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Motzkin Numbers of Higher Rank: Generating Function and Explicit Expression

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Abstract

The generating function for the (colored) Motzkin numbers of higher rank introduced recently is discussed. Considering the special case of rank one yields the corresponding results for the conventional colored Motzkin numbers for which in addition a recursion relation is given. Some explicit expressions are given for the higher rank case in the first few instances.

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1 Introduction

The classical Motzkin numbers (A001006 in [1]) count the numbers of Motzkin paths (and are also related to many other combinatorial objects, see Stanley [2]). Let us recall the definition of Motzkin paths. We consider in the Cartesian plane $\mathbb{Z} \times \mathbb{Z}$ those lattice paths starting from $(0,0)$ that use the steps $\{U, L, D\}$, where $U = (1,1)$ is an up-step, $L = (1,0)$ a level-step and $D = (1,-1)$ a down-step. Let $M(n,k)$ denote the set of paths beginning in $(0,0)$ and ending in (n,k) that never go below the x-axis. Paths in $M(n,0)$ are called *Motzkin paths* and $m_n := |M(n,0)|$ is called *n-th Motzkin number*. Sulanke showed [3] that the Motzkin numbers satisfy the recursion relation

$$(n+2)m_n = (2n+1)m_{n-1} + 3(n-1)m_{n-2} \tag{1}$$

and it is a classical fact (see Stanley [2]) that their generating function is given by

$$\sum_{n\geq 0} m_n x^n = \frac{1 - x - \sqrt{1-2x-3x^2}}{2x^2}. \tag{2}$$

Those Motzkin paths which have no level-steps are called *Dyck paths* and are enumerated by Catalan numbers (A000108 in [1]), see Stanley [2]. In recent times the above situation has been generalized by introducing colorings of the paths. For example, the *k-colored Motzkin paths* have horizontal steps colored by k colors (see [4, 5] and the references given therein). More generally, Woan introduced [6, 7] colors for each type of step. Let us denote by u the number of colors for an up-step U, by l the number of colors for a level-step L and by d the number of colors for a down-step D. (Note that if we normalize the weights as $u + l + d = 1$ we can view the paths as discrete random walks.) One can then introduce the set $M^{(u,l,d)}(n,0)$ of (u,l,d)-*colored Motzkin paths* and the corresponding (u,l,d)-*Motzkin numbers* $m_n^{(u,l,d)} := |M^{(u,l,d)}(n,0)|$. Woan has given [6] a combinatorial proof that the $(1,l,d)$-Motzkin numbers satisfy the recursion relation

$$(n+2)m_n^{(1,l,d)} = l(2n+1)m_{n-1}^{(1,l,d)} + (4d-l^2)(n-1)m_{n-2}^{(1,l,d)}. \tag{3}$$

Choosing $l = 1$ and $d = 1$ yields the recursion relation (1) of the conventional Motzkin numbers $m_n \equiv m_n^{(1,1,1)}$. Note that choosing $(u,l,d) = (1,k,1)$ corresponds to the k-colored Motzkin paths. Defining $m_{k,n} := |M^{(1,k,1)}(n,0)|$, one obtains from (3) the recursion relation $(n+2)m_{k,n} = k(2n+1)m_{k,n-1} + (4-k^2)(n-1)m_{k,n-2}$ for the number of k-colored Motzkin paths. Sapounakis and Tsikouras derived [4] the following generating function for $m_{k,n}$:

$$\sum_{n\geq 0} m_{k,n} x^n = \frac{1 - kx - \sqrt{(1-kx)^2 - 4x^2}}{2x^2}. \tag{4}$$

For $k = 1$ this identity reduces to (2) for the conventional Motzkin numbers $m_n \equiv m_{1,n}$. One of the present authors suggested [8] (as "Problem 1") that it would be interesting to find the recursion relation and generating function for the general (u,l,d)-Motzkin numbers $m_n^{(u,l,d)}$.

