Journal of Integer Sequences, Vol. 11 (2008),
Article 08.5.7

Counting Non-Isomorphic Maximal Independent Sets of the n-Cycle Graph

Raymond Bisdorff
Computer Science and Communications Research Unit
University of Luxembourg
162A, avenue de la Faïencerie
L-1511 Luxembourg
Luxembourg
raymond.bisdorff@uni.lu

Jean-Luc Marichal[1]
Mathematics Research Unit
University of Luxembourg
162A, avenue de la Faïencerie
L-1511 Luxembourg
Luxembourg
jean-luc.marichal@uni.lu

Abstract

The number of maximal independent sets of the n-cycle graph C_n is known to be the nth term of the Perrin sequence. The action of the automorphism group of C_n on the family of these maximal independent sets partitions this family into disjoint orbits, which represent the non-isomorphic (i.e., defined up to a rotation and a reflection) maximal independent sets. We provide exact formulas for the total number of orbits and the number of orbits having a given number of isomorphic representatives. We also provide exact formulas for the total number of unlabeled (i.e., defined up to a rotation) maximal independent sets and the number of unlabeled maximal independent sets having a given number of isomorphic representatives. It turns out that these formulas involve both the Perrin and Padovan sequences.

[1]Corresponding author.

1 Introduction

Let $G = (V, E)$ be a simple undirected graph, with vertex set V and edge set E. Recall that an *independent set* of G is a subset X of V such that no two vertices in X are adjacent. An independent set is *maximal* if it is not a proper subset of any other independent set.

In this paper we are concerned with maximal independent sets (MISs) of the n-cycle C_n, i.e., the graph consisting of a cycle with n vertices. Füredi [4] observed that the total number of MISs of C_n is given by the nth term $p(n)$ of the Perrin sequence (Sequence A001608 from Sloane [8]), which is defined recursively as $p(1) = 0$, $p(2) = 2$, $p(3) = 3$, and

$$p(n) = p(n-2) + p(n-3) \qquad (n \geqslant 4).$$

The action of the automorphism group $\mathrm{Aut}(C_n)$ of C_n on the family of MISs of C_n gives rise to a partition of this family into orbits, each containing *isomorphic* (i.e., identical up to a rotation and a reflection) MISs. For instance, it is easy to verify that C_6 has $p(6) = 5$ MISs, which can be grouped into two orbits: an orbit with two isomorphic MISs of size 3 and an orbit with three isomorphic MISs of size 2. Thus, C_6 has essentially two non-isomorphic MISs.

Here we give a closed form formula for the number of orbits, that is, the number of non-isomorphic MISs of C_n (see Theorem 13). We also provide a formula for the number of orbits having a given number of isomorphic representatives. Finally, by modifying the concept of isomorphic MISs, requiring that they be identical up to a rotation only (which means that their *unlabeled* versions are identical), we provide a formula for the number of these unlabeled MISs (see Theorem 14) and the number of unlabeled MISs having a given number of isomorphic representatives. Incidentally, these formulas involve both Perrin and Padovan sequences, which are constructed from the same recurrence equation.

Although many results were obtained on the maximum number of MISs in a general graph (see for instance Chang and Jou [2] and Ying et al. [10] for recent results) it seems that the searching for the exact number of distinct MISs, or non-isomorphic MISs, in special subclasses of graphs has received little attention (see however Euler [3] and Kitaev [7]). We hope that by solving the case of cycle graphs we might bring more interest to this kind of challenging enumerative combinatorics problems.

2 Some properties of $\mathrm{Aut}(C_n)$

In this section we recall and give some properties of the automorphism group $\mathrm{Aut}(C_n)$ that we will use in the rest of the paper. More information on automorphism groups of graphs can be found for instance in Godsil and Royle [5].

Denote by $V_n = \{v_0, v_1, \ldots, v_{n-1}\}$ the set of vertices of C_n labeled either clockwise or counterclockwise. The index set of the vertices is assumed to be the cyclic group $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ with addition modulo n.

It is well known that the group $\mathrm{Aut}(C_n)$ is isomorphic to the *dihedral group* of order $2n$,

$$D_{2n} = \{1, \sigma, \ldots, \sigma^{n-1}, \tau, \sigma\tau, \ldots, \sigma^{n-1}\tau\},$$

where $\sigma : 2^{V_n} \to 2^{V_n}$ is the rotation which carries every v_i into v_{i+1} and $\tau : 2^{V_n} \to 2^{V_n}$ is the reflection which carries every v_i into v_{n-i}, with the properties $\sigma^n = 1$, $\tau^2 = 1$, and $\tau\sigma = \sigma^{-1}\tau$. The subset $\{1, \sigma, \ldots, \sigma^{n-1}\}$ of rotations, also denoted $\langle\sigma\rangle$, is a cyclic subgroup of order n of D_{2n}.

Consider a family $\mathbf{X}_n \subseteq 2^{V_n}$ of subsets of V_n with the property that if $X \in \mathbf{X}_n$ then $g(X) \in \mathbf{X}_n$ for all $g \in D_{2n}$. For instance, the family of MISs of C_n fulfills this property. Indeed, independence and maximality properties, which characterize the MISs, remain stable under the action of D_{2n}.

Recall that, for any $X \in \mathbf{X}_n$, the *orbit* and the *stabilizer* of X under the action of D_{2n} are respectively defined as

$$\begin{aligned} \mathrm{Orb}(X) &= \{g(X) : g \in D_{2n}\}, \\ \mathrm{Stab}(X) &= \{g \in D_{2n} : g(X) = X\}. \end{aligned}$$

Recall also that $\mathrm{Stab}(X)$ is either a cyclic or a dihedral subgroup of D_{2n} and that, by the orbit-stabilizer theorem, we have

$$|\mathrm{Orb}(X)| \times |\mathrm{Stab}(X)| = |D_{2n}| = 2n, \tag{1}$$

which implies that both $|\mathrm{Orb}(X)|$ and $|\mathrm{Stab}(X)|$ divide $2n$.

