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Certain Sums Involving Inverses of Binomial Coefficients and Some Integrals

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Abstract

In this paper, we are concerned with sums involving inverses of binomial coefficients. We study certain sums involving reciprocals of binomial coefficients by using some integrals. Some recurrence relations related to inverses of binomial coefficients are obtained. In addition, we give the approximate values of certain sums involving the inverses of binomial coefficients.

1 Introduction

It is well known that binomial coefficients play an important role in various subjects such as combinatorics, number theory, and probability. There are many results for sums related to binomial coefficients. Sums involving inverses of binomial coefficients have been receiving much attention. For example, see [1]-[2] or [4]-[14]. In this paper, we are still interested in sums involving inverses of binomial coefficients, and we investigate these kinds of sums by using some integrals. For convenience, we first give the definition of binomial coefficients.

For nonnegative integers m and n, the binomial coefficient $\binom{n}{m}$ is defined by

$$\binom{n}{m} = \begin{cases} \dfrac{n!}{m!(n-m)!}, & \text{if } n \geq m, \\ 0, & \text{if } n < m. \end{cases}$$

We know that integral is an effective method for computing sums involving inverses of binomial coefficients (see [8, 11]). It is based on Euler's well-known Beta function defined by (see [11])

$$B(n,m) = \int_0^1 t^{n-1}(1-t)^{m-1}dt$$

for all positive integers n and m. Since $B(n,m) = \dfrac{\Gamma(n)\Gamma(m)}{\Gamma(n+m)} = \dfrac{(n-1)!(m-1)!}{(n+m-1)!}$, the binomial coefficient $\binom{n}{m}$ satisfies the equation

$$\binom{n}{m}^{-1} = (n+1)\int_0^1 t^m(1-t)^{n-m}dt. \tag{1}$$

the equation It is clear that integrals have connections with inverses of binomial coefficients. The purpose of this paper is to study sums of the following forms:

$$\sum_{n=0}^{\infty} \frac{\binom{n+k}{n} f_n}{\binom{2n}{n}} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{f_n}{\binom{2n}{n}\binom{n+k}{n}},$$

where f_n is a rational function of n. In Section 2, we derive some relations for $\sum_{n=0}^{\infty} \frac{\binom{n+k}{n} f_n}{\binom{2n}{n}}$ by means of (1) and other integrals. In the meantime, we express the series $\sum_{n=0}^{\infty} \frac{1}{\binom{2n}{n}^2}$ by some double integrations. In addition, we discuss the approximate value of the series $\sum_{n=1}^{\infty} \frac{1}{n^s\binom{2n}{n}}$, where s is a positive integer.

2 Main Results

Lemma 1. *For the integrals* $\int_0^1 \frac{dx}{[1-x(1-x)]^{k+1}}$ *and* $\int_0^1 \frac{dx}{[1+x(1-x)]^{k+1}}$, *we have*

$$\int_0^1 \frac{dx}{[1-x(1-x)]^{k+2}} = \frac{2}{3(k+1)} + \frac{2(2k+1)}{3(k+1)} \int_0^1 \frac{dx}{[1-x(1-x)]^{k+1}}, \tag{2}$$

$$\int_0^1 \frac{dx}{[1+x(1-x)]^{k+2}} = \frac{2}{5(k+1)} + \frac{2(2k+1)}{5(k+1)} \int_0^1 \frac{dx}{[1+x(1-x)]^{k+1}}. \tag{3}$$

Proof. The proofs of (2)-(3) are simple and are omitted here. □

Theorem 2. *Assume that*

$$A_k = \sum_{n=0}^{\infty} \frac{\binom{n+k}{n}}{\binom{2n}{n}} \quad \text{and} \quad B_k = \sum_{n=0}^{\infty} \frac{(-1)^n \binom{n+k}{n}}{\binom{2n}{n}},$$

where k is a nonnegative integer. Then we have the following recurrence relations:

$$A_{k+1} = -\frac{2}{3(k+1)} + \frac{2(2k+3)}{3(k+1)} A_k, \tag{4}$$

$$B_{k+1} = -\frac{2}{5(k+1)} + \frac{2(2k+3)}{5(k+1)} B_k. \tag{5}$$

Proof. It follows from (1) that

$$A_k = \sum_{n=0}^{\infty} \binom{n+k}{n} (2n+1) \int_0^1 x^n (1-x)^n dx.$$

Since $\sum_{n=0}^{\infty} \binom{n+k}{n} (2n+1) \int_0^1 x^n (1-x)^n dx$ converges uniformly for $x \in [0,1]$, we have

$$A_k = \int_0^1 \left[\sum_{n=0}^{\infty} \binom{n+k}{n} (2n+1) x^n (1-x)^n \right] dx.$$

