which covers 0. So, every topology has at least one minimal open set, and $\tau_1(n,k)$ is the subset of topologies having exactly one minimal open set.

ii) If $\tau \in \tau_2(n,k)$, then τ has at least two minimal open sets.

iii) The space $\mathbb{E}$ is a union of α minimal open sets for the topology $\tau \in \tau(n,k)$ if and only if $k = 2^\alpha$.

iv) If τ has α minimal open sets, then $k \geq 2^\alpha$.

Definition 2.4. *For $\alpha \geq 2$, we define*

$$\tau_2(n,k,\alpha) = \{\tau \in \tau_2(n,k), \quad \tau \text{ has } \alpha \text{ minimal open sets }\}.$$

Note that if $\alpha_1 \neq \alpha_2$, then $\tau_2(n,k,\alpha_1) \cap \tau_2(n,k,\alpha_2) = \varnothing$. So

$$T_2(n,k) = \sum_{\alpha \geq 2,\ \ 2^\alpha \leq\ k} T_2(n,k,\alpha).$$

The computation of $T_2(n,k)$ is then equivalent to the computation of $T_2(n,k,\alpha)$, for $\alpha \geq 2$, under the condition $2^\alpha \leq\ k$. If $k = 2^\alpha$, then

$$T_2(n, 2^\alpha, \alpha) = S(n,\alpha),$$

where $S(n,\alpha)$ is the Stirling number of the second kind.

Lemma 2.1. *Let $n \geq 1$, $\alpha \geq 2$. Then $\tau_2(n,k,\alpha)$ is empty, for $k > 2^{n-1} + 2^{\alpha-1}$. In addition, this bound is optimal:*

$$\tau_2(n, 2^{n-1} + 2^{\alpha-1}, \alpha) \neq \varnothing.$$

Proof. We argue by contradiction. Suppose that $\tau \in \tau_2(n,k,\alpha)$, and write it as

$$\tau = \{\varnothing,\ A_1, \dots, A_\alpha, \dots, \mathbb{E}\},$$

where $A_1, ..., A_\alpha$ are the α minimal open sets of τ. Put $A = \bigcup_{i=1}^{\alpha} A_i$, the topology $\tau' = \{O - A, \quad O \in \tau\}$ on A^c has at least $\lceil k\ 2^{1-\alpha} - 1 \rceil$ open sets. In the other hand, $|A^c| \leq n - \alpha$, and since τ' is at most the discrete topology, we obtain

$$k\ 2^{1-\alpha} - 1 \leq |\tau'| \leq 2^{n-\alpha}.$$

This contradiction proves that $\tau_2(n,k,\alpha)$ is empty. The second assertion will be proved in the next section. $\square$

Lemma 2.2. *Let $\tau \in \tau_2(n,k,\alpha)$, with $k > 5 \cdot 2^{n-4}$, $k \neq 6 \cdot 2^{n-4}$, and $k \neq 2^{n-1}$. Then, all the minimal open sets of τ are singletons.*

Proof. Let $\tau = \{\varnothing,\ A_1, \ldots, A_\alpha, \ldots, \mathbb{E}\} \in \tau_2(n,k,\alpha)$, where $A_1, \ldots, A_\alpha$ are its minimal open sets, and suppose that $A = \bigcup_{i=1}^{\alpha} A_i$ has more than $\alpha + 1$ elements. The same argument used in the previous Lemma gives $5 \cdot 2^{n-4} < k \leq 2^{n-2} + 2^{\alpha-1}$. This last inequality is possible only for $\alpha = n-1$ or $\alpha = n-2$. In the first case, $\mathbb{E}$ is a union of $n-1$ minimal open sets, so $k = 2^{n-1}$, which is excluded. In the second, necessarily $k = 6 \cdot 2^{n-4}$, which is also excluded. So, all the minimal open sets of τ are singletons. $\square$

3 Computation

Firstly, we compute $T_2(n,k,\alpha)$, for $k \geq 6 \cdot 2^{n-4}$ and $\alpha \geq 2$. We use the notation

$$(n)_l = n(n-1)\cdots(n-l+1),$$

and we convenient that if $l > n$, then $(n)_l = 0$. We start by the number of topologies $\tau \in \tau_2(n,k,\alpha)$, such that τ has at least one minimal open set, which is not a singleton. For this, the previous Lemma gives $k = 2^{n-1}$ or $k = 6 \cdot 2^{n-4}$. If $k = 2^{n-1}$, then $\alpha = n-1$ and the number of these topologies is