We will prove in Theorem 2.1 thet $m_n^{(u,l,d)} = m_n^{(1,l,ud)}$, yielding the desired recursion relation. Furthermore, a generating function and an explicit expression is derived for $m_n^{(u,l,d)}$. In [8] it was furthermore suggested to find the recursion relation and the generating function for the (colored) Motzkin numbers of higher rank ("Problem 2"). These numbers were introduced by Schork [9] in the context of "duality triads of higher rank" (and have also been considered before as "excursions", see, e.g., [10]). In view of this connection Schork conjectured [8] that the Motzkin numbers of rank r satisfy a recursion relation of order $2r + 1$. Very recently, Prodinger was the first to observe [11] that this conjecture does not hold by discussing explicitly the case $r = 2$. In particular, already for the first nontrivial cases $r = 2, 3$ the relations involved become very cumbersome. We will describe in Theorem 3.2 the generating function for the Motzkin numbers of higher rank and discuss then several particular cases explicitly.

2 Recursion relation and generating function for the general (colored) Motzkin numbers

Theorem 2.1. *The general (u, l, d)-Motzkin numbers satisfy the recursion relation*

$$(n + 2)m_n^{(u,l,d)} = l(2n + 1)m_{n-1}^{(u,l,d)} + (4ud - l^2)(n - 1)m_{n-2}^{(u,l,d)}. \tag{5}$$

A generating function is given by

$$\sum_{n\geq 0} m_n^{(u,l,d)} x^n = \frac{1 - lx - \sqrt{(1 - lx)^2 - 4udx^2}}{2udx^2}, \tag{6}$$

implying the explicit expression

$$m_n^{(u,l,d)} = \sum_{j=0}^{\frac{n}{2}} \frac{1}{j+1} \binom{2j}{j} \binom{n}{2j} u^j d^j l^{n-2j}. \tag{7}$$

Proof. Let us prove that $m_n^{(u,l,d)} = m_n^{(1,l,ud)}$ for all $n \geq 0$. In order to see that, let U be any up-step in a Motzkin path; we call (U, D) a *pair* if the down-step D is the first down-step on the right-hand side of U which has the same height as U. The set of pairs of a Motzkin path is uniquely determined and for each pair exist ud possible combinations of colorings. The same number of possible colorings result if the up-steps u are always colored white (i.e., $u = 1$) and if each down-step can be colored by ud colors (which we call "alternative colors" to distinguish them from the original colors). Thus, given a (u, l, d)-Motzkin path we may replace the colors (u_j, d_j) (with $1 \leq u_j \leq u$ and $1 \leq d_j \leq d$) for the j-th pair (U, D) by the combination of alternative colors $(1, c_j)$ where c_j is the $([u_j - 1]d + d_j)$-th alternative color (with $1 \leq c_j \leq ud$). Replacing the colors of all pairs in this fashion by the alternative colors yields a $(1, l, ud)$-Motzkin path. Thus, we have constructed a bijection between the set of (u, l, d)-Motzkin paths of length n and set of $(1, l, ud)$-Motzkin paths of length n, thereby showing that $m_n^{(u,l,d)} = m_n^{(1,l,ud)}$. From this identity and (3) we immediately obtain (5). An equation for the generating function $M_{(u,l,d)}(x) := \sum_{n\geq 0} m_n^{(u,l,d)} x^n$ is obtained from the "first return decomposition" of a nonempty (u,l,d)-Motzkin path M: either $M = LM'$ or $M = UM'DM''$, where M', M'' are (u,l,d)-Motzkin paths. The two possibilities give the contributions $lxM_{(u,l,d)}(x)$ and $udx^2(M_{(u,l,d)}(x))^2$. Hence, $M_{(u,l,d)}(x)$ satisfies

$$M_{(u,l,d)}(x) = 1 + lxM_{(u,l,d)}(x) + udx^2(M_{(u,l,d)}(x))^2,$$

yielding

$$M_{(u,l,d)}(x) = \frac{1 - lx - \sqrt{(1-lx)^2 - 4udx^2}}{2udx^2}$$

which is the asserted equation (6) for the generating function. Note that this may also be written as

$$M_{(u,l,d)}(x) = \frac{1}{1-lx} C\left(\frac{udx^2}{(1-lx)^2}\right),$$

where

$$C(y) = \frac{1-\sqrt{1-4y}}{2y} = \sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n} y^n$$

is the generating function for the Catalan numbers (see Stanley [2]). Thus,

$$M_{(u,l,d)}(x) = \sum_{j\geq 0} \frac{1}{j+1}\binom{2j}{j} \frac{u^j d^j x^{2j}}{(1-lx)^{2j+1}}.$$