Let $\mathcal{O}_n = \mathbf{X}_n/D_{2n}$ denote the set of orbits of $\mathbf{X}_n$ under the action of D_{2n}. We say that $X, X' \in \mathbf{X}_n$ are *isomorphic* if $\mathrm{Orb}(X) = \mathrm{Orb}(X')$, that is, if X' can be obtained from X by a rotation and/or a reflection.

For any divisor $d \geqslant 1$ of $2n$ (we write $d|2n$), denote by $\mathcal{O}_n^d$ the set of orbits of $\mathbf{X}_n$ of cardinality $\frac{2n}{d}$. By (1), $\mathcal{O}_n^d$ is also the set of orbits whose elements have a stabilizer of cardinality d. Define also the sequences

$$\begin{aligned} \mathrm{orb}(n) &= |\mathcal{O}_n|, \\ \mathrm{orb}_d(n) &= |\mathcal{O}_n^d| \qquad (d|2n), \end{aligned}$$

which obviously lead to the identity

$$\mathrm{orb}(n) = \sum_{d|2n} \mathrm{orb}_d(n). \tag{2}$$

Moreover, since the orbits partition the set $\mathbf{X}_n$ we immediately have

$$|\mathbf{X}_n| = \sum_{d|2n} \frac{2n}{d}\, \mathrm{orb}_d(n). \tag{3}$$

We now introduce a concept that will be very useful as we continue, namely the *membership function* of any set $X \in \mathbf{X}_n$.

Definition 1. The *membership* (or *characteristic*) *function* of a set $X \in \mathbf{X}_n$ is the mapping $\mathbf{1}_X : V \to \{0, 1\}$ defined as $\mathbf{1}_X(v_i) = 1$ if and only if $v_i \in X$.

The following immediate result expresses, for any set $X \in \mathbf{X}_n$, the properties of $\mathrm{Stab}(X)$ in terms of the membership function $\mathbf{1}_X$.

Proposition 2. *For any $j \in \mathbb{Z}$ and any $X \in \mathbf{X}_n$, we have*

(i) $\sigma^j \in \mathrm{Stab}(X)$ *if and only if* $\mathbf{1}_X(v_{\ell+j}) = \mathbf{1}_X(v_\ell)$ *for all* $\ell \in \mathbb{Z}$*,*

(ii) $\sigma^{2j}\tau \in \mathrm{Stab}(X)$ *if and only if* $\mathbf{1}_X(v_{j+\ell}) = \mathbf{1}_X(v_{j-\ell})$ *for all* $\ell \in \mathbb{Z}$*,*

(iii) $\sigma^{2j+1}\tau \in \mathrm{Stab}(X)$ *if and only if* $\mathbf{1}_X(v_{j+\ell}) = \mathbf{1}_X(v_{j+1-\ell})$ *for all* $\ell \in \mathbb{Z}$*.*

The conditions stated in Proposition 2 have a clear geometric interpretation. If we think of C_n as a regular n-gon centered at the origin, then condition (i) means that X is invariant under the rotation, about the origin, which carries v_0 into v_j. Then, condition (ii) means that X has a symmetry axis passing through v_j. Finally, condition (iii) means that X has a symmetry axis passing through the midpoint of v_j and v_{j+1}.

From Proposition 2 it follows that, if n is odd, all reflections fix exactly one vertex while, in the case of n even, all reflections of the form $\sigma^i\tau$ fix exactly two vertices or no vertex according to whether i is even or odd, respectively.

Definition 3. Given $X \in \mathbf{X}_n$ and an integer $d \geqslant 1$, the *d-concatenation* of X is the set $X^{(d)} \in \mathbf{X}_{dn}$ whose membership function is given by

$$\mathbf{1}_{X^{(d)}}(v_\ell) = \mathbf{1}_X(v_\ell) \qquad (\ell \in \mathbb{Z}).$$

Definition 3 can be reformulated as follows: For any $X \in \mathbf{X}_n$, the set $X^{(d)} \in \mathbf{X}_{dn}$ is characterized by the fact that the vector $\mathbf{1}_{X^{(d)}}(V_{dn})$ is obtained by concatenating d times the vector $\mathbf{1}_X(V_n)$.

Proposition 4. *For any $X \in \mathbf{X}_n$ and any $j \in \mathbb{Z}$, we have $\sigma^j \in \mathrm{Stab}(X)$ if and only if $\sigma^j \in \mathrm{Stab}(X^{(d)})$. Similarly, we have $\sigma^j\tau \in \mathrm{Stab}(X)$ if and only if $\sigma^j\tau \in \mathrm{Stab}(X^{(d)})$.*

Proof. Let us prove the second part. The first part can be proved similarly. Let $X \in \mathbf{X}_n$ and $j \in \mathbb{Z}$. Then

$$\begin{aligned}
\sigma^j\tau \in \mathrm{Stab}(X) \quad &\Leftrightarrow \quad \mathbf{1}_X(v_{j-\ell}) = \mathbf{1}_X(v_\ell) \qquad (\ell \in \mathbb{Z}) \\
&\Leftrightarrow \quad \mathbf{1}_{X^{(d)}}(v_{j-\ell}) = \mathbf{1}_{X^{(d)}}(v_\ell) \qquad (\ell \in \mathbb{Z}) \\
&\Leftrightarrow \quad \sigma^j\tau \in \mathrm{Stab}(X^{(d)}).
\end{aligned}$$

□

Proposition 5. *The number of orbits of $\mathbf{X}_n$ whose elements have a stabilizer included in $\langle\sigma\rangle$ is given by $\sum_{d|n} \mathrm{orb}_1(d)$.*

Proof. Let $d|n$ and let $X \in \mathbf{X}_n$ be such that $\mathrm{Stab}(X) = \langle\sigma^d\rangle$. Then

$$\mathbf{1}_X(v_{\ell+id}) = \mathbf{1}_X(v_\ell) \qquad (i, \ell \in \mathbb{Z})$$

and hence, there is $Y \in \mathbf{X}_d$ such that $X = Y^{(n/d)}$. By Proposition 4, we necessarily have $\mathrm{Stab}(Y) = \{1\}$ and hence $\mathrm{Orb}(Y) \in \mathcal{O}_d^1$. It is then very easy to see that the set of orbits of $\mathbf{X}_n$ whose elements have the stabilizer $\langle\sigma^d\rangle$ is

$$\Big\{ \bigcup_{Y \in O} Y^{(n/d)} : O \in \mathcal{O}_d^1 \Big\},$$

which shows that the number of orbits of $\mathbf{X}_n$ whose elements have the stabilizer $\langle\sigma^d\rangle$ is given by $\mathrm{orb}_1(d)$. Hence the result. $\square$

The next proposition shows that determining the sequences $\mathrm{orb}_d(n)$ reduces to determining only the two sequences $\mathrm{orb}_1(n)$ and $\mathrm{orb}_2(n)$.