$$S(n, n-1) = \frac{(n)_2}{2}.$$

If $k = 6 \cdot 2^{n-4}$, we have $\alpha = n-2$, and the number of these topologies is

$$2(n-2)\binom{n}{n-2}\binom{n-2}{1} = (n-2)\ (n)_3.$$

The remaining topologies of $\tau_2(n,k,\alpha)$ have the property that all their minimal open sets are singletons. For this, let $\tau \in \tau_2(n,k,\alpha)$

$$\tau = \{\varnothing,\ A_1, \ldots, A_\alpha, \ldots, \mathbb{E}\}.$$

Put $\alpha = n - i,\ \ 0 \leq i \leq n-2$, and $A = \cup_{i=1}^{\alpha} A_i$. The topology $\tau' = \{O - A, \quad O \in \tau\}$ (on A^c), can be written as follows:

$$\tau' = \{\varnothing,\ C_1, \ldots, C_m\}, \qquad m \in \left\{0, 1, 2, \ldots, 5 \cdot 2^{i-3} - 1,\ 3 \cdot 2^{i-2} - 1,\ 2^i - 1\right\}.$$

To reconstruct τ from τ', we remark that every C_j, if it exists, generates 2^{i_j} open sets in τ, with $i_j \leq n - i - 1$. So, the number k has necessarily the form:

$$k = 2^{n-i} + 2^{i_1} + 2^{i_2} + \ldots + 2^{i_m},$$

where the integers i_j , $1 \leq j \leq m$ can be dependant. Our approach is that for all α, $2 \leq \alpha \leq n$, we determine all possibilities of the number k, and next the number of all these topologies.

For $\underline{\alpha = n.}$ $A^c = \varnothing$; so $m = 0$, $k = 2^n$ and $T_2(n, 2^n, n) = 1$. This case corresponds to the discrete topology.

For $\underline{\alpha = n-1.}$ $A^c = \{x\}$; so $m = 1$, and $\tau' = \{\varnothing, C_1 = \{x\}\}$. All the possibilities of k are given by

$$k = 2^{n-1} + 2^{n-1-j}, \quad 1 \leq j \leq n-1.$$

The number of these topologies is

$$T_2(n, 2^{n-1} + 2^{n-1-j}, n-1) = n\binom{n-1}{j} = \frac{(n)_{j+1}}{j!}, \quad 1 \leq j \leq n-1.$$

For $\underline{\alpha = n-2.}$ $A^c = \{x, y\}$, $\tau' = \{\varnothing, C_1, \ldots, C_m\}$, with $m = 1, 2$ or 3.

If $m = 1$, $\tau' = \{\varnothing,\ C_1 = \{x, y\}\}$. Since we are supposing $k \geq 6 \cdot 2^{n-4}$, the unique possibility is that C_1 generates 2^{n-3} open sets. So, $k = 2^{n-2} + 2^{n-3} = 6 \cdot 2^{n-4}$, and the number of these topologies is

$$\binom{n}{n-2}\binom{n-2}{1} = \frac{(n)_3}{2}.$$

If $m = 2$, $\tau' = \{\varnothing, C_1 = \{x\}, C_2 = \{x, y\}\}$ or $\tau' = \{\varnothing, C_1 = \{y\}, C_2 = \{x, y\}\}$. Here we have two categories of solutions:

a) C_1 generates 2^{n-3} open sets, and C_2 generates $2^{n-3-j}, 0 \leq j \leq n-3$, open sets. Hence

$$k = 2^{n-2} + 2^{n-3} + 2^{n-3-j} = 6 \cdot 2^{n-4} + 2^{n-3-j}, \quad 0 \leq j \leq n-3.$$

The number of such topologies is

$$2(j+1)\binom{n}{n-2}\binom{n-2}{j+1} = \frac{(n)_{j+3}}{j!}.$$

b) C_1 generates 2^{n-4} open sets and also C_2 generates 2^{n-4}. So, $k = 2^{n-2} + 2^{n-4} + 2^{n-4} = 6 \cdot 2^{n-4}$, and the number in this case is

$$2\binom{n}{n-2}\binom{n-2}{2} = \frac{(n)_3}{2}.$$

If $m = 3$, $\tau' = \{\varnothing, C_1 = \{x\}, C_2 = \{y\}, C_3 = \{x, y\}\}$. There are 8 categories of solutions:

a) Each C_j, $j = 1, 2, 3$ generates 2^{n-3} open sets. So, $k = 2^{n-2} + 2^{n-3} + 2^{n-3} + 2^{n-3} = 10 \cdot 2^{n-4}$, and the wanted number is