It is well known that

$$\sum_{n=0}^{\infty} \binom{n+k}{n} u^n = \frac{1}{(1-u)^{k+1}}, \quad \text{for} \quad |u| < 1, \tag{6}$$

and

$$\sum_{n=0}^{\infty} n \binom{n+k}{n} u^n = \frac{(k+1)u}{(1-u)^{k+2}}, \quad \text{for} \quad |u| < 1. \tag{7}$$

From (6) and (7) we have

$$\begin{aligned} A_k &= \int_0^1 \frac{dx}{[1-x(1-x)]^{k+1}} + 2(k+1) \int_0^1 \frac{x(1-x)dx}{[1-x(1-x)]^{k+2}} \\ &= -(2k+1) \int_0^1 \frac{dx}{[1-x(1-x)]^{k+1}} + 2(k+1) \int_0^1 \frac{dx}{[1-x(1-x)]^{k+2}}. \end{aligned}$$

Using the same method, we have

$$B_k = -(2k+1) \int_0^1 \frac{dx}{[1+x(1-x)]^{k+1}} + 2(k+1) \int_0^1 \frac{dx}{[1+x(1-x)]^{k+2}}.$$

It follows from (2) and (3) that

$$A_k = \frac{4}{3} + \frac{2k+1}{3}\int_0^1 \frac{dx}{[1-x(1-x)]^{k+1}} \tag{8}$$

$$B_k = \frac{4}{5} - \frac{2k+1}{5}\int_0^1 \frac{dx}{[1+x(1-x)]^{k+1}}. \tag{9}$$

From (8) and (9) we have recurrence relations (4) and (5). □

Corollary 3. *Let m be a positive integer. Define*

$$A_{k,m} = \sum_{n=0}^{\infty} \frac{\binom{n+k}{n}}{\binom{2mn}{mn}} \quad \text{and} \quad B_{k,m} = \sum_{n=0}^{\infty} \frac{(-1)^n\binom{n+k}{n}}{\binom{2mn}{mn}}.$$

Then

$$\begin{aligned} A_{k,m} &= -(2mk+2m-1)\int_0^1 \frac{dx}{[1-x^m(1-x)^m]^{k+1}} \\ &\quad +2m(k+1)\int_0^1 \frac{dx}{[1-x^m(1-x)^m]^{k+2}} \\ B_{k,m} &= -(2mk+2m-1)\int_0^1 \frac{dx}{[1+x^m(1-x)^m]^{k+1}} \\ &\quad +2m(k+1)\int_0^1 \frac{dx}{[1+x^m(1-x)^m]^{k+2}}. \end{aligned}$$

Theorem 4. *Suppose that*

$$C_k = \sum_{n=1}^{\infty} \frac{\binom{n+k}{n}}{n\binom{2n}{n}} \quad \text{and} \quad D_k = \sum_{n=1}^{\infty} \frac{\binom{n+k-1}{k}}{n^2\binom{2n}{n}}.$$

Then we have

$$C_k = \frac{1}{2}\sum_{i=1}^{k+1}\int_0^1 \frac{dx}{[1-x(1-x)]^i}, \tag{10}$$

$$D_k = \frac{1}{2k}\sum_{i=1}^{k}\int_0^1 \frac{dx}{[1-x(1-x)]^i}, \tag{11}$$

$$C_{k+1} = \frac{1}{3(k+1)} + \frac{7k+5}{3(k+1)}C_k - \frac{4k+2}{3(k+1)}C_{k-1}, \tag{12}$$

$$D_{k+1} = \frac{1}{3k(k+1)} + \frac{7k-2}{3(k+1)}D_k - \frac{2(2k-1)(k-1)}{3k(k+1)}D_{k-1}. \tag{13}$$