Recalling $M_{(u,l,d)}(x) = \sum_n m_n^{(u,l,d)} x^n$, a comparison of coefficients shows that the number of (u,l,d)-Motzkin paths of length n is given by (7). $\square$

3 Generating function for the general (colored) Motzkin numbers of higher rank

We will now generalize the situation considered in the previous section to the case of higher rank. One of the present authors discussed [9] in the context of duality triads of higher rank (where one considers recursion relations of higher order, or equivalently, orthogonal matrix polynomials [12]) why it is interesting to consider the situation where the steps of the paths can go up or down more than one unit. The maximum number of units which a single step can go up or down will be called the rank. More precisely, let $r \geq 1$ be a natural number. The set of *admissable* steps consists of:

1. r types of up-steps $U_j = (1, j)$ with weights u_j for $1 \leq j \leq r$.

2. A level-step $L = (1, 0)$ with weight l.

3. r types of down-steps $D_j = (1, -j)$ with weights d_j for $1 \leq j \leq r$.

In the following we write $(\mathbf{u}, l, \mathbf{d}) := (u_r, \ldots, u_1, l, d_1, \ldots, d_r)$ for the vector of weights.

Definition 3.1. [9] The set $M^{(\mathbf{u},l,\mathbf{d})}(n,0)$ of $(\mathbf{u}, l, \mathbf{d})$-*colored Motzkin paths of rank* r *and length* n is the set of paths which start in $(0,0)$, end in $(n,0)$, have only admissable steps and are never below the x-axis. The corresponding number of paths, $m_n^{(\mathbf{u},l,\mathbf{d})} := |M^{(\mathbf{u},l,\mathbf{d})}(n,0)|$, will be called $(\mathbf{u}, l, \mathbf{d})$-*Motzkin number of rank* r.

Motzking paths of higher rank were considered in the literature already before [9] under the name "excursions", see, e.g., [10]. In [8] it is discussed how one may associate with each Motzkin path of rank r and length n a conventional Motzkin path of length rn in a straightforward fashion. However, it was also discussed that this association is no bijection and that the case of higher rank is more subtle. In the following we derive an equation that the generating function for the Motzkin numbers of higher rank satisfies. In order to do that we need the following notations. We denote the set of all $(\mathbf{u}, l, \mathbf{d})$-Motzkin paths of rank r that start at height s and end at height t (and have only admissable steps and are never below the x-axis) by $\mathcal{A}_{s,t}$, and we denote the subset of paths of length n by $\mathcal{A}_{s,t}(n)$. Define

$$A_{s,t} \equiv A_{s,t}(x) := \sum_{n\geq 0} |\mathcal{A}_{s,t}(n)| x^n.$$

We extend this notation by defining $A_{s,t} = 0$ for all $s < 0$ or $t < 0$. For $s, t \geq 0$ we denote the subset of paths in $\mathcal{A}_{s,t}$ that never touch the x-axis by $\mathcal{A}^*_{s,t}$. In the following we make several times use of the relation $\mathcal{A}^*_{s,t} \simeq \mathcal{A}_{s-1,t-1}$, or, $|\mathcal{A}^*_{s,t}(n)| = |\mathcal{A}_{s-1,t-1}(n)|$, implying

$$A^*_{s,t}(x) := \sum_{n\geq 0} |\mathcal{A}^*_{s,t}(n)| x^n = A_{s-1,t-1}(x). \tag{8}$$

