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Inversions of Permutations in Symmetric, Alternating, and Dihedral Groups

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Abstract

We use two methods to obtain a formula relating the total number of inversions of all permutations and the corresponding order of symmetric, alternating, and dihedral groups. First, we define an equivalence relation on the symmetric group $\mathbf{S}_n$ and consider each element in each equivalence class as a permutation of a proper subset of $\{1, 2, \ldots, n\}$. Second, we look at certain properties of a backward permutation, a permutation obtained by reversing the row images of a given permutation. Lastly, we employ the first method to obtain a recursive formula corresponding to the number of permutations with k inversions.

1 Introduction

Let n be a positive integer and A be the finite set $\{1, 2, \ldots, n\}$. The group of all permutations of A is the *symmetric group on n elements* and it is denoted by $\mathbf{S}_n$. A permutation $\sigma \in \mathbf{S}_n$ can be represented by

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix}.$$

Note that $\mathbf{S}_n$ has $n!$ elements and the identity element is given by $\iota(i) = i$ for all $i \in A$.

An *inversion* induced by a permutation σ is an ordered pair $(\sigma(i), \sigma(j))$ such that $i < j$ and $\sigma(i) > \sigma(j)$. For purposes of computations later, we represent an inversion $(\sigma(i), \sigma(j))$

just by the ordered pair (i, j). The number of inversions of a permutation is a way to measure the extent to which the permutation is "out of order". Inversions are important in sorting algorithms and have applications in computational molecular biology (see [1] for example).

If we let $I(\sigma)$ be the set of all inversions of a permutation $\sigma \in \mathbf{S}_n$, then

$$I(\sigma) = \{(i, j) : \sigma(i) > \sigma(j), 1 \leq i < j \leq n\}. \tag{1}$$

It now follows from Eq. (1) that if $N(\sigma)$ is the number of all inversions induced by $\sigma \in \mathbf{S}_n$, then $N(\sigma) = |I(\sigma)|$. Observe that the only permutation with no inversion is the identity permutation and so, $N(\sigma) = 0$ if and only if $\sigma = \iota$. Further, the number of inversions of a permutation and its inverse are equal.

In general, to determine the total number of inversions of a permutation $\sigma \in \mathbf{S}_n$, we count the number of j's such $\sigma(1) > \sigma(j)$ for $1 < j \leq n$, then the number of j's such that $\sigma(2) > \sigma(j)$ for $2 < j \leq n$, up to the number of j's such that $\sigma(n-1) > \sigma(j)$ for $n - 1 < j \leq n$, and thus, a formula for $N(\sigma)$ is given by

$$N(\sigma) = \sum_{i=1}^{n-1} \left|\{j : \sigma(i) > \sigma(j), i < j \leq n\}\right|. \tag{2}$$

Let $\beta \in \mathbf{S}_n$ be the permutation defined by

$$\beta = \begin{pmatrix} 1 & 2 & \cdots & n \\ n & n-1 & \cdots & 1 \end{pmatrix}. \tag{3}$$

Note that for $1 \leq i \leq n$ we have $\beta(i) = n - i + 1$. Thus, $i < j$ implies that $\beta(i) > \beta(j)$. It now follows that $(i, j) \in I(\beta)$ for $1 \leq i < j \leq n$ and the permutation β defined by Eq. (3) gives the maximum number of inversions in any permutation. Hence,

$$\max_{\sigma \in \mathbf{S}_n} N(\sigma) = N(\beta) = \sum_{i=1}^{n-1} (n - i) = \binom{n}{2}.$$

For each positive integer n, if we let S_n be the total number of inversions of all permutations $\sigma \in \mathbf{S}_n$ then

$$S_n = \sum_{\sigma \in \mathbf{S}_n} N(\sigma). \tag{4}$$

Using formulas (2) and (4) to determine S_n would take at most $\binom{n}{2} n!$ steps and thus inefficient for large values of n. This paper introduces two methods to determine S_n and eventually use these methods to generate explicit formulas for the total number of inversions of all permutations to two specific subgroups of $\mathbf{S}_n$, namely the alternating group $\mathbf{A}_n$ and the dihedral group $\mathbf{D}_n$.