Proof. It is evident that

$$\begin{aligned} C_k &= \frac{1}{2}\sum_{n=1}^{\infty} \frac{\binom{n+k}{n}}{(2n-1)\binom{2n-2}{n-1}} \\ D_k &= \frac{1}{2}\sum_{n=1}^{\infty} \frac{\binom{n+k-1}{k}}{n(2n-1)\binom{2n-2}{n-1}}. \end{aligned}$$

It follows from (1) that

$$\begin{aligned} C_k &= \frac{1}{2}\sum_{n=1}^{\infty}\binom{n+k}{n}\int_0^1 x^{n-1}(1-x)^{n-1}dx = \frac{1}{2}\int_0^1\sum_{n=1}^{\infty}\binom{n+k}{n}x^{n-1}(1-x)^{n-1}dx. \\ D_k &= \frac{1}{2}\int_0^1\sum_{n=1}^{\infty}\frac{\binom{n+k-1}{k}}{n}x^{n-1}(1-x)^{n-1}dx. \end{aligned}$$

Owing to (6), (10) holds.

We note that

$$\sum_{n=1}^{\infty}\frac{\binom{n+k-1}{k}}{n}u^n = \frac{1}{k}\left[\frac{1}{(1-u)^k}-1\right], \quad \text{for} \quad |u|<1. \tag{14}$$

It follows from (14) that

$$D_k = \frac{1}{2k}\int_0^1\frac{1}{x(1-x)}\left\{\frac{1}{[1-x(1-x)]^k}-1\right\}dx.$$

Hence, (11) holds.

It is clear that

$$\begin{aligned} C_{k+1}-C_k &= \frac{1}{2}\int_0^1\frac{dx}{[1-x(1-x)]^{k+2}} \\ 2(k+1)D_{k+1}-2kD_k &= \int_0^1\frac{1}{[1-x(1-x)]^{k+1}}dx \end{aligned}$$

Using (2), we can obtain (12)-(13). □

Remark: By using the same method, the reader can consider the sums

$$\sum_{n=1}^{\infty}\frac{(-1)^n\binom{n+k}{n}}{n\binom{2n}{n}} \quad \text{and} \quad \sum_{n=1}^{\infty}\frac{(-1)^n\binom{n+k-1}{k}}{n^2\binom{2n}{n}}.$$

Theorem 5. *Let*

$$E_k = \sum_{n=0}^{\infty}\frac{1}{\binom{2n}{n}\binom{n+k}{n}}, \quad F_k = \sum_{n=1}^{\infty}\frac{1}{n\binom{2n}{n}\binom{n+k}{n}}, \quad G_k = \sum_{n=1}^{\infty}\frac{1}{n^2\binom{2n}{n}\binom{n+k}{n}}.$$

Then

$$
\begin{aligned}
E_k &= 12\sum_{i=0}^{k}\binom{k}{i}(-3)^i d_{k-1,i} + 4(2k-3)\sum_{i=0}^{k}\binom{k}{i}(-3)^i d_{k,i} \\
&\quad -8k\sum_{i=0}^{k}\binom{k}{i}(-3)^i d_{k+1,i} + 24\int_0^1 \frac{(1-y)^k dy}{(4-y)^2} \\
&\quad +(4k-2)\int_0^1 \frac{(1-y)^k dy}{4-y}, \qquad (15) \\
F_k &= -8\sum_{i=0}^{k}\binom{k}{i}(-3)^i d_{k+1,i} + 8\sum_{i=0}^{k}\binom{k}{i}(-3)^i d_{k,i} + 4\int_0^1 \frac{(1-y)^k}{4-y}dy \\
&\quad -\frac{1}{k+1} + 16(k+1)\sum_{i=0}^{k}\binom{k}{i}(-3)^i d_{k+2,i+1}, \qquad (16) \\
G_k &= 16(k+1)\sum_{i=0}^{k}\binom{k}{i}(-3)^i g_{k+2,i} + 16\sum_{i=0}^{k}\binom{k}{i}(-3)^i d_{k+2,i+1}, \qquad (17)
\end{aligned}
$$