Theorem 3.2. *The generating function* $A_{0,0} \equiv M_{(\mathbf{u},l,\mathbf{d})}(x) = \sum_{n\geq 0} m_n^{(\mathbf{u},l,\mathbf{d})} x^n$ *for the general* $(\mathbf{u}, l, \mathbf{d})$-*Motzkin numbers of rank* r *satisfies*

$$A_{0,0} = 1 + lxA_{0,0} + x^2 A_{0,0} \sum_{p=1}^{r} \sum_{q=1}^{r} u_p d_q A_{p-1,q-1}, \tag{9}$$

where the generating functions $A_{i,j}$ *with* $1 \leq i \leq r-1$ *and* $0 \leq j \leq r-1$ *satisfy*

$$A_{i,j} = A_{i-1,j-1} + xA_{0,j} \sum_{q=1}^{r} d_q A_{i-1,q-1},$$

and for all $1 \leq j \leq r-1$,

$$A_{0,j} = xA_{0,0} \sum_{p=1}^{r} u_p A_{p-1,j-1}.$$

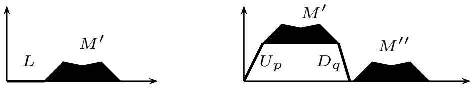

Figure 1: First return decomposition of a Motzkin path of rank r.

Proof. From the definitions, a nonempty $(\mathbf{u}, l, \mathbf{d})$-Motzkin path M can start either by a level step L or an up-step U_p with $1 \leq p \leq r$. In the case the path starts by an up-step U_p, we use the "first return decomposition‘" of M, i.e., we write M as $M = U_p M' D_q M''$ where M'' is an arbitrary $(\mathbf{u}, l, \mathbf{d})$-Motzkin path and $M' \in \mathcal{A}_{p,q}$ such that M' does not touch the height zero, see Figure 1. Thus, $M' \in \mathcal{A}^*_{p,q}$ and the generating function $A_{0,0}$ satisfies

$$A_{0,0} = 1 + lxA_{0,0} + x^2 A_{0,0} \sum_{p=1}^{r} \sum_{q=1}^{r} u_p d_q A_{p-1,q-1}$$

where we have used (8). This shows (9). Now, let us write an equation for the generating function $A_{i,j}$ (with $1 \leq i \leq r-1$ and $0 \leq j \leq r-1$) for the number of $(\mathbf{u}, l, \mathbf{d})$-Motzkin paths $Q_{ij} = U_i P_{ij} D_j$ where $P_{ij} \in \mathcal{A}_{i,j}$. In order to do that, we use the first return decomposition of such paths: either $Q_{ij} = U_i P'_{ij} D_j$ such that P'_{ij} does not touch the height zero (i.e.,

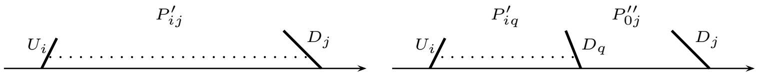

Figure 2: The case $1 \leq i \leq r-1$ and $0 \leq j \leq r-1$.

$P'_{ij} \in \mathcal{A}^*_{i,j}$), or $Q_{ij} = U_i P'_{iq} D_q P''_{0j} D_j$ such that D_q is the first down-step that touches the height zero (thus, P'_{iq} does not touch the height zero, i.e., $P'_{iq} \in \mathcal{A}^*_{i,q}$) as described in Figure 2. Using (8), the generating function $A_{i,j}$ thus satisfies

$$A_{i,j} = A_{i-1,j-1} + xA_{0,j} \sum_{q=1}^{r} d_q A_{i-1,q-1},$$

as claimed.

In order to write an equation for the generating function $A_{0,j}$ (with $0 \leq j \leq r-1$), let us consider the last up-step U_p from height zero (there must exist at least one such up-step). In this case each $(\mathbf{u}, l, \mathbf{d})$-Motzkin path can be decomposed as $P' U_p P''_{pj} D_j$, where P' is an

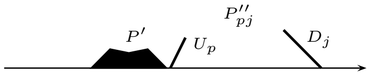

Figure 3: The case $i = 0$ and $1 \leq j \leq r-1$.