Proposition 6. *For any $d|2n$, we have*

$$\mathrm{orb}_d(n) = \begin{cases} \mathrm{orb}_1(n/d), & \text{if } d \text{ is odd;} \\ \mathrm{orb}_2(2n/d), & \text{if } d \text{ is even.} \end{cases}$$

Proof. Let $d \geqslant 1$ be an odd integer. Consider $O \in \mathcal{O}^d_{dn}$ and $Y \in O$. Then

$$\mathrm{Stab}(Y) = \langle\sigma^n\rangle = \{1, \sigma^n, \sigma^{2n}, \ldots, \sigma^{(d-1)n}\}.$$

By proceeding as in the proof of Proposition 5, we can easily see that

$$\mathcal{O}^d_{dn} = \Big\{ \bigcup_{X \in O} X^{(d)} : O \in \mathcal{O}^1_n \Big\},$$

which obviously entails $\mathrm{orb}_d(dn) = \mathrm{orb}_1(n)$.

Now, let $d \geqslant 1$ be an even integer. Consider $O \in \mathcal{O}^d_{dn/2}$ and $Y \in O$. Then either

$$\mathrm{Stab}(Y) = \langle\sigma^{\frac{n}{2}}\rangle = \{1, \sigma^{\frac{n}{2}}, \ldots, \sigma^{(d-1)\frac{n}{2}}\}$$

(assuming n even) or there is $j \in \{0, 1, \ldots, dn/2 - 1\}$ such that

$$\begin{aligned} \mathrm{Stab}(Y) &= \langle\sigma^n, \sigma^j\tau\rangle \\ &= \{1, \sigma^n, \ldots, \sigma^{(\frac{d}{2}-1)n}, \sigma^j\tau, \sigma^{n+j}\tau, \ldots, \sigma^{(\frac{d}{2}-1)n+j}\tau\}. \end{aligned}$$

In both cases, there is $X \in \mathbf{X}_n$ such that $Y = X^{(d/2)}$ and, by Proposition 4, we have

$$\mathrm{Stab}(X) = \{1, \sigma^{\frac{n}{2}}\},$$

in the first case, and

$$\mathrm{Stab}(X) = \{1, \sigma^{j \,(\mathrm{mod}\ n)}\tau\},$$

in the second case. It is then very easy to see that

$$\mathcal{O}^d_{dn/2} = \Big\{ \bigcup_{X \in O} X^{(d/2)} : O \in \mathcal{O}^2_n \Big\},$$

which obviously entails $\mathrm{orb}_d(dn/2) = \mathrm{orb}_2(n)$. $\square$

3 Counting non-isomorphic MISs of C_n

In the present section we derive formulas for the enumeration of non-isomorphic MISs of C_n. Thus, we now assume that $\mathbf{X}_n$ is the set of MISs of C_n.

We can readily see that the membership function $\mathbf{1}_X$ of any MIS X of C_n is characterized by the following two conditions:

$$\min\{\mathbf{1}_X(v_i),\mathbf{1}_X(v_{i+1})\} = 0 \qquad (i\in\mathbb{Z}), \tag{4}$$
$$\max\{\mathbf{1}_X(v_i),\mathbf{1}_X(v_{i+1}),\mathbf{1}_X(v_{i+2})\} = 1 \qquad (i\in\mathbb{Z}). \tag{5}$$

In fact, condition (4) expresses independence whereas condition (5) expresses maximality. Thus, for any MIS X of C_n, the n-tuple $\mathbf{1}_X(V_n)$ is a cyclic n-list made up of 0s and 1s, with one or two 0s between two neighboring 1s.

As mentioned in the introduction, we have $|\mathbf{X}_n| = p(n)$, the nth term of the Perrin sequence. Then, from (3), we immediately have

$$p(n) = \sum_{d|2n} \frac{2n}{d}\,\mathrm{orb}_d(n). \tag{6}$$

Example 7. The fact that the Perrin sequence counts the distinct MISs of C_n was observed without proof by Füredi [4]. A very simple bijective proof, suggested by Vatter [9], can be written as follows. Let $X\in\mathbf{X}_n$ and suppose that the vertex in X of maximal label is k, and the vertex of second-greatest label is j. Then k must be $j+2$ or $j+3$. If $k=j+2$, then $X\setminus\{v_k\}$ can be viewed as a MIS of C_{n-2}, while if $k=j+3$ then $X\setminus\{v_k\}$ can be viewed as a MIS of C_{n-3}. The inverse to this map can easily be described, thus proving that the number of MISs of C_n satisfies the Perrin recurrence. □

We can readily see that any orbit of $\mathbf{X}_n$ can be uniquely represented by a cyclic list made up of 2s and/or 3s summing up to n, where a clockwise writing is not distinguished from its counterclockwise counterpart. The bijection between this representation and the n-list representation is straightforward: put a 2 whenever only one 0 separates two neighboring 1s and put a 3 whenever two 0s separate two neighboring 1s.