2 Partitioning the symmetric group

Let $\{a_j\}_{j=1}^n$ be an increasing sequence of n distinct positive integers, that is, for $j < k$, we have $a_j < a_k$, and $\mathbf{S}(\{a_j\}_{j=1}^n)$ be the group of all permutations of $\{a_j\}_{j=1}^n$. Notice that $\mathbf{S}(\{a_j\}_{j=1}^n) \simeq \mathbf{S}_n$ and in particular $\mathbf{S}(\{a_j\}_{j=1}^n) = \mathbf{S}_n$ if $\{a_j\}_{j=1}^n = \{1, 2, \ldots, n\}$. As a consequence,

$$S_n = \sum_{\sigma \in \mathbf{S}(\{a_j\}_{j=1}^n)} N(\sigma).$$

Two permutations σ_1 and σ_2 in $\mathbf{S}_n$ are related, written as $\sigma_1 \sim \sigma_2$, if and only if $\sigma_1(1) = \sigma_2(1)$. It can be verified that $\sim$ is an equivalence relation on $\mathbf{S}_n$. The equivalence relation $\sim$ induces equivalence classes $\mathbf{O}_j = \{\sigma \in \mathbf{S}_n : \sigma(1) = j\}$, $j = 1, \ldots, n$, of $\mathbf{S}_n$. It follows that $\mathbf{O}_i \cap \mathbf{O}_j = \emptyset$ for $i \neq j$ and $\mathbf{S}_n = \bigcup_{j=1}^n \mathbf{O}_j$ and thus, the total number of inversions of all permutations in $\mathbf{S}_n$ is the same as the sum of the number of inversions of all permutations in each equivalence classes $\mathbf{O}_j$. In symbols, we have

$$S_n = \sum_{j=1}^{n} \sum_{\sigma \in \mathbf{O}_j} N(\sigma). \tag{5}$$

Let $\sigma \in \mathbf{O}_j$ and $\{a_k\}_{k=1}^{n-1}$ be an arrangement in increasing order of elements of $A - \{j\}$. The permutation τ defined by

$$\tau = \begin{pmatrix} a_1 & a_2 & \cdots & a_{n-1} \\ \sigma(2) & \sigma(3) & \cdots & \sigma(n) \end{pmatrix}$$

is an element of $\mathbf{S}(A - \{j\})$. If we define the permutation $\sigma_{\tau,j}$ by

$$\sigma_{\tau,j} = \begin{pmatrix} 1 & 2 & \cdots & n \\ j & \tau(a_1) & \cdots & \tau(a_{n-1}) \end{pmatrix},$$

then $\sigma = \sigma_{\tau,j}$ and

$$N(\sigma_{\tau,j}) = (j-1) + N(\tau). \tag{6}$$

Equations (5) and (6) give us a recursive formula for S_n and we state it as a lemma.

Lemma 1. *We have $S_1 = 0$ and*

$$S_n = \frac{n!(n-1)}{2} + nS_{n-1}, \quad n \geq 2.$$

Proof. Since $\mathbf{S}_1 = \{\iota\}$, then $S_1 = N(\iota) = 0$. Now suppose $n \geq 2$. Note that for each $j = 1, \ldots, n$

$$\begin{aligned} \sum_{\sigma \in \mathbf{O}_j} N(\sigma) &= \sum_{\tau \in \mathbf{S}(A-\{j\})} N(\sigma_{\tau,j}) \\ &= \sum_{\tau \in \mathbf{S}(A-\{j\})} [(j-1) + N(\tau)] \\ &= (j-1)|\mathbf{S}(A-\{j\})| + \sum_{\tau \in \mathbf{S}(A-\{j\})} N(\tau) \\ &= (j-1)(n-1)! + S_{n-1}. \end{aligned}$$

