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Riordan Arrays, Sheffer Sequences and "Orthogonal" Polynomials

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Abstract

Riordan group concepts are combined with the basic properties of convolution families of polynomials and Sheffer sequences, to establish a duality law, canonical forms $\rho(n,m) = \binom{n}{m} c^m F_{n-m}(m),\ c \neq 0$, and extensions $\rho(x, x-k) = (-1)^k x^{\underline{k+1}} c^{x-k} F_k(x)$, where the $F_k(x)$ are polynomials in x, holding for each $\rho(n,m)$ in a Riordan array. Examples $\rho(n,m) = \binom{n}{m} S_k(x)$ are given, in which the $S_k(x)$ are "orthogonal" polynomials currently found in mathematical physics and combinatorial analysis.

1 Introduction

We derive from basic principles in the theory of *convolution families* [10] and *Sheffer sequences* [16] of polynomials, *canonical* forms $\rho(n,m) = \binom{n}{m} c^m \mathrm{F}_{n-m}(m)$, $c \neq 0$, and *extensions* $\rho(x, x-k) = x^{\underline{k}} c^{x-k} \mathrm{F}_k(x-k)/k! = (-1)^k x^{\underline{k+1}} c^{x-k} \rho_k(x)$, holding for all elements ρ in the Riordan group. We show in Section 3 that the *extensions* are of *polynomial* type when $c = 1$. In Section 2, we define *transformation rules* and a *duality law*, that will greatly simplify algebraic manipulations. In Section 4, we give examples $\rho(n,m) = \binom{n}{m} \mathrm{S}_{n-m}(m)$, in which $\mathrm{S}_k(x)$ are "orthogonal" polynomials currently found in mathematical physics and combinatorial analysis. The concluding remarks are in Section 5.

For easy reference, we recall some definitions. The Riordan group is a set of invertible infinite lower triangular matrices $M = \{\rho(n,m)\}_{n,m\geq 0}$, called *Riordan matrices*, with entries:

$$\rho(n,m) = \left[\frac{u^n}{n!}\right] g(u)\frac{f(u)^m}{m!} = \frac{n!}{m!}[u^{n-m}]g(u)\left(\frac{f(u)}{u}\right)^m, \ \rho(0,0)=1, \tag{1}$$

$$\text{equivalently,} \qquad \sum_{n\geq m\geq 0} \rho(n,m)\frac{u^n}{n!} = g(u)\frac{f(u)^m}{m!} \qquad \text{(“exponential” Riordan group),}$$

where $g(u) = 1 + g_1 u + g_2 u^2 + \ldots$ is invertible, with $\widehat{g}(u) = 1/g(u) = \sum_{n\geq 0} \widehat{g}_n u^n$, and $f(u) = f_1 u + f_2 u^2 + \ldots$ is a *delta series* with compositional inverse $\overline{f}(u)$, $f(\overline{f}(u)) = \overline{f}(f(u)) = u$. A group element is denoted by $\rho = (g(u),\ f(u))$ and the group law $\star$ is $(l(u),\ h(u)) \star (g(u),\ f(u)) = (l(u)g(h(u)),\ f(h(u))$, with identity $I = (1,u)$ and group inverse $\rho^{-1} = (1/g(\overline{f}(u)),\ \overline{f}(u))$. A *group representation* $\rho = \rho_1 \star \rho_2$, in matrix notation, is $\rho(n,m) = \sum_i \rho_1(n,i)\rho_2(i,m)$. As we know, group representations are extensively used in the study of identities [13, 18, 14, 15] and they will play an essential role in this work. A sequence $a_k(x)$, given by $\sum_k a_k(x)u^k = a(u)^x$, is a *convolution family* of polynomials iff $a(0) = 1$ [10]. In a convolution family the polynomials $a_k(x)$ are multiples of x for $k > 0$, having degree $\leq k$, and $xa_0(x) \equiv 1$. A Sheffer sequence $\mathcal{S}_k(x)$ for $(g(u),\ f(u))$ [16] is given by $\sum_{k\geq 0} \mathcal{S}_k(x)u^k/k! = (1/g(\overline{f}(u)))e^{x\overline{f}(u)}$, where $(g(u),\ f(u))$ are as in Riordan group theory; the Sheffer sequence for $f = (1,\ f(u))$ is the *associated* sequence for $f(u)$, and the Sheffer sequence for $(g(u),\ u)$ is the *Appell* sequence for $g(u)$.

2 Transformation Rules and the Duality Law

We first define some useful *transformation rules*, noting that transformation rules and Barry's “transforms” [2] are different concepts.

1) The *duality rule* “$\sim$” for a function $f(n,m)$, n,m integers, is $\widetilde{f}(n,m) = f(-m,-n)$ and for a function $f_k(x)$, the rule is $\widetilde{f}_k(x) = f_k(k-x)$. When $x = n$ and $k = n-m$, the two rules coincide. The following special cases will be used in the sequel, without reference.

$$\widetilde{\binom{n}{m}} = \widetilde{\binom{n}{n-m}} = \binom{-m}{n-m} = (-1)^{n-m}\binom{n-1}{m-1}, \quad n\geq m\geq 0,$$

$$\widetilde{\frac{n!}{m!}} = \widetilde{n^{\underline{n-m}}} = (-m)^{\underline{n-m}} = (-1)^{n-m}(n-1)^{\underline{n-m}} = (-1)^{n-m}\frac{n!}{m!}\frac{m}{n},$$

$$\widetilde{\frac{n!}{(m-1)!}} = \widetilde{n^{\underline{n-m+1}}} = (-1)^{n-m+1}\frac{n!}{(m-1)!}.$$

The dual $\widetilde{\rho}(n,m)$ of $\rho(n,m)$ and the dual of a group representation $\rho = \rho_1 \star \rho_2$ are

$$\widetilde{\rho}(n,m) = \rho(-m,-n) = (-1)^{n-m}\frac{n!}{m!}\frac{m}{n}[u^{n-m}]g(u)\left(\frac{f(u)}{u}\right)^{-n} \qquad \text{from (1)},$$

$$\widetilde{\rho}(n,m) = \sum_i \rho_1(-m,-i)\rho_2(-i,-n) = \sum_i \widetilde{\rho}_2(n,i)\widetilde{\rho}_1(i,m), \quad \widetilde{\rho} = \widetilde{\rho}_2 \star \widetilde{\rho}_1,$$

after a change $i \to -i$; this change is legal because we are allowed to use summation $\sum_i$ over an unbounded range.