$$\binom{n}{n-2}\binom{n-2}{1} = \frac{(n)_3}{2}.$$

b) C_1 generates 2^{n-3} open sets, C_2 and C_3 each one generates 2^{n-3-j} open sets, with $1 \leq j \leq n-3$. So, $k = 2^{n-2} + 2^{n-3} + 2^{n-3-j} + 2^{n-3-j} = 6 \cdot 2^{n-4} + 2^{n-2-j}$, $\quad 1 \leq j \leq n-3$, and the number of these topologies is

$$2(j+1)\binom{n}{n-2}\binom{n-2}{j+1} = \frac{(n)_{j+3}}{j!}.$$

c) C_1 and C_2 each one generates 2^{n-3} open sets, but C_3 generates 2^{n-4} open sets. So, $k = 2^{n-2} + 2^{n-3} + 2^{n-3} + 2^{n-4} = 9 \cdot 2^{n-4}$, and the number of these topologies is

$$2\binom{n}{n-2}\binom{n-2}{2} = \frac{(n)_4}{2}.$$

d) C_1 generates 2^{n-3} open sets, C_2 generates 2^{n-2-j}, $2 \leq j \leq n-3$ open sets. So, C_3 generates 2^{n-3-j} open sets, and $k = 2^{n-2} + 2^{n-3} + 2^{n-2-j} + 2^{n-3-j} = 6 \cdot 2^{n-4} + 3 \cdot 2^{n-3-j}$, $\quad 2 \leq j \leq n-3$. The number of these topologies is

$$2(j+1)\binom{n}{n-2}\binom{n-2}{j+1} = \frac{(n)_{j+3}}{j!}.$$

e) C_1, C_2 and C_3 each one generates 2^{n-4} open sets. So, $k = 2^{n-2} + 2^{n-4} + 2^{n-4} + 2^{n-4} = 7 \cdot 2^{n-4}$, and the number of these topologies is

$$\binom{n}{n-2}\binom{n-2}{2} = \frac{(n)_4}{4}.$$

f) C_1 and C_2, each one generates 2^{n-4} open sets, but C_3 generates 2^{n-5} open sets. In this case $k = 2^{n-2} + 2^{n-4} + 2^{n-4} + 2^{n-5} = 13 \cdot 2^{n-5}$, and the number of these topologies is

$$6\binom{n}{n-2}\binom{n-2}{3} = \frac{(n)_5}{2}.$$

g) C_1 generates 2^{n-4} open sets, and each one of C_2, C_3 generates 2^{n-5}. So, $k = 2^{n-2} + 2^{n-4} + 2^{n-5} + 2^{n-5} = 6 \cdot 2^{n-4}$, and the number of these topologies is

$$6\binom{n}{n-2}\binom{n-2}{3} = \frac{(n)_5}{2}.$$

h) Each one of C_1, C_2 generates 2^{n-4} open sets, but C_3 generates 2^{n-6} . So, $k = 2^{n-2} + 2^{n-4} + 2^{n-4} + 2^{n-6} = 25 \cdot 2^{n-6}$, and the number of these topologies is

$$6\binom{n}{n-2}\binom{n-2}{4} = \frac{(n)_6}{8}.$$

All the other cases give $k < 6 \cdot 2^{n-4}$. We resume all these results in the next statement.

Theorem 3.1. *Let $n \geq 4$, and $\alpha = n-2$. Then we have*

k	$T_2(n,k,n-2)$
$6\cdot 2^{n-4}$	$(n-1)(n)_3+\frac{1}{2}(n)_5$
$6\cdot 2^{n-4}+1$	$(n)_3$
$6\cdot 2^{n-4}+2^{n-3-j}, \quad 4\le j\le n-4$	$\frac{(n-2)\ (n)_{j+3}}{(j+1)!}$
$6\cdot 2^{n-4}+3\cdot 2^{n-3-j}, \quad 5\le j\le n-3$	$\frac{(n)_{j+3}}{j!}$
$25\cdot 2^{n-6}$	$\frac{7}{24}(n)_6+\frac{1}{24}(n)_7$
$51\cdot 2^{n-7}$	$\frac{1}{24}(n)_7$
$13\cdot 2^{n-5}$	$(n)_5+\frac{1}{6}(n)_6$
$27\cdot 2^{n-6}$	$\frac{1}{6}(n)_6$
$7\cdot 2^{n-4}$	$\frac{5}{4}(n)_4+\frac{1}{2}(n)_5$
$15\cdot 2^{n-5}$	$\frac{1}{2}(n)_5$
2^{n-1}	$(n)_3+(n)_4$
$9\cdot 2^{n-4}$	$\frac{1}{2}(n)_4$
$10\cdot 2^{n-4}$	$\frac{1}{2}(n)_3$

All other topologies in $\tau_2(n,k,n-2)$ have $k<6\cdot 2^{n-4}$ open sets.