This representation of orbits immediately leads to the following result:

Proposition 8. $\mathrm{orb}(n)$ *is the number of cyclic compositions of n in which each term is either 2 or 3, where a clockwise writing is not distinguished from its counterclockwise counterpart.*

Consider the Padovan sequence $q=(q(n))_{n\in\mathbb{N}}$ (a shifted version of Sloane's A000931), which is defined as $q(1)=0$, $q(2)=1$, $q(3)=1$, and

$$q(n) = q(n-2)+q(n-3) \qquad (n\geqslant 4), \tag{7}$$

and let $r=(r(n))_{n\in\mathbb{N}}$ be the sequence defined by

$$r(n) = \begin{cases} q(k), & \text{if } n=2k-1;\\ q(k+2), & \text{if } n=2k.\end{cases}$$

Note that, from (7) it follows that $r(n)$ fulfills the recurrence equation

$$r(n) = r(n-4) + r(n-6) \qquad (n \geqslant 7).$$

The following immediate proposition shows that $q(n)$ is the number of ways of writing n as an ordered sum in which each term is either 2 or 3. For example, $q(8) = 4$, and there are 4 ways of writing 8 as an ordered sum of 2s and 3s: $2+2+2+2$, $2+3+3$, $3+2+3$, $3+3+2$.

Proposition 9. *$q(n)$ is the number of compositions of n in which each term is either 2 or 3.*

Proof. The values of $q(n)$ for $n \leqslant 3$ can be easily calculated. Now, fix $n \geqslant 4$. From among all the possible compositions of n, $q(n-2)$ of them begin with a 2 and $q(n-3)$ of them begin with a 3, which establishes (7). $\square$

As the following proposition shows, $r(n)$ shares the same property as $q(n)$ but with the additional condition that the terms form a cyclic and palindromic composition. For example, $r(8) = 2$, and there are only 2 ways of writing 8 as an ordered sum of 2s and 3s in a cyclic and palindromic way: $2+2+2+2$, $3+2+3$.

Proposition 10. *$r(n)$ is the number of cyclic and palindromic compositions of n in which each term is either 2 or 3.*

Proof. Let $T \in \{2,3\}^m$ be a cyclic and palindromic composition of n, with m terms placed counterclockwise and uniformly on the unit circle. Let X be a representative of the corresponding orbit of $\mathbf{X}_n$. Clearly, each symmetry axis of X induces a symmetry axis of the cyclic list T. The number of symmetry axes, say $d \geqslant 1$, necessarily divides n since $\mathrm{Stab}(X)$ is a dihedral subgroup of D_{2n}.

Each of the d axes intersects the unit circle at two points. Label these $2d$ intersection points as $a_0, a_1, \ldots, a_{2d-1}$ counterclockwise. These points constitute the vertices of a regular $2d$-gon. Let S be the sublist of T consisting of the terms located on the arc $[a_0, a_1]$ inclusive, with the property that the terms located at a_0 or a_1, if any, are divided by two, and let $\overline{S}$ be the reversed version of S. According to the kaleidoscope principle (see for instance [6] for an expository note on dihedral kaleidoscopes and Coxeter groups), the sublist S is located on each of the d arcs of the form $[a_{2i}, a_{2i+1}]$ $(i \in \mathbb{Z}_d)$ whereas the sublist $\overline{S}$ is located on each of the d arcs of the form $[a_{2i+1}, a_{2i+2}]$ $(i \in \mathbb{Z}_d)$. In particular, the elements of $S \cup \overline{S}$ sum up to n/d.

Suppose first that $n = 2k-1$ is odd, which necessarily implies that d is odd (since $d|n$). By symmetry, there is a point a_i $(i \in \mathbb{Z}_{2d})$ at which we have a 3 and, by rotating the composition, we can assume that $i = 0$. At the opposite point a_d we have either a 2 or no term. As this feature holds for every symmetry axis, we immediately observe that the only possible palindromic writing of T whose two extreme terms are $\frac{3}{2}$s is obtained by the alternating concatenation of $2d$ sublists S and $\overline{S}$, that is,

$$W = S\overline{S}S\overline{S}\cdots S\overline{S}.$$

Let L be the first half of W and, except for its first element (which is $\frac{3}{2}$) and its last element (which can be 1, 2, or 3), replace in L each sublist $(\frac{3}{2}, \frac{3}{2})$ with 3 and each sublist $(1, 1)$ with 2.

Let $\mathbf{T}_n$ be the set of cyclic and palindromic compositions of n in which each term is either 2 or 3. Clearly, we can assume that each of these compositions is written so that its two extreme terms are $\frac{3}{2}$s. For any $T \in \mathbf{T}_n$, denote by L^T the sublist as defined above. Then, it is easy to see that the disjoint union

$$\bigcup_{T \in \mathbf{T}_n} L^T,$$

which has cardinality $|\mathbf{T}_n|$, is also the set of compositions of $\frac{3}{2} + \frac{n-3}{2} = \frac{3}{2} + (k-2)$ in which each term is either 2 or 3, except for the first term, which is $\frac{3}{2}$, and the last term which can be 1, or 2, or 3. By calculating the cardinality of this latter set, we obtain

$$|\mathbf{T}_n| = q(k-2) + q(k-3).$$

Indeed, by Proposition 9, there are $q(k-2)$ compositions whose last term is either 2 or 3, plus $q(k-3)$ compositions whose last term is 1. Finally, using (7) we obtain $|\mathbf{T}_n| = q(k) = r(n)$, which proves the result for $n = 2k-1$.

Let us now assume that $n = 2k$ is even. Then the two possible palindromic writings of T are obtained by the alternating concatenation of $2d$ sublists S and $\overline{S}$, that is,

$$\begin{aligned} W_1 &= S\overline{S}S\overline{S}\cdots S\overline{S}, \\ W_2 &= \overline{S}S\overline{S}S\cdots\overline{S}S. \end{aligned}$$

(Note that if S contains at least two elements then it cannot be palindromic for otherwise we would have more than d symmetry axes.)

Let L_1 (resp. L_2) be the first half of W_1 (resp. W_2) and, without modifying its two extreme elements, replace in L_1 (resp. L_2) each sublist $(\frac{3}{2}, \frac{3}{2})$ with 3 and each sublist $(1, 1)$ with 2. If S contains at least two elements then L_1 and L_2 are two distinct sublists. If S contains only one element then we simply set $L_1 = (1, 2, \ldots, 2, 1)$ and $L_2 = (2, \ldots, 2)$ in case $S = (1)$ and $L_1 = (\frac{3}{2}, 3, \ldots, 3, \frac{3}{2})$ and $L_2 = (3, \ldots, 3)$ in case $S = (\frac{3}{2})$.