2) For any number μ, the *scaling rule* "(μ)" is a group automorphism:

$$(\mu)\rho = (1,\ \mu u) \star \rho \star \left(1,\ \frac{u}{\mu}\right) = \left(g(\mu u),\ \frac{f(\mu u)}{\mu}\right), \quad \lim_{\mu \to 0} (\mu)(g(u),\ f(u)) = (1,\ u) = I,$$
$$((\mu)\rho)(n,m) = \mu^{n-m}\rho(n,m), \quad ((\mu)\rho)^{-1} = (\mu)\rho^{-1}, \quad (\mu)(\rho_1 \star \rho_2) = (\mu)\rho_1 \star (\mu)\rho_2.$$

3) The *negation rule* "(-)" is the special *scaling* $\mu = -1$:

$$(-)\rho = (g(-u),\ -f(-u)) : \quad ((-)\rho)(n,m) = (-1)^{n-m}\rho(n,m).$$

Duality and *negation* applied to (1) and Lagrange's inversion formula yield, respectively,

$$\begin{aligned} (-1)^{n-m}\widetilde{f}(n,m) &= (-1)^{n-m} f(-m,-n) = \frac{n!}{m!}\frac{m}{n}[u^{n-m}]\left(\frac{f(u)}{u}\right)^{-n}, \\ f^{-1}(n,m) &= \frac{n!}{m!}[u^n]\overline{f}(u)^m = \frac{n!}{m!}\frac{m}{n}[u^{n-m}]\left(\frac{f(u)}{u}\right)^{-n}. \end{aligned}$$

Combining the two formulas, we find a *duality law* in the *associated* subgroup $\{(1,\ f(u))\}$:

$$f^{-1}(n,m) = (-1)^{n-m}\widetilde{f}(n,m), \quad f^{-1} = (-)\widetilde{f}, \tag{2}$$

For g in the *Appell* subgroup $\{(g(u),\ u)\}$ and $\rho^{-1} = f^{-1} \star g^{-1} = (-)\widetilde{f} \star g^{-1}$, (1) yields:

$$g(n,m) = \frac{n!}{m!}g_{n-m} = \binom{n}{m}(n-m)!g_{n-m}, \quad g^{-1}(n,m) = \binom{n}{m}(n-m)!\widehat{g}_{n-m},$$
$$\rho^{-1}(n,m) = \sum_{i=0}^{n-m}(-1)^i \widetilde{f}(n,n-i)(n-i)^{\underline{n-m-i}}\widehat{g}_{n-m-i}.$$

3 Canonical Forms and Extensions

Let us write the *delta series* $f(u)$ in $f = (1,\ f(u))$ and the defining relation (1) as:

$$f(u) = uca(u) = uc(1 + a_1 u + \cdots), \quad c \neq 0,$$
$$\rho(n,m) = \binom{n}{m}c^m\left[\frac{u^{n-m}}{(n-m)!}\right]g(u)\left(\frac{f(u)}{cu}\right)^m = \binom{n}{m}c^m F_{n-m}(m), \tag{3}$$

and call $c \neq 0$ the *weight* of f, $\{ca_n\}$ the *f-reference* sequence (f-refseq) and $\binom{n}{m}c^m F_{n-m}(m)$ the *canonical* form of $\rho(n,m)$. The remarkable structure of *canonical* forms, namely, a binomial coefficient multiplied by a factor $c^m F_{n-m}(m)$, implies the following.

Proposition 1. *A numerical array with entries $\rho(n,m) = \binom{n}{m}c^m F_{n-m}(m)$ is a Riordan matrix iff the $F_k(x)$ are polynomials forming a Sheffer sequence.*

Proof. If the $\rho(n,m)$ are numbers related to an element $\rho = (g(u),\ f(u))$, then according to (3) we can write

$$\sum_{n-m} \frac{F_{n-m}(m)}{(n-m)!} u^{n-m} = g(u) e^{m \ln\left(\frac{f(u)}{cu}\right)} = g(u) \left(\frac{f(u)}{cu}\right)^m,$$

thus the $F_k(x)$ form a Sheffer sequence given by $\sum_k F_k(x) u^k/k! = g(u) e^{x \ln(f(u)/cu)}$. Conversely, if the above Sheffer sequence is given, then, by letting $x = m$, we obtain (3), proving that the $\rho(n,m)$ define a Riordan matrix. □

Corollary 2. *Each $\rho(n,m)$ in a Riordan matrix has an extension*

$$\rho(x, x-k) = (-1)^k x^{\underline{k+1}} \rho_k(x), \quad \rho_k(x) = c^{x-k} \frac{(-1)^k}{k!} \frac{F_k(x-k)}{x-k}, \tag{4}$$

which is of polynomial type when $c = 1$.

Proof. Since $F_{n-m}(m)$ is a polynomial in m, we can put in the canonical form $n - m = k$, $n = x$ and $m = x - k$, thus obtaining immediately well-defined *extensions* (4), which are clearly of *polynomial* type when $c = 1$. □

Since $f(u)/cu = a(u)$ is invertible and $a(0) = 1$, the coefficients $(-1)^k x f_k(x+k)$ in the power series expansion of $(f(u)/cu)^x$, written as $(f(u)/cu)^x = x \sum_k (-1)^k f_k(x+k) u^k$, form a *convolution family*. Similarly, (1) written for $f^{-1}(n,m) = (-1)^{n-m} \widetilde{f}(n,m)$, with $u \to cu$ and $m = -x$, implies:

$$\begin{aligned}
\left(\frac{c\overline{f}(u)}{u}\right)^m &= -m \sum_{n-m \geq 0} (-1)^{n-m} f_{n-m}(-m) \frac{u^{n-m}}{c^{n-m}}, \\
\left(\frac{\overline{f}(cu)}{u}\right)^{-x} &= \left(\frac{\overline{f(u)/c}}{u}\right)^{-x} = x \sum_{k \geq 0} (-1)^k f_k(x) u^k, \quad c\overline{f}\left(f\left(\frac{u}{c}\right)\right) = u,
\end{aligned}$$

showing that the $(-1)^{k+1} x f_k(-x)$ form a *convolution family* for $(\overline{f(u)/c}/u)^x$. Moreover,

$$\begin{aligned}
f(n,m) &= (-1)^{n-m} \frac{n!}{(m-1)!} c^m f_{n-m}(n), \\
f^{-1}(n,m) &= (-1)^{n-m} \widetilde{f}(n,m) = (-1)^{n-m+1} \frac{n!}{(m-1)!} c^{-n} f_{n-m}(-m) \\
f(x, x-k) &= c^{x-k} (-1)^k x^{\underline{k+1}} f_k(x), \quad f^{-1}(x, x-k) = c^{-x} (-1)^{k+1} x^{\underline{k+1}} f_k(k-x).
\end{aligned}$$

The $f_k(x),\ k > 0$, will be called the f-polynomial sequence (f-polseq).
One can verify that if f has weight c, then f^{-1} has *weight* $1/c$, and that changes of *scale* keep *weights* invariant, hence, they cannot modify the characteristics of an *extension*.