arbitrary $(\mathbf{u}, l, \mathbf{d})$-Motzkin path and $P''_{pj} \in \mathcal{A}_{p,j}$ such that it does not touch the height zero (i.e., $P''_{pj} \in \mathcal{A}^*_{p,j}$), as described in Figure 3. Thus, using again (8), the generating function $A_{0,j}$ satisfies

$$A_{0,j} = xA_{0,0} \sum_{p=1}^{r} u_p A_{p-1,j-1},$$

which completes the proof. ☐

Example 3.3. *Let us consider as an example the case $r = 1$. It follows from Theorem 3.2 that*

$$0 = 1 + (lx - 1)A_{0,0} + u_1 d_1 x^2 A_{0,0}^2, \tag{10}$$

yielding

$$A_{0,0} \equiv M_{(u_1,l,d_1)}(x) = \frac{1 - lx - \sqrt{(1 - lx)^2 - 4u_1 d_1 x^2}}{2u_1 d_1 x^2}$$

as described in the proof of Theorem 2.1. This implies - according to (7) - that

$$m_n^{(u_1,l,d_1)} = \sum_{j=0}^{n/2} \frac{1}{1+j} \binom{2j}{j} \binom{n}{2j} l^{n-2j} (u_1 d_1)^j.$$

Example 3.4. *As another example, it follows from Theorem 3.2 that one has for $r = 2$ that*

$$\begin{cases} A_{0,0} &= 1 + lxA_{0,0} + x^2 A_{0,0}(u_1 d_1 A_{0,0} + u_1 d_2 A_{0,1} + u_2 d_1 A_{1,0} + u_2 d_2 A_{1,1}), \\ A_{0,1} &= xA_{0,0}(u_1 A_{0,0} + u_2 A_{1,0}), \\ A_{1,0} &= xA_{0,0}(d_1 A_{0,0} + d_2 A_{0,1}), \\ A_{1,1} &= A_{0,0} + xA_{0,1}(d_1 A_{0,0} + d_2 A_{0,1}). \end{cases}$$

Solving the above system of equations we obtain that

$$\begin{aligned} 0 \;=\;& 1 + (lx-1)A_{0,0} - x^2(d_2 u_2 - d_1 u_1)A_{0,0}^2 + x^2(xu_1^2 d_2 + xu_2 d_1^2 - 2xlu_2 d_2 + 2d_2 u_2)A_{0,0}^3 \\ & -u_2 d_2 x^4 (d_2 u_2 - d_1 u_1)A_{0,0}^4 + x^4 u_2^2 d_2^2 (lx-1)A_{0,0}^5 + x^6 u_2^3 d_2^3 A_{0,0}^6. \end{aligned}$$

Note that, if we set $u_2 = d_2 = 0$ in the above expression then we get (10). It is also interesting to consider the case $(\mathbf{u}, l, \mathbf{d}) = (\mathbf{1}, 1, \mathbf{1})$ of non-colored Motzkin paths of rank 2, representing the most natural generalization of the conventional Motzkin paths. The above equation reduces in this case to

$$\begin{aligned} 0 &= 1 + (x-1)A_{0,0} + 2x^2 A_{0,0}^3 + x^4(x-1)A_{0,0}^5 + x^6 A_{0,0}^6 \\ &= (1 + xA_{0,0})^2 (1 - (x+1)A_{0,0} + x(x+2)A_{0,0}^2 - x^2(x+1)A_{0,0}^3 + x^4 A_{0,0}^4). \end{aligned}$$

Setting $A_{0,0} \equiv M_{(\mathbf{1},1,\mathbf{1})}(x)$ and using that $A_{0,0}$ is a formal power series (thus, $A_{0,0} \neq -\frac{1}{x}$), this is equivalent to

$$0 = 1 - (x+1)M_{(\mathbf{1},1,\mathbf{1})}(x) + x(x+2)M^2_{(\mathbf{1},1,\mathbf{1})}(x) - x^2(x+1)M^3_{(\mathbf{1},1,\mathbf{1})}(x) + x^4 M^4_{(\mathbf{1},1,\mathbf{1})}(x),$$

as described already in [10, 11]. To obtain a recursion relation for $m_n \equiv m_n^{(\mathbf{1},1,\mathbf{1})}$, one can use the MAPLE program package gfun *written by Salvy et al. [13]. Prodinger has done this [11] to obtain*

$$\begin{aligned}
&625(n+3)(n+2)(n+1)m_n - 125(n+3)(n+2)(7n+27)m_{n+1} - 50(n+3)(5n^2+24n+23)m_{n+2}\\
&+(41890+30860n+7540n^2+610n^3)m_{n+3} - (6844+5151n+1214n^2+91n^3)m_{n+4}\\
&\quad -(n+7)(23n^2+301n+976)m_{n+5} + 2(2n+13)(n+8)(n+7)m_{n+6} = 0
\end{aligned}$$