From Eq. (5), we get

$$\begin{aligned} S_n &= \sum_{j=1}^{n}[(j-1)(n-1)! + S_{n-1}] \\ &= (n-1)!\sum_{j=1}^{n}(j-1) + nS_{n-1} \\ &= \frac{n!(n-1)}{2} + nS_{n-1}. \end{aligned}$$

□

Theorem 2. *For $n \geq 1$, we have*

$$S_n = \frac{|\mathbf{S}_n|}{2}\binom{n}{2} = \frac{n!}{2}\binom{n}{2}.$$

Proof. The case where $n = 1$ is trivial. Assuming that the formula holds for some fixed integer k, we go on to show that it must hold for $k+1$ too. Using Lemma 1 and the induction hypothesis,

$$\begin{aligned} S_{k+1} &= \frac{(k+1)!k}{2} + (k+1)S_k \\ &= \frac{(k+1)!k}{2} + (k+1)\frac{k!}{2}\binom{k}{2} \\ &= \frac{(k+1)!}{2}\left(k + \frac{k(k-1)}{2}\right) \\ &= \frac{(k+1)!}{2}\binom{k+1}{2}, \end{aligned}$$

which is the formula in the case $n = k+1$. This establishes the theorem. □

A permutation σ is said to be *even* if $N(\sigma)$ is even, otherwise it is said to be *odd*. Let $\mathbf{A}_n$ be the set of all even permutations in $\mathbf{S}_n$. Note that $\mathbf{A}_n$ is a subgroup of index 2 of $\mathbf{S}_n$ called the *alternating group of degree* n. Similarly, we let $\mathbf{A}(\{a_j\}_{j=1}^n)$ be the corresponding alternating group of all even permutations of $\{a_j\}_{j=1}^n$. If we denote A_n to be the number of inversions of all permutations in $\mathbf{A}_n$ then

$$A_n = \sum_{\sigma\in\mathbf{A}_n} N(\sigma) = \sum_{\sigma\in\mathbf{A}(\{a_j\}_{j=1}^n)} N(\sigma) \tag{7}$$

For small values of n, A_n can easily be determined using Eq. (7). Indeed, $A_1 = A_2 = 0$ and $A_3 = 4$. A drawback of counting, however, occurs when n is large.

Because $\mathbf{A}_n$ is a subset of $\mathbf{S}_n$, if $\sigma \in \mathbf{A}_n$, then $\sigma \in \mathbf{O}_j$ for some j. Thus, the method used to determine S_n can as well be extended to determine A_n. It should be noted, however, that if an even permutation σ is an element of $\mathbf{O}_j$, it is not true that all other permutations in $\mathbf{O}_j$ are also even. Thus, some minor modifications in counting are necessary.

Theorem 3. *For all $n \geq 4$ we have*

$$A_n = \frac{|\mathbf{A}_n|}{2}\binom{n}{2} = \frac{n!}{4}\binom{n}{2}.$$

Proof. Recall that every permutation $\sigma \in \mathbf{S}_n$ can be uniquely represented by $\sigma_{\tau,j}$ for some $\tau \in \mathbf{S}(A - \{j\})$, where $1 \leq j \leq n$. It follows from Eq. (6) that $\sigma_{\tau,j}$ is even if and only if j and $N(\tau)$ have different parity. For simplicity, we let $\mathbf{A}(j) = \mathbf{A}(A - \{j\})$ and $(\mathbf{A}(j))^c$ be the complement of $\mathbf{A}(A - \{j\})$ with respect to $\mathbf{S}(A - \{j\})$, in other words, it is the set of permutations of $A - \{j\}$ with an odd number of inversions.