For $g \in \{(g(u),\ u)\}$, we have simply:

$$\begin{aligned}
g(n,m) &= \binom{n}{n} (n-m)! g_{n-m}, \quad g^{-1}(n,m) = \binom{n}{n} (n-m)! \widehat{g}_{n-m}, \\
g(x, x-k) &= g_k x^{\underline{k}}, \quad g^{-1}(x, x-k) = \hat{g}_k x^{\underline{k}}.
\end{aligned}$$

For $\rho(n,m)$, in addition to (3) and (4), and for $\rho^{-1}(n,m)$ we write relations that, when $g(u)\equiv 1$, reduce to the corresponding expressions for $f(n,m)$ and $f^{-1}(n,m)$:

$$
\begin{aligned}
\rho(n,m) &= (-1)^{n-m}\frac{n!}{(m-1)!}\rho_{n-m}(n),\\
\rho^{-1}(n,m) &= \binom{n}{m}c^{-n}G_{n-m}(m) = (-1)^{n-m+1}\frac{n!}{(m-1)!}\rho^{-1}_{n-m}(-m),\\
\rho^{-1}(x,x-k) &= (-1)^{k+1}x^{\underline{k+1}}\rho^{-1}_k(k-x), \quad \rho^{-1}_k(x) = c^{-x}\frac{(-1)^k}{k!}\frac{G_k(-x)}{x}.
\end{aligned}
$$

When the nonzero entries in a f-refseq are of the same sign, we say that $\rho = (g(u),\ f(u))$ is of the *second* kind, otherwise, of the *first* kind. In a pair $\{\rho,\ \rho^{-1}\}$, at most one element can be of the 2nd kind; when such an element exists, it will be denoted ρ^{-1}. Similarly, capital letters in an inverse pair $\{\phi,\ \Phi\}$ will, in general, indicate elements of the 2nd kind.

Duality defines a *dual* element $\widetilde{\rho}$ that extends $\rho(n,m)$ to all integers $n,\ m$, and since, as we have seen, $\widetilde{\rho}(n,m)$, $\rho(n,m)$ and $\rho^{-1}(n,m)$ are tied together, it is natural to include these numbers in a single extended ρ-*array* that will represent the pair $\{\rho,\ \rho^{-1}\}$.

We adopt the term *generalized* numbers for the $\rho(n,m)$ in the sense that these numbers extend to all integer values $n,m,$. The ρ-refseq: $c\sum_{i=0}^{n} g_i a_{n-i}$ given by

$$
\frac{\rho(n+1,1)}{(n+1)!} = c[u^n]g(u)\frac{f(u)}{cu} = c[u^n]g(u)a(u) = c\sum_{i=0}^{n} g_i a_{n-i}.
$$

4 Riordan Arrays and "Orthogonal" Polynomials

Consider a Sheffer sequence $\mathcal{S}_k(x)$ for $(g(u),\ f(u))$ and the elements $\mathrm{U} = \left(1/g(\overline{f}(u)),\ ue^{\overline{f}(u)}\right)$ and $C = (1/g(\overline{f}(u)),\ \overline{f}(u))$, then we can write:

$$
\begin{aligned}
&\sum_{k\geq 0}\frac{\mathcal{S}_k(x)}{k!}u^k = \frac{e^{x\overline{f}(u)}}{g(\overline{f}(u))} = \sum_{i\geq 0} x^i\frac{1}{i!}\frac{\overline{f}(u)^i}{g(\overline{f}(u))} = \sum_{i\geq 0} x^i \sum_{k\geq 0} C(k,i)\frac{u^k}{k!},\\
&\mathcal{S}_k(x) = \sum_{i=0}^{k} C(k,i)x^i, \quad C(n,m) = \binom{n}{m}\left[\frac{u^{n-m}}{(n-m)!}\right]\frac{\left(\frac{\overline{f}(u)}{u}\right)^m}{g(\overline{f}(u))},\\
&\mathrm{U}(n,m) = \binom{n}{m}\left[\frac{u^{n-m}}{(n-m)!}\right]\frac{e^{m\overline{f}(u)}}{g(\overline{f}(u))} = \binom{n}{m}\mathcal{S}_{n-m}(m) = \frac{(-1)^{n-m+1}n!}{(m-1)!}\mathrm{U}_{n-m}(-m),\\
&\mathrm{U}(x,x-k) = (-1)^{k+1}x^{\underline{k+1}}\mathrm{U}_k(k-x), \quad \mathrm{U}_k(x) = \frac{(-1)^k}{k!}\frac{\mathcal{S}_k(-x)}{x},\ k>0,\ \mathrm{U}_0(x) = \frac{1}{x}.
\end{aligned}
$$

An immediate consequence of these equations is the following important relationship between Riordan group elements and Sheffer sequences.

Proposition 3. *$S_k(x) = \sum_{i=0}^{k} C(k,i)x^i$ are polynomials forming a Sheffer sequence for $(g(u),\ f(u))$ iff the coefficients $C(n,m)$ are* generalized *numbers for $C = (1/g(\overline{f}(u)),\ \overline{f}(u))$.*

We now give examples where the $S_k(x)$ are classical "orthogonal" polynomials, currently found in mathematical physics and combinatorial analysis.

Example 1. In the typical <u>*Stirling*-array</u> $\{s = (1,\ \ln(1+u)),\ S = (1,\ e^u - 1)\}$, we have:

$$s(n,m) = (-1)^{n-m}\frac{n!}{(m-1)!}\sigma_{n-m}(n), \quad S(n,m) = (-1)^{n-m+1}\frac{n!}{(m-1)!}\sigma_{n-m}(-m);$$
$$s(x, x-k) = (-1)^k x^{\underline{k+1}}\sigma_k(x), \quad S(x, x-k) = (-1)^{k+1}x^{\underline{k+1}}\sigma_k(k-x);$$
$$\mathit{Stirling}\text{-polseq} : \sigma_k(x);\ s\text{-refseq} : \frac{(-1)^n}{n+1},\ S\text{-refseq} : \frac{1}{(n+1)!};$$

$$\left(\frac{\ln(1+u)}{u}\right)^x = x\sum_{k\geq 0}(-1)^k\sigma_k(x+k)u^k,\ \left(\frac{e^u-1}{u}\right)^{-x} = x\sum_{k\geq 0}(-1)^k\sigma_k(x)u^k.$$

The <u>*Stirling*-array</u> is related to the **Exponential Polynomials** $\phi_k(x)$ [12, p. 63] forming the *associated* sequence for $f(u) = \ln(1+u)$, $\overline{f}(u) = e^u - 1$, $\gamma^{-1} = (1,\ e^u - 1)$:

$$\sum_k \frac{\phi_k(x)}{k!}u^k = e^{x\overline{f}(u)} = e^{x(e^u-1)}, \quad C(n,m) = S(n,m) : \text{ Stirling numbers (2d kind),}$$

$$\Phi = \mathrm{U} = \left(1,\ ue^{e^u-1}\right), \quad \phi_k(x) = \sum_{i=0}^{k} S(k,i)x^i : \text{ Touchard polynomials;}$$

$$\Phi(n,m) = \binom{n}{m}\phi_{n-m}(m) = \frac{(-1)^{n-m+1}n!}{(m-1)!}\Phi_{n-m}(-m), \quad \Phi_k(x) = \frac{(-1)^k}{k!}\frac{\phi_k(-x)}{x};$$

$$\phi_k(1) = \varpi_n = \sum_{i=0}^{k} S(k,i) : \text{ Bell numbers, given by } e^{e^u-1} = \sum_k \varpi_n u^k/k!.$$