Let $\mathbf{T}_n$ be the set of cyclic and palindromic compositions of n in which each term is either 2 or 3 and, for any $T \in \mathbf{T}_n$, denote by L_1^T and L_2^T the two sublists as defined above. Then, it is easy to see that the disjoint union

$$\bigcup_{T \in \mathbf{T}_n} L_1^T \cup \bigcup_{T \in \mathbf{T}_n} L_2^T,$$

which has cardinality $2|\mathbf{T}_n|$, is also the set of compositions of $\frac{n}{2} = k$ in which each term is either 2 or 3, except for the two extreme terms which can be 1, or $\frac{3}{2}$, or 2, or 3. By calculating the cardinality of this latter set, we obtain

$$\begin{aligned} 2\,|\mathbf{T}_n| &= q(k) + 2q(k-1) + q(k-2) + q(k-3) \\ &= 2q(k) + 2q(k-1) \\ &= 2q(k+2). \end{aligned}$$

Indeed, by Proposition 9, there are $q(k)$ compositions whose two extreme terms are 2s or 3s, plus $2q(k-1)$ compositions whose only one extreme term is 1, plus $q(k-2)$ compositions whose two extreme terms are 1s, plus $q(k-3)$ compositions whose two extreme terms are $\frac{3}{2}$s. Finally, we have $|\mathbf{T}_n| = q(k+2) = r(n)$, which completes the proof. $\square$

The following example illustrates the proof of Proposition 10 in the case $n = 16$.

Example 11. There are 7 cyclic and palindromic compositions of 16 in which each term is either 2 or 3. Table 1 gives these compositions together with the number d of symmetry axes and the sublists L_1 and L_2, as defined in the proof of Proposition 10.

T	d	L_1	L_2
$(2,2,2,2,2,2,2,2)$	8	$(2,2,2,2)$	$(1,2,2,2,1)$
$(3,2,3,3,2,3)$	2	$(3,2,3)$	$(1,3,3,1)$
$(2,2,3,2,3,2,2)$	1	$(2,2,3,1)$	$(1,3,2,2)$
$(2,3,2,2,2,3,2)$	1	$(2,3,2,1)$	$(1,2,3,2)$
$(2,3,3,3,3,2)$	1	$(2,3,3)$	$(3,3,2)$
$(3,2,2,2,2,2,3)$	1	$(3,2,2,1)$	$(1,2,2,3)$
$(\frac{3}{2},2,3,3,3,2,\frac{3}{2})$	1	$(\frac{3}{2},2,3,\frac{3}{2})$	$(\frac{3}{2},3,2,\frac{3}{2})$

Table 1: Cyclic and palindromic compositions of 16

Through the identification of the orbits of $\mathbf{X}_n$ whose elements have at least one symmetry axis with the cyclic and palindromic compositions of n made up of 2s and/or 3s, Proposition 10 can be immediately rewritten as follows:

Proposition 12. *The number of orbits of $\mathbf{X}_n$ whose elements have a stabilizer not included in $\langle\sigma\rangle$ is given by $r(n)$.*

We are now able to state our main result, which gives explicit expressions for the sequences orb, orb_1, and orb_2. To this end, recall that the *Dirichlet convolution product* of two sequences $f = (f(n))_{n\in\mathbb{N}}$ and $g = (g(n))_{n\in\mathbb{N}}$ is the sequence $f * g = ((f * g)(n))_{n\in\mathbb{N}}$ defined as

$$(f * g)(n) = \sum_{d|n} f(d)\, g(\frac{n}{d}).$$

Moreover, in addition to the sequences $p(n)$, $q(n)$, and $r(n)$ introduced above, consider the following integer sequences:

$$\begin{aligned}
\mu(n) &= \text{A008683}(n) \qquad (\textit{Möbius function}),\\
\text{A113788}(n) &= \frac{1}{n}\,(p * \mu)(n),\\
\mathbf{1}(n) &= 1,\\
e_k(n) &= \begin{cases} 1, & \text{if } n = k;\\ 0, & \text{else,}\end{cases} \qquad (k \in \mathbb{N}).
\end{aligned}$$

Theorem 13. *The following equalities hold:*

$$\begin{aligned}
\mathrm{orb} &= r + \mathrm{orb}_1 * \mathbf{1}, &(8)\\
2\mathrm{orb}_1 &= \underline{A113788} - r * \mu, &(9)\\
\mathrm{orb}_2 &= r * \mu + \mathrm{orb}_1 * e_2. &(10)
\end{aligned}$$

Proof. Combining Propositions 5 and 12 immediately leads to (8). Next, combining (2) and (8), we can write

$$r(n) = \sum_{d|2n} \mathrm{orb}_d(n) - \sum_{d|n} \mathrm{orb}_1(d),$$

that is, using Proposition 6,

$$\begin{aligned}
r(n) &= \sum_{\substack{d|2n\\ d \text{ even}}} \mathrm{orb}_d(n) + \sum_{\substack{d|2n\\ d \text{ odd}}} \mathrm{orb}_d(n) - \sum_{d|n} \mathrm{orb}_1(d)\\
&= \sum_{\substack{d|2n\\ \frac{2n}{d} \text{ even}}} \mathrm{orb}_2(d) + \sum_{\substack{d|n\\ \frac{n}{d} \text{ odd}}} \mathrm{orb}_1(d) - \sum_{d|n} \mathrm{orb}_1(d)\\
&= \sum_{d|n} \mathrm{orb}_2(d) - \sum_{\substack{d|n\\ \frac{n}{d} \text{ even}}} \mathrm{orb}_1(d).
\end{aligned}$$