First, consider the case where $n \geq 4$ is even, and so

$$\begin{aligned}
A_n &= \sum_{j=1}^{n/2} \sum_{\tau \in \mathbf{A}(2j-1)} [(2j-2) + N(\tau)] + \sum_{j=1}^{n/2} \sum_{\tau \in (\mathbf{A}(2j))^c} [(2j-1) + N(\tau)] \\
&= \sum_{j=1}^{n/2} \left[(2j-2)|\mathbf{A}_{n-1}| + \sum_{\sigma \in \mathbf{A}_{n-1}} N(\sigma) + (2j-1)|\mathbf{A}_{n-1}^c| + \sum_{\sigma \in \mathbf{A}_{n-1}^c} N(\sigma) \right] \\
&= \sum_{j=1}^{n/2} \left[\frac{(4j-3)(n-1)!}{2} + A_{n-1} + (S_{n-1} - A_{n-1}) \right] \\
&= \frac{(n-1)!}{2} \sum_{j=1}^{n/2} \left[(4j-3) + \binom{n-1}{2} \right] \\
&= \frac{n!}{4}\binom{n}{2}.
\end{aligned}$$

Now suppose $n \geq 5$ is odd so that $n-1$ is even. From the previous result, we have

$$A_{n-1} = \frac{(n-1)!}{4}\binom{n-1}{2}.$$

Similarly, we compute as follows

$$\begin{aligned}
A_n &= \sum_{j=1}^{(n-1)/2} \sum_{\tau \in \mathbf{A}(2j-1)} [(2j-2) + N(\tau)] + \sum_{\tau \in \mathbf{A}(n)} [(n-1) + N(\tau)] \\
&\quad + \sum_{j=1}^{(n-1)/2} \sum_{\tau \in (\mathbf{A}(2j))^c} [(2j-1) + N(\tau)] \\
&= \sum_{j=1}^{(n-1)/2} \left[\frac{(4j-3)(n-1)!}{2} + S_{n-1} \right] + \frac{(n-1)(n-1)!}{2} + A_{n-1} \\
&= \frac{(n-1)!}{2} \sum_{j=1}^{(n-1)/2} \left[(4j-3) + \binom{n-1}{2} \right] + \frac{(n-1)!}{2} \left[(n-1) + \frac{(n-1)(n-2)}{4} \right] \\
&= \frac{n!}{4}\binom{n}{2}.
\end{aligned}$$

$\square$

The following corollary, which relates S_n and A_n, follows immediately from the previous theorems.

Corollary 4. *If $n \geq 1$, then $A_n = \lfloor S_n/2 \rfloor$.*

3 Backward permutations

A *backward inversion* of a permutation $\sigma \in \mathbf{S}_n$ is a pair $(\sigma(i), \sigma(j))$ such that $1 \leq i < j \leq n$ and $\sigma(i) < \sigma(j)$. Again, for computation purposes, we represent a backward inversion $(\sigma(i), \sigma(j))$ just by the ordered pair (i, j). If we let $M(\sigma)$ denotes the total number of backward inversions of a permutation σ, then

$$\begin{aligned} M(\sigma) &= |\{(i,j) : \ \sigma(i) < \sigma(j)\ , 1 \leq i < j \leq n\}| \\ &= |\{(i,j) : \ 1 \leq i < j \leq n\}| - |\{(i,j) : \ \sigma(i) > \sigma(j)\ , 1 \leq i < j \leq n\ \}| \\ &= \binom{n}{2} - N(\sigma). \end{aligned}$$

Therefore, for any permutation $\sigma \in \mathbf{S}_n$, the sum of the total number of inversions and backward inversions is $\binom{n}{2}$, that is,

$$N(\sigma) + M(\sigma) = \binom{n}{2}. \tag{8}$$

An immediate consequence of Eq. (8) is stated as a theorem which characterizes a permutation in terms of backward inversions.

Theorem 5. *Let $\sigma \in \mathbf{S}_n$.*

(i) If $n \equiv 0, 1 \pmod 4$, then $\sigma \in \mathbf{A}_n$ if and only if $M(\sigma)$ is even.

(ii) If $n \equiv 2, 3 \pmod 4$, then $\sigma \in \mathbf{A}_n$ if and only if $M(\sigma)$ is odd.