The **Iterated Exponential Polynomials** $\phi_k^{[q]}(x)$ form the *associated* sequence for $f(u) = s^{[q]}(u) = s(s^{[q-1]}(u)),\ s^{[0]} = u;\ \ \overline{f}(u) = S^{[q]}(u) = S(S^{[q-1]}(u)),\ S^{[0]} = u$:

$$\sum_k \frac{\phi_k^{[q]}(x)}{k!}u^k = e^{xS^{[q]}}, \quad \phi_k^{[q]}(x) = \sum_{i=0}^{k} S^{[q]}(k,i)x^i; \quad S^{[q]} = \underbrace{S\star S\star\ldots\star S}_{q\ terms},$$

$$\Phi^{[q]} = \mathrm{U}^{[q]} = \left(1,\ ue^{S^{[q]}(u)}\right), \quad \Phi^{[1]} = \Phi, \quad C^{[q]} = (1,\ S^{[q]}(u))$$

$$S^{[q]}(n,m) = \binom{n}{m}\phi_{n-m}^{[q]}(m), \quad \Phi_k^{[q]}(x) = \frac{(-1)^k}{k!}\frac{\phi_k^{[q]}(-x)}{x}.$$

The <u>*Stirling*-array</u> corresponds to $q = 1$, and for $q = 2$, we have the <u>$\mathit{Stir}^{[2]}$-array</u> $\{\beta =$

$(1,\ \beta(u)),\ \mathcal{B}\}, \quad \beta(u) = \ln(1+\ln(1+u)), \quad \mathcal{B}(u) = e^{e^u-1} - 1, \quad \mathcal{B}\text{-refseq} : \varpi_{n+1}/(n+1)!,$

$$\beta(n,m) = \sum_{i=m}^{n} s(n,i)s(i,m), \quad \mathcal{B}(n,m) = \sum_{i=m}^{n} S(n,i)S(i,m),$$

$$\sum_k \frac{\phi_k^{[2]}(x)}{k!}u^k = e^{x\ln(1+\ln(1+u))} = (1+\ln(1+u))^x, \quad \phi_k^{[2]}(x) = \sum_{j\geq i\geq 0}^{k} s(k,j)s(j,i)x^i,$$

$$Stir^{[2]}\text{-polseq} : \sigma_k^{[2]}(x) = \sum_{i=0}^{k}(x-i)\sigma_{k-i}(x-i)\sigma_i(x).$$

Example 2. For the $\underline{Lah\text{-array}}$ $\{\lambda,\ \Lambda = (1,\ \Lambda(u))\}$[1], $\quad \Lambda(u) = -\lambda(-u) = u/(1-u/2)$:

$$\Lambda(n,m) = \frac{(-1)^{n-m+1}n!}{(m-1)!}Lah_{n-m}(-m) = \binom{n}{m}\frac{(n-m)!}{2^{n-m}}\binom{n-1}{m-1} : \text{scaled Lah numbers},$$

and for the $\underline{Pascal\text{-array}}$ $\{p,\ P = (1/(1-u), u/(1-u))\}, \quad p = P^{-1} = (-)P$:

$$P(n,m) = \frac{(-1)^{n-m+1}n!}{(m-1)!}Pas_k(-m) = \frac{n!}{m!}[u^{n-m}]\frac{1}{(1-u)^{m+1}} = \binom{n}{m}\frac{n!}{m!}, \text{ by [8, (5.56)]},$$

$$Pascal\text{-polseq} : Pascal_k(x) = \frac{(-1)^k}{k!}(x-1)^{\underline{k-1}}.$$

The above arrays are related to the **Laguerre Polynomials** $L_k^{(a)}(x)$ **of order** a [12, p. 31, p. 108] forming the Sheffer sequence for $(g(u) = (1+u)^{-a-1},\ f(u) = u/(u-1) = \overline{f}(u))$, given by $\sum_k L_k^{(a)}(x)u^k/k! = (1-u)^{-a-1}e^{xu/(u-1)}$; $\quad L_k^{\{0\}}(x)$: (simple) Laguerre polynomials.

$$\begin{aligned}
\mathrm{Lag}^{(a)} = \mathrm{U} &= ((1-u)^{-a-1},\ ue^{\overline{f}(u)}), \quad C^{(a)} = \left((1-u)^{-a-1},\ \frac{u}{u-1}\right), \\
\mathrm{Lag}^{(a)}(n,m) &= \binom{n}{m}L_{n-m}^{\{a\}}(m) = (-1)^{n-m+1}\frac{n!}{(m-1)!}\mathrm{Lag}_{n-m}^{(a)}(-m), \\
C^{(a)}(n,m) &= (-1)^m\frac{n!}{m!}[u^{n-m}](1-u)^{-a-1-m} = (-1)^m\frac{n!}{m!}\binom{a+n}{n-m}, \text{ by [8, (5.56)]}, \\
L_k^{(a)}(x) &= \sum_{i=0}^{k} C^{(a)}(k,i)x^i = \sum_{i=0}^{k}\frac{k!}{i!}\binom{a+k}{k-i}(-x)^i. \qquad (5)
\end{aligned}$$

The $\underline{Lah\text{-array}}$ corresponds to $a = -1$ and the $\underline{Pascal\text{-array}}$ to $a = 0$.

[1] Erratum: in Della Riccia [5, p. 3, line before last]: $\Lambda(u) = u/(1+u/2)$ should be $\Lambda(u) = u/(1-u/2)$.

Example 3. Let us consider the <u>*Tanh*-array</u> $\{\theta = (1,\ \theta(u)),\ \Theta = (1,\ \Theta(u))\}$, with

$$\theta(u) = 2\frac{e^u - 1}{e^u + 1} = 2\tanh\frac{u}{2},\ \Theta(u) = \ln\frac{1 + u/2}{1 - u/2} = 2\arg\tanh\frac{u}{2},\ \Theta = \Lambda\star s,\ S = \Lambda\star\theta:$$
$$\theta(n,m) = (-1)^{n-m}\frac{n!}{(m-1)!}\delta_{n-m}(n),\quad \Theta(n,m) = (-1)^{n-m+1}\frac{n!}{(m-1)!}\delta_{n-m}(-m),$$
$$\delta_k(x) = \sum_{i=0}^{k}\frac{(-1)^{i+1}}{2^i}\frac{(k-x-1)^{\underline{i}}}{i!}\sigma_{k-i}(k-x-i),\quad \sigma_k(x) = -\sum_{i=0}^{k}\frac{(x-1)^{\underline{i}}}{2^i i!}\delta_{k-i}(k-x),$$
$$Tanh\text{-polseq}:\ \delta_k(x).$$

The <u>*Tanh*-array</u> and the <u>*Tangent*-array</u> $\{\arctan = (1,\ \arctan u),\ \tan = (1,\ \tan u)\}$ are related by the elementary trigonometric formulas:

$$\tan u = \frac{1}{\imath}\tanh(\imath u) = \frac{1}{2\imath}2\tanh\left(2\imath\frac{u}{2}\right) = \frac{\theta(2\imath u)}{2\imath},\quad \arctan u = \frac{1}{\imath}\arg\tanh(\imath u) = \frac{\Theta(2\imath u)}{2\imath},$$

which look like a scaling of θ and Θ with $\mu = 2\imath,\ \imath^2 = -1$: $tan(n,m) = (2\imath)^{n-m}\theta(n,m)$, $arctan(n,m) = (2\imath)^{n-m}\Theta(n,m)$. The numbers $tan(n,m)$ and $\arctan(n,m)$ appear in Comtet [4, p.p.259-260] as $T(n,k)$ and $t(n,k)$. Putting $T(n,m) = (2\imath)^{n-m}(n!/(m-1)!)\delta_{n-m}(n)$ in the recursion for $T(n,k)$ and after common factors are divided out, we are left with

$$(x+1)\delta_k(x+1) = (x-k)\delta_k(x) - \frac{x-k+2}{4}\delta_{k-2}(x),\quad \delta_k(x)\equiv 0,\ k<0,\ x\delta_0(x)\equiv 1,$$
$$\text{(compare with}(x+1)\sigma_k(x+1) = (x-k)\sigma_k(x) + x\sigma_{k-1}(x)\ \text{[8, Exercise 6.18]}. \tag{6}$$