Hence, introducing $g_2(n) = \frac{1+(-1)^n}{2}$, we obtain

$$r = \mathrm{orb}_2 * \mathbf{1} - \mathrm{orb}_1 * g_2,$$

that is, using Möbius inversion formula and the immediate identity $g_2 = e_2 * \mathbf{1}$,

$$r * \mu = \mathrm{orb}_2 - \mathrm{orb}_1 * e_2. \tag{11}$$

On the other hand, combining (6) with Proposition 6, we obtain

$$\begin{aligned}
p(n) &= \sum_{\substack{d|2n\\ d \text{ odd}}} \frac{2n}{d} \mathrm{orb}_1\left(\frac{n}{d}\right) + \sum_{\substack{d|2n\\ d \text{ even}}} \frac{2n}{d} \mathrm{orb}_2\left(\frac{2n}{d}\right)\\
&= \sum_{\substack{d|n\\ d \text{ odd}}} \frac{2n}{d} \mathrm{orb}_1\left(\frac{n}{d}\right) + \sum_{\substack{d|2n\\ \frac{2n}{d} \text{ even}}} d\,\mathrm{orb}_2(d)\\
&= \sum_{\substack{d|n\\ \frac{n}{d} \text{ odd}}} 2d\,\mathrm{orb}_1(d) + \sum_{d|n} d\,\mathrm{orb}_2(d).
\end{aligned}$$

Hence, introducing $f_1(n) = n\,\mathrm{orb}_1(n)$, $f_2(n) = n\,\mathrm{orb}_2(n)$, and $g_1(n) = \frac{1-(-1)^n}{2}$, we obtain

$$p = 2f_1 * g_1 + f_2 * \mathbf{1},$$

that is, using Möbius inversion formula and the fact that $g_1 = \mathbf{1} - g_2 = \mathbf{1} - e_2 * \mathbf{1}$,

$$p * \mu = 2f_1 - 2f_1 * e_2 + f_2,$$

which implies

$$\text{A113788} = 2\text{orb}_1 - \text{orb}_1 * e_2 + \text{orb}_2. \tag{12}$$

Combining (11) and (12) immediately leads to formulas (9) and (10). □

It is noteworthy that the sequence $\text{orb}(n)$ can be rewritten as

$$2\text{orb}(n) = r(n) + \frac{1}{n}(p * \phi)(n),$$

where $\phi =$ A000010 is the *Euler totient function*. In fact, combining (8) and (9), we obtain

$$\begin{aligned} 2\text{orb}(n) &= 2r(n) + 2(\text{orb}_1 * \mathbf{1})(n) \\ &= r(n) + (\text{A113788} * \mathbf{1})(n) \end{aligned}$$

and the latter term also writes

$$(\text{A113788} * \mathbf{1})(n) = \sum_{d|n} \frac{d}{n}(p * \mu)\left(\frac{n}{d}\right) = \frac{1}{n}(p * \mu * \text{Id})(n) = \frac{1}{n}(p * \phi)(n), \tag{13}$$

where $\text{Id}(n) = n$ is the identity sequence.

4 Counting unlabeled MISs of C_n

We now investigate the simpler problem of enumerating the *unlabeled* MISs of C_n, that is, the MISs defined up to a rotation. In fact, these unlabeled MISs are nothing less than the orbits of $\mathbf{X}_n$ under the action of $\langle\sigma\rangle$.

Let $\text{orb}^\sigma(n)$ denote the number of unlabeled MISs of C_n and, for any $d|n$, let $\text{orb}^\sigma_d(n)$ denote the number of unlabeled MISs of C_n that have $\frac{n}{d}$ isomorphic representatives or, equivalently, those that have a stabilizer of cardinality d.

Then, we clearly have

$$\text{orb}^\sigma(n) = \sum_{d|n} \text{orb}^\sigma_d(n), \tag{14}$$

$$p(n) = \sum_{d|n} \frac{n}{d}\, \text{orb}^\sigma_d(n), \tag{15}$$

and, by proceeding as in the proof of Proposition 6, we can easily show that

$$\text{orb}^\sigma_d(n) = \text{orb}^\sigma_1(n/d) \qquad (d|n). \tag{16}$$

We then obtain the following formulas:

Theorem 14. *The following equalities hold:*

$$\begin{aligned} \mathrm{orb}^\sigma &= \mathrm{orb}_1^\sigma * \mathbf{1}, \qquad (17)\\ \mathrm{orb}_1^\sigma &= \textit{A113788}. \qquad (18)\end{aligned}$$

Proof. Define the sequence $f_1^\sigma(n) = n\,\mathrm{orb}_1^\sigma(n)$. Combining (14) and (16) immediately leads to (17). On the other hand, combining (15) and (16), we obtain

$$p(n) = \sum_{d|n} \frac{n}{d} \mathrm{orb}_1^\sigma(n/d) = (f_1^\sigma * \mathbf{1})(n)$$

and by using Möbius inversion formula we are immediately led to (18). □

It is noteworthy that, combining (13), (17), and (18), we also obtain

$$\mathrm{orb}^\sigma(n) = \frac{1}{n}(p * \phi)(n).$$

Finally, we also have the immediate result:

Proposition 15. $\mathrm{orb}^\sigma(n)$ *is the number of cyclic compositions of* n *in which each term is either* 2 *or* 3.

Example 16. Just as for the results obtained in Section 2, the results stated in the present section do not essentially lie on the intrinsic properties of MISs. Up to the left-hand side of (15), formulas (14)–(16) remain valid for any other definition of $\mathbf{X}_n$.