where

$$
d_{m,i} = \int_0^{\frac{1}{\sqrt{3}}} \frac{u^{2i}\arctan u}{(1+u^2)^m}du, \quad g_{m,i} = \int_0^{\frac{1}{\sqrt{3}}} \frac{u^{2i+1}(\arctan u)^2}{(1+u^2)^m}du,
$$

and they satisfy the equations

$$
d_{m,i} + d_{m,i+1} = d_{m-1,i}, \qquad (18)
$$

$$
g_{m,i} + g_{m,i+1} = g_{m-1,i}. \qquad (19)
$$

Proof. It follows from (1) that

$$
\begin{aligned}
E_k &= \sum_{n=0}^{\infty}(2n+1)(n+k+1)\int_0^1 x^n(1-x)^n dx \int_0^1 y^n(1-y)^k dy \\
&= \sum_{n=0}^{\infty}(2n+1)(n+1+k)\int_0^1\int_0^1 x^n(1-x)^n y^n(1-y)^k dxdy \\
&= 2\sum_{n=0}^{\infty} n^2 \int_0^1\int_0^1 x^n(1-x)^n y^n(1-y)^k dxdy \\
&\quad +(2k+3)\sum_{n=0}^{\infty} n \int_0^1\int_0^1 x^n(1-x)^n y^n(1-y)^k dxdy \\
&\quad +(k+1)\sum_{n=0}^{\infty}\int_0^1\int_0^1 x^n(1-x)^n y^n(1-y)^k dxdy.
\end{aligned}
$$

By using (6)-(7) and

$$
\sum_{n=0}^{\infty} n^2 u^n = \frac{1}{1-u} - \frac{3}{(1-u)^2} + \frac{2}{(1-u)^3}, \qquad (20)
$$

we obtain

$$\begin{aligned} E_k &= 4\int_0^1\int_0^1 \frac{(1-y)^k}{[1-x(1-x)y]^3}dxdy + (2k-3)\int_0^1\int_0^1 \frac{(1-y)^k}{[1-x(1-x)y]^2}dxdy \\ &\quad -k\int_0^1\int_0^1 \frac{(1-y)^k}{1-x(1-x)y}dxdy. \end{aligned}$$

When $|y|\le 1$, put $I_n(y)=\displaystyle\int_0^1 \frac{dx}{[1-x(1-x)y]^n}$. One can verify that

$$I_{n+1}(y) = \frac{2(2n-1)}{n(4-y)}I_n(y) + \frac{2}{n(4-y)}. \tag{21}$$

From (21), we have

$$\begin{aligned} E_k &= 4\int_0^1 (1-y)^k\left[\frac{6}{(4-y)^2}I_1(y) + \frac{6}{(4-y)^2} + \frac{1}{4-y}\right]dy \\ &\quad +(2k-3)\int_0^1 (1-y)^k\left[\frac{2}{4-y}I_1(y) + \frac{2}{4-y}\right]dy - k\int_0^1 (1-y)^k I_1(y)dy. \end{aligned}$$

Noting that

$$I_1(y) = \frac{4}{\sqrt{y(4-y)}}\arctan\sqrt{\frac{y}{4-y}},$$

$$\begin{aligned} E_k &= 96\int_0^1 \frac{(1-y)^k}{(4-y)^2\sqrt{y(4-y)}}\arctan\sqrt{\frac{y}{4-y}}dy + 24\int_0^1 \frac{(1-y)^k}{(4-y)^2}dy \\ &\quad +(4k-2)\int_0^1 \frac{(1-y)^k}{4-y}dy + 8(2k-3)\int_0^1 \frac{(1-y)^k}{(4-y)\sqrt{y(4-y)}}\arctan\sqrt{\frac{y}{4-y}}dy \\ &\quad -4k\int_0^1 \frac{(1-y)^k}{\sqrt{y(4-y)}}\arctan\sqrt{\frac{y}{4-y}}dy. \end{aligned}$$