One can prove by induction that the $\delta_k(x),\ k = 2j > 0$, have degree $j-1$ and $\delta_k(x)\equiv 0$ when k is odd; hence factors $(-1)^k$ may be omitted, leaving us with simplified formulas:

$$\left(\frac{2\tanh\frac{u}{2}}{u}\right)^x = x\sum_{k\ge 0}\delta_k(x+k)u^k,\quad \left(\frac{1}{u}\ln\frac{1+\frac{u}{2}}{1-\frac{u}{2}}\right)^{-x} = x\sum_{k\ge 0}\delta_k(x)u^k;\quad \Theta = \widetilde{\theta}.$$

The <u>*Tanh*-array</u> is related to the **Mittag-Leffler Polynomials** $M_k(x)$ [12, p.75] forming the *associated* sequence for

$$f(u) = \frac{e^u-1}{e^u+1} = \tanh\frac{u}{2},\quad \overline{f}(u) = \ln\frac{1+u}{1-u} = 2\arg\tanh u,\quad \sum_k\frac{M_k(x)}{k!}u^k = \left(\frac{1+u}{1-u}\right)^x.$$

$$\begin{aligned}\text{MiLef} = \text{U} &= \left(1,\ u\frac{1+u}{1-u}\right),\quad C = (1,\ 2\arg\tanh u),\quad \text{MiLef}_k(x) = \frac{(-1)^k}{k!}\frac{M_k(-x)}{x},\\ \text{MiLef}(n,m) &= \binom{n}{m}M_{n-m}(m) = \frac{(-1)^{n-m+1}n!}{(m-1)!}\text{MiLef}_{n-m}(-m),\end{aligned}$$

$$M_k(x) = 2^k\sum_{i=0}^{k}\Theta(k,i)x^i = 2^k\sum_{i=0}^{k}\sum_j\Lambda(k,j))s(j,i)x^i = \sum_{j=0}^{k}\frac{k!}{j!}\binom{k-1}{k-j}2^j x^{\underline{j}}, \tag{7}$$

where we used the identity $\sum_i s(j,i)x^i = x^{\underline{j}}$.

Example 4. The simple $\underline{Binom\text{-array}}$ $\{bin = (\exp^{-u},\ u),\ Bin\}$,

$$bin(n,m) = (-1)^{n-m}\binom{n}{m},\ Bin(n,m) = \binom{n}{m},\quad Bin(x, x-k) = (-1)^k bin(x, x-k) = \frac{x^{\underline{k}}}{k!},$$

is related to the **Poisson-Charlier Polynomials** $c_k^{\{a\}}(x) = a^{-k}L_k^{\{x-k\}}(a)$ [12, p.119], since the egf of $c_k^{\{a\}}(0) = a^{-k}L_k^{\{-k\}}(a) = (-1)^k$ is e^{-u}, by using $L_k^{\{-k\}}(a) = (-1)^k a^k$ from (5). The $\underline{Tree\text{-array}}$ $\{r = (1,\ ue^{-u}),\ R = (1,\ \overline{ue^{-u}})\}$,

$$\begin{aligned} r(n,m) &= (-1)^{n-m}\binom{n}{m}m^{n-m}, \quad R(n,m) = \binom{n}{m}\frac{m}{n}n^{n-m}, \\ \left(\frac{r(u)}{u}\right)^x &= e^{-xu},\ \left(\frac{R(u)}{u}\right)^{-x} = x\sum_k \frac{(-1)^k}{k!}(x-k)^{k-1}u^k, \quad Tree_k(x) = \frac{1}{k!}(x-k)^{k-1}, \\ R(n,1) &= n^{n-1}:\ \text{number of rooted trees of } n \text{ vertices [4, p.152]}. \end{aligned}$$

can be related to the case $a = -1$ of the **Abel Polynomials** $A_k(x;a)$**,** $a \neq 0$, [12, p.73], forming the *associated* sequence for $f(u) = ue^{au}$; $\overline{f}(u) = \overline{ue^{au}}$, $A_k(x;a) = x(x-ak)^{k-1}$.

$$\begin{aligned} &\mathrm{Abel}^{(a)} = \mathrm{U}^{(a)} = (1,\ ue^{\overline{ue^{au}}}), \quad \mathrm{Abel}_k^{(a)}(x) = \frac{(-1)^k}{k!}\frac{A_k(-x;a)}{x}, \quad C^{(a)} = (1,\ \overline{ue^{au}}); \\ &\mathrm{Abel}^{(a)}(n,m) = (-1)^{n-m+1}\frac{n!}{(m-1)!}\mathrm{Abel}_{n-m}^{(a)}(-m) = \binom{n}{m}A_{n-m}(m;a). \\ &\text{For } a=-1,\ \mathrm{Abel}_k^{(-1)}(x) = (-1)^k\sum_{i=0}^k \binom{k}{i} i^{k-i}(-x)^i, \quad \binom{k}{i} i^{k-i}:\ \text{idempotent numbers.} \end{aligned}$$

Example 5. The $\underline{Sinh\text{-array}}$ $\{argshin = (1,\ 2\arg\sinh(u/2)/b),\ shin = (1,\ 2\sinh(bu/2))\}$ is related to the **Gould Polynomials** $G_k(x;a,b)$**,** $b \neq 0$, [12, p.67] forming the *associated* sequence for $f(u) = e^{au}(e^{bu}-1)$, $b \neq 0$. In fact, when $a = -b/2$, $G_k(x;b) = G_k(x;-b/2,b)$, $f(u) = e^{bu/2} - e^{-bu/2} = 2\sinh(bu/2)$, $c = b$, $\overline{f}(u) = 2\arg\sinh(u/2)/b$,

$$\sum_k G_k(x;b)u^k/k! = e^{x(2/b)\arg\sinh(u/2)} = \mathcal{B}_{1/2}(u)^{x/b} = (u/2 + \sqrt{1+u^2/4})^{2x/b}\ \text{[10, p.71]},$$

where $\mathcal{B}_{1/2}(u) = (u/2 + \sqrt{1+u^2/4})^2$ is a *generalized binomial series* [8, p.203].

$$\begin{aligned} \mathrm{Gould}^{(b)} = \mathrm{U}^{(b)} &= \left(1,\ ue^{\frac{2}{b}\arg\sinh\frac{u}{2}}\right), \quad \mathrm{Gould}_k^{(b)}(x) = \frac{(-1)^k}{k!}\frac{G_k(-x;b)}{x}, \\ \mathrm{Gould}^{(b)}(n,m) &= (-1)^{n-m+1}\frac{n!}{(m-1)!}\mathrm{Gould}_{n-m}^{(b)}(-m) = \binom{n}{m}G_{n-m}(m;b); \\ G_k(x;1) &= x\left(x + \frac{1}{2}k - 1)\right)^{\underline{k-1}}:\ \text{central factorial polynomials [12, p.68].} \end{aligned}$$