5 Summary

Starting from the Perrin (A001608) and Padovan (shifted A000931) sequences, respectively denoted $p(n)$ and $q(n)$ in the present paper, we have introduced the following new integer sequences, with an explicit expression for each of them:

- $r(n) =$ A127682(n) is the number of non-isomorphic MISs of C_n having at least one symmetry axis (see Proposition 12), where two MISs are isomorphic if they are identical up to a rotation and a reflection. It is also the number of cyclic and palindromic compositions of n in which each term is either 2 or 3 (see Proposition 10). Recall that this sequence is defined as

$$r(n) = \begin{cases} q(k), & \text{if } n = 2k-1;\\ q(k+2), & \text{if } n = 2k.\end{cases}$$

- For any $d|2n$, $\mathrm{orb}_d(n)$ gives the number of non-isomorphic MISs of C_n having $\frac{2n}{d}$ isomorphic representatives. This sequence can always be expressed from one of the sequences $\mathrm{orb}_1(n) =$ A127683(n) and $\mathrm{orb}_2(n) =$ A127684(n) (see Proposition 6), which in turn can be directly calculated from the formulas

$$\begin{aligned} \mathrm{orb}_1(n) &= \frac{1}{2}\big(\text{A113788}(n) - (r*\mu)(n)\big)\\ \mathrm{orb}_2(n) &= (r*\mu)(n) + (\mathrm{orb}_1 * e_2)(n).\end{aligned}$$

- $\mathrm{orb}(n) = $ A127685(n) gives the number of non-isomorphic MISs of C_n. It is also the number of cyclic compositions of n in which each term is either 2 or 3, where a clockwise writing is not distinguished from its counterclockwise counterpart (see Proposition 8). This sequence is given explicitly by

$$\mathrm{orb}(n) = r(n) + (\mathrm{orb}_1 * \mathbf{1})(n) = \frac{1}{2}\Big(r(n) + \frac{1}{n}(p * \phi)(n)\Big).$$

- $(\mathrm{orb}_1 * \mathbf{1})(n) = $ A127686(n) is the number of non-isomorphic MISs of C_n having no symmetry axis (see Proposition 5). It is also the number of cyclic and non-palindromic compositions of n in which each term is either 2 or 3, where a clockwise writing is not distinguished from its counterclockwise counterpart. This sequence is given explicitly by

$$(\mathrm{orb}_1 * \mathbf{1})(n) = \mathrm{orb}(n) - r(n).$$

- $\mathrm{orb}^\sigma(n) = $ A127687(n) is the number of unlabeled MISs of C_n. It is also the number of cyclic compositions of n in which each term is either 2 or 3 (see Proposition 15). This sequence is given explicitly by

$$\mathrm{orb}^\sigma(n) = (\text{A113788} * \mathbf{1})(n) = \frac{1}{n}(p * \phi)(n) = 2\mathrm{orb}(n) - r(n).$$

- For any $d|n$, $\mathrm{orb}^\sigma_d(n)$ gives the number of unlabeled MISs of C_n having $\frac{n}{d}$ isomorphic representatives. This sequence can always be expressed from the sequence $\mathrm{orb}^\sigma_1(n)$ (see (16)), which in turn can be directly calculated from the formulas

$$\mathrm{orb}^\sigma_1(n) = \text{A113788}(n) = (\mathrm{orb}^\sigma * \mu)(n).$$

The first 40 values of the main sequences considered in this paper are listed in Table 2. We observe that, when n is prime, we have $\mathrm{orb}(n) = \mathrm{orb}_1(n) + \mathrm{orb}_2(n)$, $\mathrm{orb}_2(n) = r(n)$, and $\mathrm{orb}^\sigma(n) = \mathrm{orb}^\sigma_1(n) = \mathrm{orb}(n) + \mathrm{orb}_1(n) = \frac{1}{n}p(n)$, which can be easily verified.

6 A musical application

In classical tonal music (see for instance Benward and Saker [1]), MISs on C_{7k} for $k = 1, 2, \ldots$ may be seen as musical chords built on thirds and fourths, where k denotes the number of octaves on some major or minor scale. For instance, in C major the Cmaj chord, namely C-E-G, is a MIS on C_7 and the thirteenth chord C13maj, namely C-E-G-B-d-f-a, is a MIS on C_{14}.

On C_7 there exists only $\mathrm{orb}(7) = \mathrm{orb}^\sigma(7) = 1$ non-isomorphic MIS, with $p(7) = 7$ elements giving all possible chords built on generic thirds and fourths. However, on C_{14}, that is a two octave scale, we observe $\mathrm{orb}(14) = \mathrm{orb}^\sigma(14) = 5$ non-isomorphic MIS, with $p(14) = 51$ elements giving all possible chords on generic (unlabeled) thirds and fourths.

Figure 1 shows the 5 possible chord sets in C major built from generic thirds and fourths on two octaves. MIS (a) represents the above-mentioned thirteenth chord C13maj, in which