Put $\sqrt{\dfrac{y}{4-y}}=u$. Then we get $y=\dfrac{4u^2}{1+u^2}$, $dy=\dfrac{8udu}{(1+u^2)^2}$, and

$$\begin{aligned} E_k &= 12\int_0^{\frac{1}{\sqrt3}} \frac{(1-3u^2)^k\arctan u}{(1+u^2)^{k-1}}du + 4(2k-3)\int_0^{\frac{1}{\sqrt3}} \frac{(1-3u^2)^k\arctan u}{(1+u^2)^k}du \\ &\quad -8k\int_0^{\frac{1}{\sqrt3}} \frac{(1-3u^2)^k\arctan u}{(1+u^2)^{k+1}}du + 24\int_0^1 \frac{(1-y)^k dy}{(4-y)^2} + (4k-2)\int_0^1 \frac{(1-y)^k dy}{4-y}. \end{aligned}$$

Hence (15) holds.

Using a similar approach, we have

$$\begin{aligned}
F_k &= \sum_{n=1}^{\infty} 2n \int_0^1 \int_0^1 (xy)^n (1-x)^n (1-y)^k dxdy \\
&\quad +(2k+3) \sum_{n=1}^{\infty} \int_0^1 \int_0^1 (xy)^n (1-x)^n (1-y)^k dxdy \\
&\quad +(k+1) \sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 \int_0^1 (xy)^n (1-x)^n (1-y)^k dxdy \\
&= 2 \int_0^1 \int_0^1 \frac{(1-y)^k xy(1-x)}{[1-x(1-x)y]^2} dxdy + (2k+3) \int_0^1 \int_0^1 \frac{(1-y)^k xy(1-x)}{1-xy(1-x)} dxdy \\
&\quad -(k+1) \int_0^1 \int_0^1 (1-y)^k \ln[1-xy(1-x)] dxdy \\
&= -\int_0^1 (1-y)^k I_1(y) dy + 4 \int_0^1 \frac{(1-y)^k}{4-y} I_1(y) dy + 4 \int_0^1 \frac{(1-y)^k}{4-y} dy - \frac{1}{k+1} \\
&\quad + \frac{k+1}{2} \int_0^1 (1-y)^k y I_1(y) dy \\
&= -8 \int_0^{\frac{1}{\sqrt{3}}} \frac{(1-3u^2)^k \arctan u}{(1+u^2)^{k+1}} du + 8 \int_0^{\frac{1}{\sqrt{3}}} \frac{(1-3u^2)^k \arctan u}{(1+u^2)^k} du \\
&\quad +4 \int_0^1 \frac{(1-y)^k}{4-y} dy - \frac{1}{k+1} + 16(k+1) \int_0^{\frac{1}{\sqrt{3}}} \frac{u^2(1-3u^2)^k \arctan u}{(1+u^2)^{k+2}} du.
\end{aligned}$$

Then (16) holds.

Now we prove (17). After calculus, we have

$$G_k = \frac{k+1}{2} \int_0^1 (1-y)^k dy \int_0^1 \frac{\ln[1-x(1-x)y]}{-x(1-x)} dx + \frac{1}{2} \int_0^1 y(1-y)^k I_1(y) dy.$$

Let

$$h(y) = \int_0^1 \frac{\ln[1-x(1-x)y]}{-x(1-x)} dx, \quad 0 \le y \le 1.$$

We can verify that

$$\begin{aligned}
h'(y) &= \int_0^1 \frac{1}{1-x(1-x)y} dx \\
&= \frac{4}{y} \sqrt{\frac{y}{4-y}} \arctan \sqrt{\frac{y}{4-y}}, \\
h(y) &= 4 \left(\arctan \sqrt{\frac{y}{4-y}} \right)^2.
\end{aligned}$$

Then

$$G_k = 2(k+1) \int_0^1 (1-y)^k \left(\arctan \sqrt{\frac{y}{4-y}} \right)^2 dy + \frac{1}{2} \int_0^1 y(1-y)^k I_1(y) dy.$$

By the proof of (15), we get

$$G_k = 16(k+1)\int_0^{\frac{1}{\sqrt{3}}} \frac{(1-3u^2)^k u(\arctan u)^2}{(1+u^2)^{k+2}}du + 16\int_0^{\frac{1}{\sqrt{3}}} \frac{(1-3u^2)^k u^2 \arctan u}{(1+u^2)^{k+2}}du.$$

Then (17) holds. The proofs of (18)-(19) are omitted here. □

Now, we express the series $\sum_{n=0}^{\infty} \frac{1}{\binom{2n}{n}^2}$ by means of some double integrations.