Proof. If $n \equiv 0, 1 \pmod 4$ then $\binom{n}{2}$ is even, and it follows from Eq. (8) that $M(\sigma)$ is even if and only if $N(\sigma)$ is even. If $n \equiv 2, 3 \pmod 4$ then $\binom{n}{2}$ is odd, and so $M(\sigma)$ is odd if and only if $N(\sigma)$ is even. $\square$

Given a permutation $\sigma \in \mathbf{S}_n$, the *backward permutation* of σ, denoted by $\overline{\sigma}$, is defined as

$$\overline{\sigma} = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ \sigma(n) & \sigma(n-1) & \cdots & \sigma(2) & \sigma(1) \end{pmatrix}. \tag{9}$$

It is clear from the definition that any backward permutation $\overline{\sigma}$ is also in $\mathbf{S}_n$.

Let B be the bijective mapping $B : \mathbf{S}_n \to \mathbf{S}_n$ that sends every permutation onto its backward permutation, that is, $B(\sigma) = \overline{\sigma}$. Thus, $\mathbf{S}_n = \{\overline{\sigma} : \sigma \in \mathbf{S}_n\}$ and $B(\overline{\sigma}) = \overline{\overline{\sigma}} = \sigma$. It now follows that $N(\overline{\sigma}) = M(\sigma)$ and from Eq. (8), we have

$$N(\overline{\sigma}) + N(\sigma) = \binom{n}{2}. \tag{10}$$

The power of backward permutations and backward inversions can be best illustrated by offering an alternative proof of Theorem 2. Because $\mathbf{S}_n = \{\overline{\sigma} : \sigma \in \mathbf{S}_n\}$, we have

$$\begin{aligned} 2S_n \;=\; 2\sum_{\sigma\in\mathbf{S}_n} N(\sigma) \;&=\; \sum_{\sigma\in\mathbf{S}_n} N(\sigma) + \sum_{\sigma\in\mathbf{S}_n} N(\overline{\sigma}) \\ &=\; \sum_{\sigma\in\mathbf{S}_n} [N(\sigma) + N(\overline{\sigma})] \\ &=\; \sum_{\sigma\in\mathbf{S}_n} \binom{n}{2} \;=\; n!\binom{n}{2}, \end{aligned}$$

and the result follows.

Using Theorem 5, one can check that if $n \equiv 0, 1 \pmod 4$ then $B[\mathbf{A}_n] = \mathbf{A}_n$ and if $n \equiv 2, 3 \pmod 4$ then $B[\mathbf{A}_n] = \mathbf{A}_n^c$. Thus, the concept of backward permutations seems inappropriate for computing A_n for any values of n. It will, however, be most useful in the next section.

4 Backward permutations in dihedral groups

Consider the regular n-gon, with $n \geq 3$. Label successive vertices of the n-gon by $1, 2, \ldots, n$. The *Dihedral group* $\mathbf{D}_n$ of isometries of the plane which map a regular n-gon onto itself can be considered as a subgroup of $\mathbf{S}_n$. To see this, first let us represent the elements of $\mathbf{D}_n$ as permutations. A $(360/n)^{\text{o}}$ clockwise rotation (about the center of the n-gon) is represented by the permutation

$$\rho = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ 2 & 3 & \cdots & n & 1 \end{pmatrix}.$$

Thus, for each $1 \leq k < n$, a $(360k/n)^{\text{o}}$ clockwise rotation can be represented as the permutation ρ^k given by

$$\rho^k = \begin{pmatrix} 1 & 2 & \cdots & n-k & n-k+1 & \cdots & n \\ k+1 & k+2 & \cdots & n & 1 & \cdots & k \end{pmatrix}, \tag{11}$$

and $\rho^n = \iota$. Note that $\langle\rho\rangle$ is the subgroup of $\mathbf{D}_n$ consisting of all rotations. Further, for each $k = 1, \ldots, n$, one can see from Eq. (11) that $N(\rho^k) = k(n-k)$ and so

$$\sum_{\sigma\in\langle\rho\rangle} N(\sigma) = \sum_{k=1}^{n} N(\rho^k) = \sum_{k=1}^{n} k(n-k) = \frac{n+1}{3}\binom{n}{2}.$$