We use the same reasoning as above, to show the following theorem.

Theorem 3.2. *Let $n\ge 5$, and $\alpha=n-i, \quad 3\le i\le\ n-2$. Then, the following results hold.*
For $\alpha=n-3$, if $n=5$, we have

k	12	13	14	15	18
$T_2(5,k,2)$	360	60	180	60	20

If $n\ge 6$, we have

k	$6\cdot 2^{n-4}$	$25\cdot 2^{n-6}$	$13\cdot 2^{n-5}$	$27\cdot 2^{n-6}$	$7\cdot 2^{n-4}$	$15\cdot 2^{n-5}$	$9\cdot 2^{n-4}$
$T_2(n,k,n-3)$	$(n)_4+\frac{5}{2}(n)_5$ $+\frac{5}{4}(n)_6$	$\frac{1}{4}(n)_6$	$\frac{1}{2}(n)_5$	$\frac{1}{6}(n)_6$	$(n)_4+\frac{1}{2}(n)_5$	$\frac{1}{2}(n)_5$	$\frac{1}{6}(n)_4$

For $\alpha=n-4$, and $n\ge 6$

k	$25\cdot 2^{n-6}$	$13\cdot 2^{n-5}$	$27\cdot 2^{n-6}$	$17\cdot 2^{n-5}$
$T_2(n,k,n-4)$	$\frac{1}{8}(n)_6$	$\frac{1}{2}(n)_5+\frac{1}{6}(n)_6$	$\frac{1}{6}(n)_6$	$\frac{1}{24}(n)_5$

For $\alpha = n - i, \quad 5 \leq i \leq n-2,$ *and* $n \geq 7$

k	$6 \cdot 2^{n-4} + 2^{n-i-1}$	$6 \cdot 2^{n-4} + 3 \cdot 2^{n-i-2}$	$2^{n-1} + 2^{n-i-1}$
$T_2(n,k,n-i)$	$\frac{(n-2)}{(i-1)!}\,(n)_{i+1}$	$\frac{1}{(i-1)!}\,(n)_{i+2}$	$\frac{(n)_{i+1}}{i!}$

All other topologies in $\tau_2(n,k,n-i), \quad 3 \leq i \leq n-2,$ *have* $k < 6 \cdot 2^{n-4}$ *open sets.*

Now, we compute $T_1(n,k)$, for $k > 5 \cdot\ 2^{n-4}$.

Theorem 3.3. *For all* $n \geq 5$, *and* $k > 5 \cdot\ 2^{n-4}$, *we have:*

$$\begin{aligned} T_1(n, 2^{n-1}+1) &= n, \\ T_1(n, 3\cdot 2^{n-3}+1) &= (n)_3, \\ T_1(n, 5\cdot 2^{n-4}+1) &= (n)_4, \\ T_1(n,k) &= 0, \mathit{otherwise}. \end{aligned}$$

Proof. Obviously, we have $T_1(n, 2^{n-1}+1) = nT(n-1, 2^{n-1}) = n$, $T_1(n, 3\cdot 2^{n-3}+1) = nT(n-1, 3\cdot 2^{n-3}) = n\,(n-1)_2 = (n)_3$, and $T_1(n, 5\cdot 2^{n-4}+1) = nT(n-1, 5\cdot 2^{n-4}) = n(n-1)_3 = (n)_4$. If $k > 2^{n-1}+1$, we have $T(l, k-1) = 0$, for $1 \leq l \leq n-1$, so $T_1(n,k) = 0$. If $5\cdot 2^{n-4}+1 < k < 2^{n-1}+1$, and $k \neq 3\cdot 2^{n-3}+1$, the Theorem 2.1 yields $T_1(n,k) = nT(n-1,k-1)$. But we know that $T(n-1,k-1) = 0$, for $5\cdot 2^{n-4} < k-1 < 2^{n-1}$, and $k \neq 3\cdot 2^{n-3}+1$; so we deduce $T_1(n,k) = 0$, and the proof is complete. □

Now, we can give the number of all labelled topologies with $k \geq 6\cdot 2^{n-4}$ open sets.