and mentions that Salvy has informed him that this recursion of order 6 *is minimal. Thus, the Motzkin numbers of rank $r = 2$ satisfy a* 7*-term recursion relation and not a $2 \cdot 2 + 1 = 5$-term relation as the conjecture of Schork [8] implies. Thus, the conjecture does not hold! This was first observed by Prodinger [11]. The first few values of $m_n \equiv m_n^{(\mathbf{1},1,\mathbf{1})}$ are given in Table 1; this sequence is sequence A104184 in [1].*

	$n=1$	2	3	4	5	6	7	8	9	10
$m_n^{(\mathbf{1},1,\mathbf{1})}$	1	3	9	32	120	473	1925	8034	34188	147787

Table 1: The first few values of the non-colored Motzkin numbers of rank two.

The above example shows that the general case (where the weights are not restricted to 1) seems to be extremely complicated (see Theorem 3.2). Thus, from now on let us consider the case where all weights are equal to 1, i.e., $u_i = l = d_i = 1$ for $1 \leq i \leq r$. In this case we denote the set of paths by $\mathcal{B}_{s,t}$ or $\mathcal{B}_{s,t}(n)$ and the generating function by $B_{i,j}$ (instead of $A_{i,j}$). Then one has

$$B_{s,t} = B_{t,s} \tag{11}$$

which can be easily seen since to each path $P \in \mathcal{B}_{s,t}(n)$ one can associate a path $P' \in \mathcal{B}_{t,s}(n)$ by traversing P in opposite direction. Since this is clearly a bijection the above equation follows. Theorem 3.2 shows that one has to solve in the general case a system of r^2 equations for the r^2 unknowns $A_{i,j}$. Due to the symmetry (11) this reduces in the case where all weights are equal to 1 to a system of $\frac{r(r+1)}{2}$ equations in the $\frac{r(r+1)}{2}$ unknowns $B_{i,j}$ (where $r-1 \geq i \geq j \geq 0$).

Example 3.5. *Theorem 3.2 with (11) gives for $r = 3$ the following set of $\frac{3 \cdot 4}{2} = 6$ equations*

$$\begin{cases}
B_{0,0} &= 1 + xB_{0,0} + x^2 B_{0,0}(B_{0,0} + 2B_{1,0} + 2B_{2,0} + 2B_{2,1} + B_{1,1} + B_{2,2}),\\
B_{1,0} &= xB_{0,0}(B_{0,0} + B_{1,0} + B_{2,0}),\\
B_{2,0} &= xB_{0,0}(B_{1,0} + B_{1,1} + B_{2,1}),\\
B_{1,1} &= B_{0,0} + xB_{1,0}(B_{0,0} + B_{1,0} + B_{2,0}),\\
B_{2,1} &= B_{1,0} + xB_{1,0}(B_{1,0} + B_{1,1} + B_{2,1}),\\
B_{2,2} &= B_{1,1} + xB_{2,0}(B_{1,0} + B_{1,1} + B_{2,1}).
\end{cases}$$

Solving the above system of equations we obtain that the generating function $B_{0,0} \equiv M_{(\mathbf{1},1,\mathbf{1})}(x)$ *for the non-colored Motzkin numbers of rank 3 satisfies*

$$0 = 1 - (1+x)B_{0,0} + 2xB_{0,0}^2 + x^2(1-2x)B_{0,0}^4 + 2x^5B_{0,0}^6 - x^6(1+x)B_{0,0}^7 + x^8B_{0,0}^8.$$

Now, we would like to give a recursion relation for the sequence $m_n^{(\mathbf{1},1,\mathbf{1})}$. *This can be automatically done with MAPLE's program* gfun*: The procedure "*algeqtodiffeq*" translates the (algebraic) equation for* $B_{0,0} \equiv M_{(\mathbf{1},1,\mathbf{1})}(x)$ *into an equivalent differential equation of order* 7 *(it is too long to present here) and then the procedure "*diffeqtorec*" translates the differential equation into a* 28*-term recursion relation. The first few values of* $m_n^{(\mathbf{1},1,\mathbf{1})}$ *are given in Table 2; this sequence seems not to be listed in [1].*

	$n=1$	2	3	4	5	6	7	8	9	10
$m_n^{(\mathbf{1},1,\mathbf{1})}$	1	4	16	78	404	2208	12492	72589	430569	2596471

Table 2: The first few values of the non-colored Motzkin numbers of rank three.