If n is odd, for each $1 \leq k \leq n$, let μ_k be the mirror reflection whose axis bisects the angle corresponding to the vertex k of the n-gon.
Case 1. If $2k-1 \leq n$ then

$$\mu_k = \begin{pmatrix} 1 & 2 & \cdots & k & \cdots & 2k-2 & 2k-1 & 2k & \cdots & n \\ 2k-1 & 2k-2 & \cdots & k & \cdots & 2 & 1 & n & \cdots & 2k \end{pmatrix}.$$

It follows that

$$\overline{\mu_k} = \begin{pmatrix} 1 & 2 & \cdots & n-2k+1 & n-2k+2 & n-2k+3 & \cdots & n \\ 2k & 2k+1 & \cdots & n & 1 & 2 & \cdots & 2k-1 \end{pmatrix},$$

and so $\overline{\mu_k} = \rho^{2k-1}$.
Case 2. If $2k-1 > n$ then

$$\mu_k = \begin{pmatrix} 1 & \cdots & 2k-n-1 & 2k-n & \cdots & k & k+1 & \cdots & n \\ 2k-n-1 & \cdots & 1 & n & \cdots & k & k-1 & \cdots & 2k-n \end{pmatrix}$$

and

$$\overline{\mu_k} = \begin{pmatrix} 1 & 2 & \cdots & 2n-2k+1 & 2n-2k+2 & \cdots & n \\ 2k-n & 2k-n+1 & \cdots & n & 1 & \cdots & 2k-n-1 \end{pmatrix}.$$

and thus, $\overline{\mu_k} = \rho^{2k-n-1}$.

Suppose now that n is even. For $1 \leq k \leq n$, we have $\mu_k = \mu_{k+n/2}$ and thus we only need to consider those reflections μ_k for $1 \leq k \leq n/2$. Similarly, it can be shown that $\overline{\mu_k} = \rho^{2k-1}$ for all $1 \leq k \leq n/2$.

Aside from the mirror reflection μ_k defined above, there are mirror reflections whose axis bisects two parallel sides of the n-gon, when n is even. For each $k = 1, 2, \ldots, n/2$, denote $\mu_{k,k+1}$ be the mirror reflection whose axis bisects the two sides of the n-gon, one having vertices k and $k+1$. Then

$$\mu_{k,k+1} = \begin{pmatrix} 1 & \cdots & 2k & 2k+1 & \cdots & n \\ 2k & \cdots & 1 & n & \cdots & 2k+1 \end{pmatrix}.$$

Taking the backward permutation yields

$$\overline{\mu_{k,k+1}} = \begin{pmatrix} 1 & \cdots & n-2k & n-2k+1 & \cdots & n \\ 2k+1 & \cdots & n & 1 & \cdots & 2k \end{pmatrix}.$$

and thus $\overline{\mu_{k,k+1}} = \rho^{2k}$.

Hence, the backward permutation of any mirror reflection is a rotation. If M be the set of all mirror reflections in $\mathbf{D}_n$, it follows that $B[M]$ is the set of all rotations in $\mathbf{D}_n$. We state these results as a theorem.

Theorem 6. *If $n \geq 3$ and M is the set of all mirror reflections in $\mathbf{D}_n$, then $\{M, B[M]\}$ partitions $\mathbf{D}_n$.*

Similarly, we define

$$D_n = \sum_{\sigma \in \mathbf{D}_n} N(\sigma)$$

and the following theorem gives an explicit formula for D_n.