Theorem 3.4. *Suppose that* $n \geq 7$, *then the total number of labelled topologies, with* $k \geq 6\cdot 2^{n-4}$ *open sets, is given by*

k	$T_2(n,k)$	$T_1(n,k)$	$T(n,k)$
$6\cdot 2^{n-4}$	$(n-1)(n)_3+(n)_4+$ $3\ (n)_5+\frac{5}{4}(n)_6$	0	$(n-1)(n)_3+(n)_4$ $+3\ (n)_5+\frac{5}{4}(n)_6$
$6\cdot 2^{n-4}+1$	$(n)_3$	$(n)_3$	$2\ (n)_3$
$6\cdot 2^{n-4}+2^{n-3-j},\ 4\leq j\leq n-4$	$\frac{2\ (n-2)\ (n)_{j+3}}{(j+1)!}$	0	$\frac{2\ (n-2)(n)_{j+3}}{(j+1)!}$
$6\cdot 2^{n-4}+3\cdot 2^{n-3-j},\ 5\leq j\leq n-3$	$\frac{2}{j!}(n)_{j+3}$	0	$\frac{2\ (n)_{j+3}}{j!}$
$25\cdot 2^{n-6}$	$\frac{n+14}{24}(n)_6+\frac{1}{24}(n)_7$	0	$\frac{(n+14)\ (n)_6}{24}+\frac{(n)_7}{24}$
$51\cdot 2^{n-7}$	$\frac{1}{12}(n)_7$	0	$\frac{1}{12}(n)_7$
$13\cdot 2^{n-5}$	$2\ (n)_5+\frac{1}{3}(n)_6$	0	$2\ (n)_5+\frac{1}{3}(n)_6$
$27\cdot 2^{n-6}$	$\frac{1}{2}(n)_6$	0	$\frac{1}{2}(n)_6$
$7\cdot 2^{n-4}$	$\frac{9}{4}(n)_4+(n)_5$	0	$\frac{9}{4}(n)_4+(n)_5$
$15\cdot 2^{n-5}$	$(n)_5$	0	$(n)_5$
2^{n-1}	$\frac{1}{2}(n)_2+(n)_3+(n)_4$	0	$\frac{1}{2}(n)_2+(n)_3+(n)_4$
$2^{n-1}+1$	n	n	$2\ n$
$2^{n-1}+2^{n-j-1},\ 5\leq j\leq n-2$	$\frac{2}{j!}(n)_{j+1}$	0	$\frac{2}{j!}(n)_{j+1}$
$17\cdot 2^{n-5}$	$\frac{1}{12}(n)_5$	0	$\frac{1}{12}(n)_5$
$9\cdot 2^{n-4}$	$\frac{5}{6}(n)_4$	0	$\frac{5}{6}(n)_4$
$10\cdot 2^{n-4}$	$(n)_3$	0	$(n)_3$
$3\cdot 2^{n-2}$	$(n)_2$	0	$(n)_2$
2^n	1	0	1

For $n=6$, the total number of labelled topologies having $k\geq 24$ open sets is given by

k	$\|\tau_2(6,k)\|$	$\|\tau_1(6,k)\|$	$\|\tau(6,k)\|$
24	4020	0	4020
25	480	120	600
26	1680	0	1680
27	360	0	360
28	1530	0	1530
30	720	0	720
32	495	0	495
33	6	6	12
34	60	0	60
36	300	0	300
40	120	0	120
48	30	0	30
64	1	0	1

For $n = 5$, the total number of labelled topologies having $k \geq 12$ open sets is given by

k	$\|\tau_2(5,k)\|$	$\|\tau_1(5,k)\|$	$\|\tau(5,k)\|$
12	660	0	660
13	180	60	240
14	390	0	390
15	120	0	120
16	190	0	190
17	5	5	10
18	100	0	100
20	60	0	60
24	20	0	20
32	1	0	1

For $n = 4$, the total number of labelled topologies having $k \geq 6$ open sets is given by

$k =$	6	7	8	9	10	12	16
$\|\tau_2(4,k)\|$	72	30	54	16	24	12	1
$\|\tau_1(4,k)\|$	0	24	0	4	0	0	0
$\|\tau(4,k)\|$	72	54	54	20	24	12	1

All the others topologies on $\mathbb{E}$ have $k < 6 \cdot 2^{n-4}$ open sets.

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2000 *Mathematics Subject Classification*: Primary 05A15; Secondary 06A07, 06A99.
Keywords: binary relation, enumeration, finite set, finite topology, partial order, posets.

(Concerned with sequences A000798, A001930, and A008277.)

Received April 19 2006; revised version received February 28 2007. Published in *Journal of Integer Sequences*, March 19 2007.