Example 6. The **Bernoulli Polynomials** $B_k^{(a)}(x)$ **of order** a**,** $a \neq 0$, [12, p.93] form the *Appell* sequence for $g(u) = ((e^u - 1)/u)^a$, given by $\sum_k B_k^{(a)}(x)u^k/k! = (u/(e^u - 1))^a e^{xu}$. $B_k^{(m)} = B_k^{(m)}(0)$ are the higher order Bernoulli numbers and $B_k = B_k^{(1)}$ are the Bernoulli numbers given by $\sum_k B_k u^k/k! = u/(e^u - 1) = B(u)$.

$$\mathrm{Bern}^{(a)} = \mathrm{U} = \left(\left(\frac{u}{e^u - 1}\right)^a,\ ue^u\right), \quad \mathrm{Bern}^{(a)}(n,m) = \binom{n}{m} B_{n-m}^{(a)}(m)$$
$$C^{(a)} = \left(\left(\frac{u}{e^u - 1}\right)^a,\ u\right), \quad C^{(a)}(n,m) = \binom{n}{m} B_{n-m}^{(a)},$$
$$B_k^{(a)}(x) = \sum_{i=0}^{k} \binom{k}{i} B_{k-i}^{(a)} x^i, \quad B_k(x) = \sum_{i=0}^{k} \binom{k}{i} B_{k-i} x^i :\ \text{Bernoulli polynomials.}$$

The *Nörlund polynomials* $B_k^{(x)} = B_k^{(x)}(0)$ [1] form the *associated* sequence for $\overline{\ln B(u)}$,

$$\sum_k \frac{B_k^{(x)}}{k!} u^k = e^{x \ln B(u)} = \left(\frac{e^u - 1}{u}\right)^{-x} = x \sum_{k \geq 0} (-1)^k \sigma_k(x) u^k,$$
$$\frac{B_k^{(x)}}{xk!} = (-1)^k \sigma_k(x) = \frac{s(x, x-k)}{x^{\underline{k+1}}}, \quad \frac{B_k^{(-x)}}{xk!} = (-1)^{k+1} \sigma_k(-x) = \frac{S(x+k, x)}{(x+k)^{\underline{k+1}}}, \tag{8}$$
$$\frac{B_k}{k!} = \sigma_k(1) - [k=1] = -k\sigma_k(0),\ k > 0, \quad \text{by the recursion (6).} \tag{9}$$

For the <u>*Bern*-array</u> $\{ber,\ Ber = (1,\ uB(u))\}$, Ber-refseq: $B_n/n!$, we have

$$Ber(n,m) = \binom{n}{m} B_{n-m}^{(m)}, \quad Bern_k(x) = \frac{(-1)^k}{k!} \frac{B_k^{(-x)}}{x} = -\sigma_k(-x),\ k \geq 0.$$

In passing, we remark that the *associated* sequence $\mathcal{U}_k(x)$ for $f(u) = \overline{uB(u)}$ is given by

$$\sum_k \frac{\mathcal{U}_k(x)}{k!} u^k = e^{xuB(u)} = \sum_i x^i \frac{(uB(u))^i}{i!}, \quad \mathcal{U}_k(x) = \sum_{i=0}^{k} \binom{k}{i} B_{k-i}^{(i)} x^i.$$

Now consider the Stirling polynomials convolution formula [8, (6.46)], written with $t = 1$, $i \to i - m, r \to r + m$ and $s \to s + m$:

$$-(r+m)(s+n) \sum_{i=m}^{n} \sigma_{i-m}(r+i) \sigma_{n-i}(-s-i) = (r - s + m - n) \sigma_{n-m}(r-s).$$

Using (8), we can write $\sigma_{n-m}(r-s)$, $\sigma_{i-m}(r+i)$ and $\sigma_{n-i}(-s-i)$ in terms of Stirling numbers and *Nörlund* polynomials; after substitution in the convolution relation, we get

$$\sum_{i=m}^{n} \frac{s(r+i, r+m)}{(r+i)^{\underline{i-m}}} \frac{S(s+n, s+i)}{(s+n)^{\underline{n-i+1}}} = \frac{r - s + m - n}{s+n} \frac{B_{n-m}^{(r-s)}}{(r-s)(n-m)!}.$$

For $a = r - s$, integers $r, s \geq 0$, $\dfrac{1}{(r+i)^{\underline{i-m}}} = \dfrac{(r+m)!}{(r+i)!}$, $\dfrac{1}{(s+n)^{\underline{n-i+1}}} = \dfrac{(s+i-1)!}{(s+n)!}$:

$$\sum_{i=m}^{n} s(r+i, r+m)S(s+n, s+i)\frac{(s+i-1)^{\underline{i-1}}}{(r+i)!} = \frac{a+m-n}{s+n}\frac{(s+n)!}{(r+m)!}\frac{B_{n-m}^{(a)}}{a(n-m)!}.$$

With $r = s$, $a = 0$, we get from (8): and (9),

$$\sum_{i=m}^{n} s(r+i, r+m)S(r+n, r+i)\frac{1}{i} = \frac{(r+n)^{\underline{n-m}}}{r+n}\left(\frac{B_{n-m}}{(n-m)!} + [m = n-1]\right),\ m > 0,$$

which is a generalization of the case $r = s = 0$, that corresponds to the known identity:

$$\sum_{i=m}^{n} s(i, m)S(n, i)\frac{1}{i} = \frac{1}{n}\binom{n}{m}\frac{B_{n-m}}{(n-m)!} + [m = n-1],\ m > 0 \quad [8, (6.100)], \tag{10}$$

The identity (10), in turn, generalizes the identity $B_n = \sum_{i=0}^{n}(-1)^i i! S(n,i)/(i+1)$ appearing in Comtet [4, p.220] and which is the special case $m = 1$ of (10) and, at the same time, of Kaneko's identity $B_{n-m} = (-1)^{n-m+i}S(n-m, i)i!/(i+1)^m$ [9, Theorem 1].