Theorem 7. *For all $n \geq 3$ we have*

$$D_n = \frac{|\boldsymbol{D}_n|}{2}\binom{n}{2} = n\binom{n}{2}.$$

Proof. Using the previous theorem, we get

$$\begin{aligned} D_n = \sum_{\sigma \in \mathbf{D}_n} N(\sigma) &= \sum_{\sigma \in M} N(\sigma) + \sum_{\sigma \in B[M]} N(\sigma) \\ &= \sum_{\sigma \in M} N(\sigma) + \sum_{\sigma \in M} N(\overline{\sigma}) \\ &= \sum_{\sigma \in M} [N(\sigma) + N(\overline{\sigma})] \\ &= |M|\binom{n}{2} \quad = \quad n\binom{n}{2}. \end{aligned}$$

□

Corollary 8. *If t_n denotes the nth triangular number, then for all $n \geq 3$ we have*

$$D_n = \sum_{t_{n-1} < i < t_n} i.$$

Using Theorems 2, 3 and 7, we can generate the following table :

n	1	2	3	4	5	6	7	8	9
S_n	0	1	9	72	600	5400	52920	564480	6531840
A_n	0	0	4	36	300	2700	26460	282240	3265920
D_n			9	24	50	90	147	224	324

5 Number of permutations with k inversions

Let $I_n(k)$ denotes the number of permutations in $\mathbf{S}_n$ having k inversions. It was shown that the sequence $\{I_n(k) : 0 \leq k \leq \binom{n}{2}\}$ has the generating function

$$\sum_{k=0}^{\binom{n}{2}} I_n(k) x^k = \prod_{j=1}^{n} \frac{1-x^j}{1-x},$$

and using this polynomial, one can find the value of $I_n(k)$ for $k = 0, 1, \ldots, \binom{n}{2}$, (see [3]). Also, asymptotic formulas of the sequence $\{I_{n+k}(n) : n \in \mathbb{N}\}$ for a fixed integer $k \geq 0$ were discussed in [2] and [3]. In this paper, we employ the partitioning method to provide a recursive formula in finding these inversion numbers.

Lemma 9. *For each $0 \leq k \leq \binom{n}{2}$, there exists $\sigma \in \mathbf{S}_n$ such that $N(\sigma) = k$.*

Proof. If $n = 1, 2$, the statement clearly holds. Assume that the lemma is true for $n-1$. Let $0 \leq k \leq \binom{n}{2}$. For all $1 \leq j \leq n$ and $0 \leq l \leq \binom{n-1}{2}$ there exists $\tau \in \mathbf{S}(A - \{j\})$ such that $N(\sigma_{\tau,j}) = (j-1) + N(\tau)$ and $N(\tau) = l$. Observe that the possible values of $(j-1) + N(\tau)$ are $0, 1, \ldots, \binom{n}{2}$. Therefore, one can find a j and a τ such that $\sigma_{\tau,j} \in \mathbf{S}_n$ and $N(\sigma_{\tau,j}) = (j-1) + N(\tau) = k$. □

We note that S_n and A_n can be represented by the inversion numbers. Indeed, we have

$$S_n = \sum_{k=0}^{\binom{n}{2}} kI_n(k) \quad \text{and} \quad A_n = \sum_{k=0}^{\lfloor \frac{1}{2}\binom{n}{2} \rfloor} 2kI_n(2k).$$

Theorem 10. *For $0 \leq k \leq \binom{n}{2}$ where $n \geq 2$, we have the following recurrence relation*

$$I_1(0) = I_2(0) = I_2(1) = 1 \tag{12}$$

and

$$I_n(k) = \sum_{i=\max\{0,k-n+1\}}^{\min\left\{k,\binom{n-1}{2}\right\}} I_{n-1}(i), \quad n \geq 3. \tag{13}$$

Proof. Eq. (12) is clear. Now suppose $n \geq 3$. Recall that $N(\sigma_{\tau,j}) = (j-1) + N(\tau)$. Let $N(\sigma_{\tau,j}) = k$ and $N(\tau) = i$ then $0 \leq i \leq \binom{n-1}{2}$ and $0 \leq j-1 \leq n-1$. We find those i such that given j, $N(\sigma_{\tau,j}) = k$ and $I_n(k)$ can be formed by adding $I_{n-1}(i)$ for all values of i that we found. We have $j - 1 + i = k$ or equivalently $i = k - j + 1 \leq k$. Because $i \leq \binom{n-1}{2}$, then $i \leq \min\left\{k, \binom{n-1}{2}\right\}$. Now $i = k - j + 1 \geq k - n + 1$. But i must be nonnegative and thus $i \geq \max\{0, k - n + 1\}$. Hence, we have Eq. (13). □