Example 3.6. *Theorem 3.2 with (11) gives for* $r=4$ *a set of* $\frac{4\cdot 5}{2} = 10$ *equations for the* $B_{i,j}$. *As above, it is possible to solve this system and obtain that the generating function* $B_{0,0} \equiv M_{(\mathbf{1},1,\mathbf{1})}(x)$ *for the non-colored Motzkin numbers of rank 4 satisfies the following equation*

$$\begin{aligned}0 \;=\;& 1 + (x-1)B_{0,0} - 2xB_{0,0}^2 - x(x+2)(x-1)B_{0,0}^3 - x^2(x-2)(x+2)B_{0,0}^4 \\ &+x^2(x-1)B_{0,0}^5 + x^3(x-2)(x+1)^2B_{0,0}^6 + x^4(x+1)(x-1)^2B_{0,0}^7 - x^5(2x^2-3x-4)B_{0,0}^8 \\ &+x^6(x+1)(x-1)^2B_{0,0}^9 + x^7(x-2)(x+1)^2B_{0,0}^{10} + x^8(x-1)B_{0,0}^{11} - x^{10}(x-2)(x+2)B_{0,0}^{12} \\ &-x^{11}(x+2)(x-1)B_{0,0}^{13} - 2x^{13}B_{0,0}^{14} + x^{14}(x-1)B_{0,0}^{15} + x^{16}B_{0,0}^{16}.\end{aligned}$$

The first few values of the corresponding Motzkin numbers $m_n^{(\mathbf{1},1,\mathbf{1})}$ *of rank 4 are given in Table 3; this sequence seems not to be listed in [1].*

	$n=1$	2	3	4	5	6	7	8	9	10
$m_n^{(\mathbf{1},1,\mathbf{1})}$	1	5	25	155	1025	7167	51945	387000	2944860	22791189

Table 3: The first few values of the non-colored Motzkin numbers of rank four.

Remark. Let us consider the non-colored Motzkin numbers of rank r (i.e., all weights are equal to 1). They satisfy a $\tau(r)$-term recursion relation where $\tau(r)$ is defined by this property (and is minimal). This yields a well-defined sequence $\{\tau(r)\}_{r\in\mathbb{N}}$ starting - according to the above examples - with $3, 7, 28, \ldots$. The original conjecture $\tau(r) = 2r + 1$ proved to be too naive, but it seems that an exact formula for $\tau(r)$ will be very difficult to obtain. Thus, approximations or bounds would also be interesting. Considering the equations for the generating functions $B_{0,0} \equiv M_{(\mathbf{1},1,\mathbf{1})}(x)$ of non-colored Motzkin numbers given in the above examples for rank $r \leq 4$, one observes that they all have the form $f(x, B_{0,0}) = 0$ where $f(x, y)$ is a polynomial of degree 2^r in y with coefficients in $\mathbb{Z}[x]$ (and with constant coefficient 1); more precisely, the coefficient $a_i(x)$ of y^i is a polynomial of degree at most i in x over the integers $\mathbb{Z}$. Thus, we can write

$$0 = 1 + \sum_{i=1}^{2^r} \sum_{j=0}^{i} a_{i,j} x^j (B_{0,0})^i \tag{12}$$

where $a_{i,j} \in \mathbb{Z}$. It would be interesting to find out whether this representation holds for all ranks (which is what we expect) or whether it is confined to $r \leq 4$.

We would like to close this paper by stressing that it is still an open problem to derive a recursion relation for the Motzkin numbers of higher rank. As the explicit examples $r = 2, 3, 4$ and the complicated set of equations for the generating function given in Theorem 3.2 show this will be a rather daunting task (see also the last remark).

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