Example 7. The **Euler Polynomials** $E_k^{(a)}(x)$ **of order** a**,** $a \neq 0$, [12, p.100] form the *Appell* sequence for $g(u) = ((e^u+1)/2)^a$,

$$\sum_k \frac{E_k^{(a)}(x)}{k!}u^k = \left(\frac{2}{e^u+1}\right)^a e^{xu} = \left(\frac{e^{-\frac{u}{2}}}{\cosh\frac{u}{2}}\right)^a e^{xu},\ E_k(x) = E_k^{(1)}(x) : \text{Euler polynomials},$$

$$\text{for } u \to 2u,\ x = \frac{a}{2} : \sum_k \frac{2^k E_k^{(a)}(\frac{a}{2})}{k!}u^k = \left(\frac{1}{\cosh u}\right)^a,\ E_k = 2^k E_k\left(\frac{1}{2}\right) : \text{Euler numbers}.$$

$$\begin{aligned}
&\mathrm{Euler}^{(a)} = \mathrm{U} = \left(\left(\frac{2}{e^u+1}\right)^a,\ ue^u\right), \quad \mathrm{Euler}^{(a)}(n,m) = \binom{n}{m}E_{n-m}^{(a)}(m),\\
&C^{(a)} = \left(\left(\frac{2}{e^u+1}\right)^a,\ u\right), \quad C^{(a)}(n,m) = \binom{n}{m}E_{n-m}^{(a)}(0),\\
&E_k^{(a)}(x) = \sum_{i=0}^{k} C^{(a)}(k,i)x^i = \sum_{i=0}^{k}\binom{k}{i}E_{k-i}^{(a)}(0)x^i.
\end{aligned}$$

For the <u>*Euler*-array</u> $\{eul,\ Eul = (1,\ uE(u))\}$, $E(u) = 1/\cosh u$, Eul-refseq: $E_n/n!$, we find

$$\begin{aligned}
Eul(n,m) &= \frac{n!}{m!}[u^{n-m}]E(u)^m = \binom{n}{m}2^{n-m}E_{n-m}^{(m)}\left(\frac{m}{2}\right)\\
2^{n-m}E_{n-m}^{(m)}\left(\frac{m}{2}\right) &= \sum_{i=0}^{n-m}\binom{n-m}{i}2^i m^i E_i^{(m)}(0), \quad \frac{Eul(n+1,1)}{n+1} = E_n = \sum_{i=0}^{n}\binom{n}{i}2^i E_i(0).
\end{aligned}$$

The egf of Euler polynomials evaluated at $x = 0$ and the gf $G(u)$ for the Genocchi numbers G_n are related by

$$\sum_n \frac{E_n(0)}{n!} u^n = \left(1 - \tanh \frac{u}{2}\right) = \frac{2}{e^u + 1} = \sum_{n\geq 0} \frac{G_{n+1}}{(n+1)!} u^n = \frac{G(u)}{u} \quad [4, \text{p.49}],$$

thus, for the <u>*Geno*-array</u> $\{gen,\ Gen = (1,\ G(u))\}$, *Gen*-refseq: $G_{n+1}/(n+1)!$, we have

$$Gen(n,m) = \frac{n!}{m!}[u^n]\left(\frac{G(u)}{u}\right)^m = \binom{n}{m} E_{n-m}^{(m)}(0), \quad Gen_k(x) = \frac{(-1)^k}{k!}\frac{E_k^{(-x)}}{x}.$$

$$\begin{aligned}
\text{With } G(u) &= u\left(1 - \tanh\frac{u}{2}\right) = u\left(1 - \frac{\theta(u)}{2}\right) \text{ and } \frac{\theta(u)^i}{i!} = \sum_j \theta(j,i)\frac{u^j}{j!}, \text{ we find :} \\
Gen(n,m) &= \frac{n!}{m!}[u^{n-m}]\sum_i \binom{m}{i}\frac{(-1)^i}{2^i}\theta(u)^i = \binom{n}{m}\sum_{i=0}^{n-m}\binom{m}{i}\frac{(-1)^i i!}{2^i}\theta(n-m,i) \\
&= \binom{n}{m}(-1)^{n-m}(n-m)!\sum_{i=0}^{n-m}\binom{m}{i}\frac{i}{2^i}\delta_{n-m-i}(n-m), \\
Gen(n+1,1) &= G_{n+1} = (-1)^n\sum_{i=0}^{n-m}\binom{1}{i}\frac{i}{2^i}\delta_{n-i}(n) = [n=0] + \frac{(-1)^n(n+1)!}{2}\delta_{n-1}(n).
\end{aligned}$$

$$\begin{aligned}
\text{With } G(u) &= \frac{2u}{e^u+1} = \frac{2u(e^u-1)}{e^{2u}-1} = B(2u)S(u), \text{ we get the binomial convolution} \\
Gen(n,m) &= \sum_{i=0}^{n-m}\binom{n}{i}2^i B_i^{(m)} S(n-i,m). \\
\text{With } \frac{Ber(2u)}{2} &= u\frac{2u}{e^{2u}-1} = \frac{u}{e^u-1}\frac{2u}{e^u+1} = B(u)G(u), \text{ we get :} \\
2^{n-m}Ber(n,m) &= \sum_{i=0}^{n-m}\binom{n}{i}B_i^{(m)}Gen(n-i,m) = \binom{n}{m}\sum_{i=0}^{n-m}\binom{n-m}{i}B_i^{(m)}E_{n-m-i}^{(m)}(0);
\end{aligned}$$

$$Gen(n+1,1) = G_{n+1} = (n+1)E_n(0) = \sum_{i=0}^{n}\binom{n+1}{i}2^i B_i,$$

$$Ber(n+1,1) = (n+1)B_n = \frac{1}{2^n}\sum_{i=0}^{n}\binom{n+1}{i}B_i G_{n+1-i}.$$

Writing $G(2u)/2 = 2u/(e^{2u}+1) = e^{-u}u/\cosh u = e^{-u}Eul(u)$, and applying the binomial coefficients *inversion formula*, we obtain a pair of *inverse relations*, presumably original,

$$\begin{aligned}
\frac{2^{n-m}}{m^n}Gen(n,m) &= \sum_i \binom{n}{i}(-1)^{n-i}\frac{Eul(i,m)}{m^i} \leftrightarrow \frac{Eul(n,m)}{m^n} = \sum_i\binom{n}{i}\frac{2^{i-m}Gen(i,m)}{m^i}, \\
2^n G_{n+1} &= \sum_{i=0}^{n}\binom{n+1}{i+1}(-1)^{n-i}(i+1)E_i \ \leftrightarrow\ (n+1)E_n = \sum_{i=0}^{n}\binom{n+1}{i+1}2^i G_{i+1}.
\end{aligned}$$

Example 8. We consider the $\underline{Harm1}$-array $\{har1,\ Har1 = (1,\ Har1(u))\}$ and the $\underline{Harm2}$-array $\{har2,\ Har2 = (1,\ Har2(u))\}$, where $Har1(u)$ and $Har2(u)$ are gf's related to the harmonic numbers H_n:

$$\sum_{k>0} H_n u^n = \frac{-\ln(1-u)}{1-u} = Har1(u) \quad [8, (7.43)],$$

$$2\int \frac{-\ln(1-u)}{1-u}du = \frac{\ln^2(1-u)}{u} = u\left(\frac{\ln(1-u)}{-u}\right)^2 = \sum_{n>0} \frac{2H_n}{n+1}u^n = Har2(u),$$

$$Har1(n,m) = \frac{n!}{m!}[u^{n-m}]\left(\frac{-\ln(1-u)}{u(1-u)}\right)^m, \ Har2(n,m) = \frac{n!}{m!}[u^{n-m}]\left(\frac{\ln(1-u)}{-u}\right)^{2m}.$$