Theorem 6. *For the series* $\sum_{n=0}^{\infty} \frac{1}{\binom{2n}{n}^2}$, *we have*

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{1}{\binom{2n}{n}^2} &= \int_0^1\int_0^1 \frac{dxdy}{1-x(1-x)y(1-y)} - 8\int_0^1\int_0^1 \frac{1}{[1-x(1-x)y(1-y)]^2}dxdy \\ &\quad +8\int_0^1\int_0^1 \frac{1}{[1-x(1-x)y(1-y)]^3}dxdy. \end{aligned} \tag{22}$$

Proof. It follows from (1) that

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{1}{\binom{2n}{n}^2} &= \sum_{n=0}^{\infty}(4n^2+4n+1)\int_0^1 x^n(1-x)^n dx \int_0^1 y^n(1-y)^n dy \\ &= \sum_{n=0}^{\infty}(4n^2+4n+1)\int_0^1\int_0^1 x^n(1-x)^n y^n(1-y)^n dxdy.\end{aligned}$$

From (6)-(7) and (20) we can prove that (23) holds. □

Finally, we discuss the computation of the series $\sum_{n=1}^{\infty} \frac{1}{n^s\binom{2n}{n}}$.

Let

$$\psi(s) = \sum_{n=1}^{\infty} \frac{1}{n^s\binom{2n}{n}}.$$

One knows that (see [3])

$$\psi(1) = \frac{\sqrt{3}\pi}{9}, \quad \psi(2) = \frac{\pi^2}{18}, \quad \text{and} \quad \psi(4) = \frac{17\pi^4}{3240}.$$

But we do not know the accurate value of $\psi(s)$ ($s = 3$ or $s \geq 5$). Now we give the approximate value of $\psi(s)$. It is clear that

$$\begin{aligned}\psi(s) - \frac{1}{2} - \frac{1}{6\times 2^s} &= \sum_{n=3}^{\infty} \frac{1}{n^s\binom{2n}{n}} < \sum_{n=3}^{\infty}\frac{1}{n^s} = 3^{-s}\left(1 + \frac{3^s}{4^s} + \frac{3^s}{5^s} + \cdots\right) \\ &< 3^{-s}\left(1 + \int_3^{\infty}\frac{3^s}{x^s}dx\right) = 3^{-s}\left(1 + \frac{3}{s-1}\right).\end{aligned}$$

Then we have

$$\lim_{s\to+\infty} s^3\left(\psi(s)-\frac{1}{2}-\frac{1}{6\times 2^s}\right)=0. \tag{23}$$

Using a similar approach, we have

$$\lim_{s\to+\infty} s^3\left(\psi(s-\psi(s+1)-\frac{1}{6\times 2^{s+1}}\right)=0. \tag{24}$$

(23)-(24) provide the following simple approximate

$$\psi(s)\approx\frac{1}{2}+\frac{1}{6\times 2^s}, \tag{25}$$

$$\psi(s+1)\approx\psi(s)-\frac{1}{6\times 2^{s+1}}. \tag{26}$$

By (25)-(26), we obtain

$$\psi(6)\approx\frac{193}{384},\quad \psi(3)\approx\frac{\pi^2}{18}-\frac{1}{48},\quad \psi(5)\approx\frac{17\pi^4}{3240}-\frac{1}{192}.$$

Similarly, we get

$$\sum_{n=1}^{\infty}\frac{1}{n^s\binom{2n}{n}^2}\approx\frac{1}{4}+\frac{1}{36\times 2^s},\quad \sum_{n=1}^{\infty}\frac{1}{n^s\binom{2n}{n}^r}\approx\frac{1}{2^r}+\frac{1}{2^s\times 6^r},\quad r\geq 3.$$

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