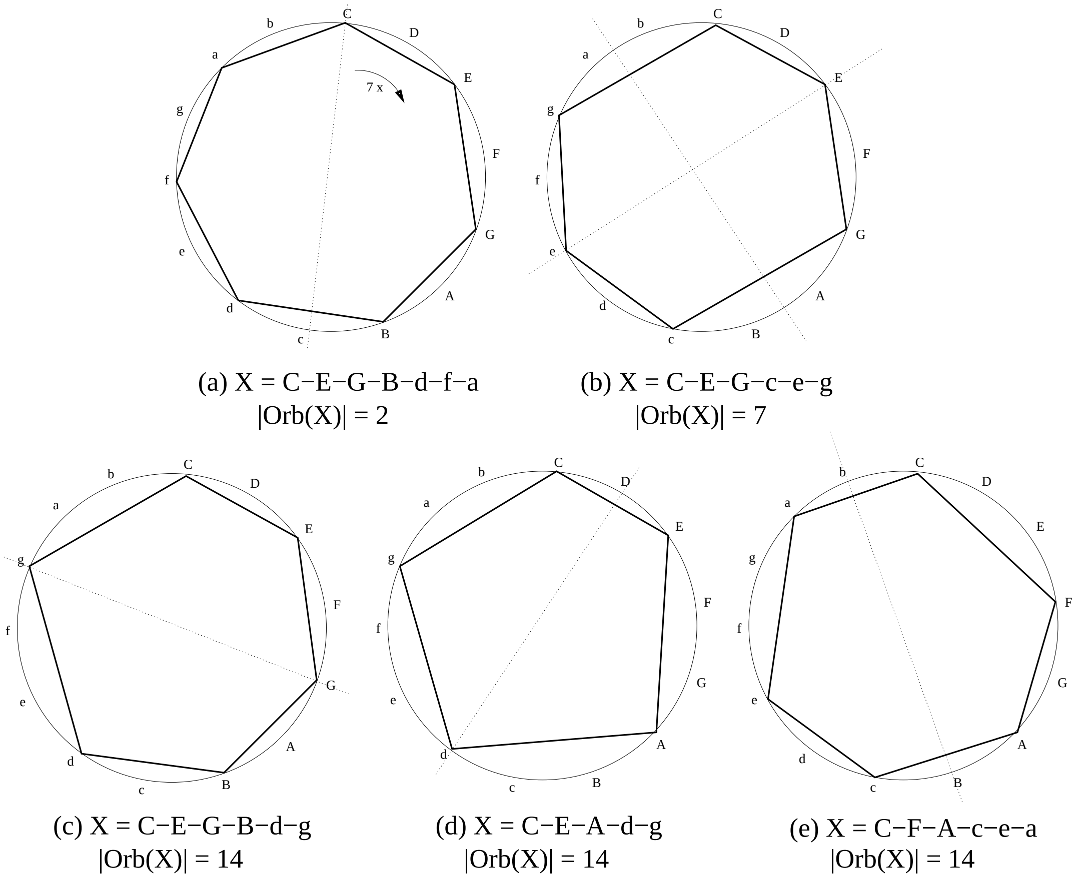

Figure 1: Chord sets in C major built from generic thirds and fourths on two octaves

we observe 7 symmetry axes passing through every note of the chord. In this case, the orbit contains only 2 representatives. MIS (b) represents a repetition of the Cmaj chord in each octave, in which we have 2 symmetry axes, and the orbit contains 7 representatives. In the three remaining MISs (c), (d), and (e), we have only one symmetry axis passing respectively through 2, 1, and 0 notes and each of the corresponding orbits contains 14 representatives.

Our results provide useful tools for analyzing the symmetries of third and fourth chords on more than two octaves.

n	$p(n)$	$q(n)$	$r(n)$	$\mathrm{orb}(n)$	$\mathrm{orb}_1(n)$	$\mathrm{orb}_2(n)$	$\mathrm{orb}^\sigma(n)$	$\mathrm{orb}_1^\sigma(n)$
1	0	0	0	0	0	0	0	0
2	2	1	1	1	0	1	1	1
3	3	1	1	1	0	1	1	1
4	2	1	1	1	0	0	1	0
5	5	2	1	1	0	1	1	1
6	5	2	2	2	0	0	2	0
7	7	3	1	1	0	1	1	1
8	10	4	2	2	0	1	2	1
9	12	5	2	2	0	1	2	1
10	17	7	3	3	0	1	3	1
11	22	9	2	2	0	2	2	2
12	29	12	4	4	0	2	4	2
13	39	16	3	3	0	3	3	3
14	51	21	5	5	0	3	5	3
15	68	28	4	5	1	2	6	4
16	90	37	7	7	0	5	7	5
17	119	49	5	6	1	5	7	7
18	158	65	9	10	1	6	11	8
19	209	86	7	9	2	7	11	11
20	277	114	12	14	2	9	16	13
21	367	151	9	14	5	7	19	17
22	486	200	16	20	4	13	24	21
23	644	265	12	20	8	12	28	28
24	853	351	21	30	9	16	39	34
25	1130	465	16	31	15	15	46	45
26	1497	616	28	44	16	24	60	56
27	1983	816	21	48	27	19	75	73
28	2627	1081	37	67	30	32	97	92
29	3480	1432	28	74	46	28	120	120
30	4610	1897	49	104	54	44	159	151
31	6107	2513	37	117	80	37	197	197
32	8090	3329	65	161	96	58	257	250
33	10717	4410	49	188	139	46	327	324
34	14197	5842	86	254	167	81	422	414
35	18807	7739	65	302	237	63	539	537
36	24914	10252	114	407	292	104	700	687
37	33004	13581	86	489	403	86	892	892
38	43721	17991	151	654	501	145	1157	1145
39	57918	23833	114	801	687	110	1488	1484
40	76725	31572	200	1064	862	189	1928	1911

Table 2: First 40 values of the main sequences

References

[1] B. Benward and M. Saker. *Music in Theory and Practice*, Vol 1. McGraw-Hill Humanities, 8th edition, 2008.

[2] G. J. Chang and M.-J. Jou. The number of maximal independent sets in connected triangle-free graphs. *Discrete Math.* **197/198** (1999), 169–178.

[3] R. Euler. The Fibonacci number of a grid graph and a new class of integer sequences. *J. Integer Seq.* **8** (2005), Article 05.2.6.

[4] Z. Füredi. The number of maximal independent sets in connected graphs. *J. Graph Theory* **11** (1987), 463–470.

[5] C. Godsil and G. Royle. *Algebraic Graph Theory*, Vol. 207 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2001.

[6] R. Goodman. Alice through looking glass after looking glass: the mathematics of mirrors and kaleidoscopes. *Amer. Math. Monthly*, **111** (2004), 281–298.

[7] S. Kitaev. Independent sets on path-schemes. *J. Integer Seq.* **9** (2006), Article 06.2.2.

[8] N. J. A. Sloane (editor). The On-Line Encyclopedia of Integer Sequences. `http://www.research.att.com/~njas/sequences/`

[9] V. R. Vatter. Private communication, 2006. School of Mathematics and Statistics, University of St Andrews, Scotland.

[10] G. C. Ying, K. K. Meng, B. E. Sagan, and V. R. Vatter. Maximal independent sets in graphs with at most r cycles. *J. Graph Theory* **53** (2006), 270–282.

2000 *Mathematics Subject Classification*: Primary 05C69; Secondary 05C38, 05A15, 05A17, 05C25, 1155.
Keywords: maximal independent set, cycle graph, combinatorial enumeration, dihedral group, group action, cyclic and palindromic composition of integers, Perrin sequence, Padovan sequence.

(Concerned with sequences A000010, A000931, A001608, A008683, A113788, A127682, A127683, A127684, A127685, A127686, and A127687.)

Received May 26 2008; revised version received November 25 2008. Published in *Journal of Integer Sequences*, December 14 2008.

Return to Journal of Integer Sequences home page.