Using the inverse pair of group representations $Har1 = (-)s\star(-)r$ and $(-)r = (-)S\star Har1$, and $Har1(n+1,1)/(n+1)! = H_{n+1}$, we get:

$$Har1(n,m) = \sum_i (-1)^{n-i}s(n,i)\binom{i}{m}m^{i-m} \leftrightarrow \binom{n}{m}m^{n-m} = \sum_i (-1)^{n-i}S(n,i)Har1(i,m),$$

$$(n+1)!H_{n+1} = \sum_{i=0}^{n}(-1)^{n-i}s(n+1,i+1)(i+1) \leftrightarrow n+1 = \sum_{i=0}^{n}(-1)^{n-i}S(n+1,i+1)(i+1)!H_{i+1}.$$

Similarly, with $(-)Har2 = s\star Ber$ and $Ber = S\star(-)Har2$, and $Har2(n+1,1)/(n+1)! = 2H_{n+1}/(n+2)$, we find:

$$(-1)^{n-m}Har2(n,m) = \sum_{i=m}^{n} s(n,i)\binom{i}{m}B_{i-m}^{(m)} = \frac{n!}{(m-1)!}2\sigma_{n-m}(n+m)$$

$$\leftrightarrow \quad \binom{n}{m}B_{n-m}^{(m)} = \sum_{i=m}^{n} S(n,i)(-1)^{i-m}Har2(i,m);$$

$$\frac{2H_{n+1}}{n+2} = \frac{(-1)^n}{(n+1)!}\sum_{i=0}^{n} s(n+1,i+1)(i+1)B_i = 2\sigma_n(n+2), \tag{11}$$

$$B_n = \frac{2}{n+1}\sum_{i=0}^{n} S(n+1,i+1)\frac{(-1)^i(i+1)!}{i+2}H_{i+1}.$$

The identity (11) appears in [12, p.100], written in a different form. Finally:

$$Har1_k(x) = (-1)^k\sum_{i=0}^{k}(x-i)\sigma_{k-i}(k-x)(-x)^{i-1}, \quad Har2_k(x) = (-1)^{k+1}2\sigma_k(k-2x).$$

The elements $Har1$ and $Har2$ are related to the **Narumi Polynomials** $Nar_k^{(a)}(x)$ [12, p.127], forming the Sheffer sequence for $(g(u) = (u/(e^u-1))^a,\ f(u) = e^u-1), \overline{f}(u) = \ln(1+u)$:

$$\sum_k \frac{Nar_k^{(a)}(x)}{k!}u^k = \left(\frac{u}{\ln(1+u)}\right)^a(1+u)^x. \tag{12}$$

In fact, from (12) and the definitions of $Har1$ and $Har2$, we derive the *canonical* forms:

$$Har1(n,m) = \binom{n}{m}(-1)^{n-m} Nar_{n-m}^{(-m)}(-m), \quad Har2(n,m) = \binom{n}{m}(-1)^{n-m} Nar_{n-m}^{(-2m)}(0).$$

$$\mathrm{Nar}^{(a)} = \mathrm{U}^{(a)} = \left(\left(\frac{u}{\ln(1+u)}\right)^a,\ u(1+u)\right), \quad C^{(a)} = \left(\left(\frac{u}{\ln(1+u)}\right)^a,\ \ln(1+u)\right);$$

$$\mathrm{Nar}^{(a)}(n,m) = \binom{n}{m} Nar_{n-m}^{(a)}(m), \quad Nar_k^{(a)}(x) = \sum_{i=0}^{k} C(k,i)x^i,$$

$$C^{(a)}(n,m) = \frac{n!}{m!}[u^{n-m}]\left(\frac{\ln(1+u)}{u}\right)^{m-a} = (-1)^{n-m}\frac{n!}{m!}(m-a)\sigma_{n-m}(n-a).$$

In addition, with $u \to -u$, $v = -\ln(1-u)$ and $x = -a = m$, (12) yields:

$$\begin{aligned} \frac{(-1)^k Nar_k^{(-m)}(m)}{k!} &= [u^k]\left(\frac{\ln(1-u)(1-u)}{-u}\right)^m = [u^k]\left(\frac{v}{e^v-1}\right)^m = [u^k]\sum_i B_i^{(m)}\frac{v^i}{i!} \\ &= \sum_{i=1}^{k}\frac{(-1)^{k-i}}{k!}B_i^{(m)}s(k,i); \end{aligned} \tag{13}$$

when $m = 1$, $\displaystyle\sum_{i=1}^{k}(-1)^{k-i}s(k,i)B_i = \left[\frac{u^k}{k!}\right]\frac{\ln(1-u)(1-u)}{-u} = k!\left(\frac{1}{k+1} - \frac{1}{k}\right) = \frac{-(k-1)!}{k+1}$,

that is an identity which appears in Wilf [17, (4.3.21)].

Consider the $\underline{\mathcal{T}au\text{-array}}$ $\{\tau,\ \mathcal{T}\}$, $\quad \tau = s \star (-)r, \quad \tau(u) = -r(-s(u)) = (1+u)\ln(1+u)$:

$$\tau(n,m) = \mathrm{Nar}^{(-m)}(n,m) = \binom{n}{m} Nar_{n-m}^{(-m)}(m) = \sum_{i=m}^{n} s(n,i)\binom{i}{i-m} m^{i-m},$$

$$\mathcal{T}(n,m) = (-1)^{n-m}\widetilde{\tau}(n,m) = \sum_{i=m}^{n} S(i,m)\binom{n-1}{n-i} n^{n-i}.$$

The numbers $\tau(n,m)$ are denoted $b(n,m)$ in Comtet [4, pp. 139–140], wherein they are used in the computation of the n-th derivative of x^{ax}, $x > 0$, a real $\neq 0$. For completeness, we also mention the group representation $\tau = (2)\lambda \star Har1$.

5 Concluding remarks

In this paper we obtained several known or original identities between sequences by setting $m = 1$ in identities between generalized numbers, for example, (11). Conversely, we found identities between generalized numbers, extending identities between numerical sequences like, for instance, (13) which is an original generalization of Wilf's identity [17, (4.3.21)]. The Akiyama-Tanigawa algorithm and the Euler-Seidel construction are also based on extensions of numerical sequences, but the matrices have properties different from those of Riordan matrices and, at any rate, the purpose there is to develop efficient methods for calculating special sequences. The interested reader may consult the literature on the subject, for instance, [6, 11, 3, 7], in order to compare the various approaches.

Canonical forms implied Corollary 2, readily proving the existence of *extensions* $\rho(x, x-k)$, that are of *polynomial* type when ρ has *weight* $c = 1$, and they established a connection with the family of "orthogonal" polynomials, but some care should be exercised because certain "orthogonal" polynomials may have different names, for instance, Mittag-Leffler polynomials and Meixner polynomials of the second kind $M(x; 0, 0)$ [12, p. 126] coincide, similarly, Bernoulli polynomials of the second kind $b_k(x)$ [12, p. 113] and Narumi polynomials $Nar_k^{(-1)}(x)$ are the same. Recall also that, applying Proposition (3), we easily obtained the formula (5) for $L_k^{(\alpha)}(x)$ and (7) for $M_k(x)$, which are found in [12, p. 109 and p. 76] after longer algebraic manipulations (and a printing error in the last expression of $M_k(x)$).

Finally, we remember that *transformation rules* greatly simplified algebraic manipulations, and that the *duality law* played an important role in the computation of inverse numbers, especially when one of the two gf's: $f(u)$, $\overline{f}(u)$, was not available in closed form.

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