With the aid of the previous theorem, we can generate the following table:

$I_n(k)$																
$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1															
2	1	1														
3	1	2	2	1												
4	1	3	5	6	5	3	1									
5	1	4	9	15	20	22	20	15	9	4	1					
6	1	5	14	29	49	71	90	101	101	90	71	49	29	14	5	1

Let $n > 1$. Using Theorem 10, if the value of $I_{n-1}(k)$ for $0 \le k \le \binom{n-1}{2}$ is known, then the values of $I_n(k)$ for $0 \le k \le \binom{n}{2}$ can be determined by the following formula

$$I_n(k) = \begin{cases} 1, & \text{if } k = 0; \\ I_n(k-1) + I_{n-1}(k), & \text{if } 1 \le k \le n-1; \\ I_n(k-1) + I_{n-1}(k) - I_{n-1}(k-n), & \text{if } n \le k \le \binom{n-1}{2}; \\ I_n(k-1) - I_{n-1}(k-n), & \text{if } \binom{n-1}{2} < k \le \binom{n}{2}. \end{cases} \tag{14}$$

Theorem 11. *For all* $0 \le k \le \binom{n}{2}$ *we have* $I_n\left(\binom{n}{2} - k\right) = I_n(k)$.

Proof. Let $K_1 = \{\sigma \in \mathbf{S}_n : N(\sigma) = k\}$ and $K_2 = \left\{\sigma \in \mathbf{S}_n : N(\sigma) = \binom{n}{2} - k\right\}$, and by Lemma 5.1, K_1 and K_2 are both nonempty. The mapping $B_1 : K_1 \to K_2$ defined by $B_1(\sigma) = \overline{\sigma}$ is clearly bijective. Therefore $|K_1| = |K_2|$ and so $I_n\left(\binom{n}{2} - k\right) = I_n(k)$. $\square$

Corollary 12. *If* $n \equiv 2, 3 \pmod 4$ *and* $C = \frac{1}{2}\left[\binom{n}{2} - 1\right]$ *then* $\displaystyle\sum_{k=0}^{C} I_n(k) = \frac{n!}{2}$.

As an application of Eq. (14), we will consider the sequence $\{I_{n+k}(k) : n \ge 0\}$, where $k \ge 1$ is fixed. One can verify, using the second case in Eq. (14), that $I_{n+1}(1) = n$, $I_{n+2}(2) = n(n+3)/2$ and $I_{n+3}(3) = (n+3)(n^2+6n+2)/6$, for all $n \ge 0$. Suppose $I_{n+k}(k) = \sum_{i=0}^{k} a_{ki} n^i$, where $a_{kk} \ne 0$ so that $deg\ I_{n+k}(k) = k$. Thus

$$\begin{aligned} I_{n+k+1}(k+1) &= I_{k+1}(k+1) + \sum_{j=1}^{n} I_{j+1+k}(k) \\ &= C_{k+1} + \sum_{j=1}^{n} \sum_{i=0}^{k} a_{ki}(j+1)^i \\ &= C_{k+1} + \sum_{j=1}^{n} \sum_{i=0}^{k} \sum_{h=0}^{i} \binom{i}{h} a_{ki} j^h \\ &= C_{k+1} + \sum_{i=0}^{k} \sum_{h=0}^{i} \binom{i}{h} a_{ki} P_{h+1}(n), \end{aligned}$$

for all $n \ge 0$, where $C_{k+1} = I_{k+1}(k+1)$ and $P_{h+1}(n) = \sum_{j=1}^{n} j^h$. It can be shown that $P_{h+1}(n)$ is a polynomial of degree $h+1$. From these, it follows that $I_{n+k+1}(k+1)$ is a polynomial of degree $k+1$. Thus we have shown that, $I_{n+k}(k)$ is a polynomial of degree k for all $k \ge 1$. This result implies that $I_n(k) = O(n^k)$ for all $k \ge 1$.

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