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A Note on the Hankel Transform of the Central Binomial Coefficients

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Abstract

We show that the $n \times n$ Hankel matrix formed from the successive even central binomial coefficients $\binom{2l}{l}$, $l = 0, 1, \ldots$ arises naturally when considering the trace form in the number ring of the maximal real subfield of suitable cyclotomic fields. By considering the trace form in two different integral bases of the number ring we get a factorization of this matrix which immediately yields the well-known zeroth and first Hankel transforms of the sequence.

1 Introduction

Given a sequence a_l, $l = 0, 1, \ldots$, the $n \times n$ Hankel matrix $H_n^{(k)}$, $k = 0, 1, \ldots$, formed from this sequence is the matrix

$$\begin{pmatrix} a_k & a_{k+1} & a_{k+2} & \cdots & a_{k+n-1} \\ a_{k+1} & a_{k+2} & a_{k+3} & \cdots & a_{k+n} \\ a_{k+2} & a_{k+3} & a_{k+4} & \cdots & a_{k+n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{k+n-1} & a_{k+n} & a_{k+n+1} & \cdots & a_{k+2n-2} \end{pmatrix}$$

The *k-th Hankel Transform* of the sequence a_l, $l = 0, 1, \ldots$ is the sequence of determinants $d_n^{(k)}$ of the matrices $H_n^{(k)}$ above, $n = 1, 2, \ldots$. It is worth mentioning that some authors refer to the Hankel transform only as the sequence $d_n = \det H_n^{(0)}$ (see, for example, [7]).

Hankel matrices have been studied extensively, and connections between Hankel matrices and other areas of mathematics are well known (see [13] for a very nice survey of Hankel matrices especially in relation to combinatorics and coding theory). The term Hankel transform was introduced in Sloane's sequence A055878 and first studied in [7]. Since then, there have been numerous further studies of Hankel transforms of sequences, for instance, [1, 3, 4, 5, 12].

Consider the particular sequence $a_l = \binom{2l}{l}$, $l = 0, 1, \ldots$, and denote $H_n^{(0)}$ by simply H_n. The main purpose of this paper is to show that the matrices $H_n := H_n^{(0)}$ and $H_n^{(1)}$ that define the zeroth and first Hankel transforms of this particular sequence arise very naturally when considering the trace form $\mathrm{Tr}(x, y) = \mathrm{Tr}_{K/\mathbb{Q}}(xy)$ on the number ring $\mathcal{O}_K$, where K is the maximal real subfield of the 2^N-th cyclotomic field, for any N such that $2^N \geq 8n$. By considering the same trace form with respect to two different integral bases of $\mathcal{O}_K$, we obtain in a very natural way a factorization of H_n as

$$H_n = B_n D_n B_n^T \tag{1}$$

where

$$B_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \binom{2}{1} & 1 & 0 & \cdots & 0 \\ \binom{4}{2} & \binom{4}{1} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{2(n-1)}{n-1} & \binom{2(n-1)}{n-2} & \binom{2(n-1)}{n-3} & \cdots & 1 \end{pmatrix} \tag{2}$$

and

$$D_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2 \end{pmatrix} \tag{3}$$

Similarly, we get the factorization

$$H_n^{(1)} = 2C_n C_n^T \tag{4}$$

where

$$C_n = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ \binom{3}{1} & 1 & 0 & \dots & 0 \\ \binom{5}{2} & \binom{5}{1} & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{2n-1}{n-1} & \binom{2n-1}{n-2} & \binom{2n-1}{n-3} & \dots & 1 \end{pmatrix} \tag{5}$$

These factorizations yield at once the following well-known result (see, for example, [1, 10, 11]):

Corollary 1. *The zeroth Hankel transform $d_n^{(0)}$ of the sequence $\binom{2l}{l}$, $l = 0, 1, \dots$ is the sequence 2^{n-1}, $n = 1, 2, \dots,$ and the first Hankel transform $d_n^{(1)}$ is the sequence 2^n, $n = 1, 2 \dots.$*

Of course, once the factorization has been guessed at, Eq. (1) and Eq. (4) can be proved by elementary means: our point here is only to show that the Hankel matrices H_n and $H_n^{(1)}$ and the factorizations above arise completely naturally in number theory.

2 Trace Calculations in $\mathcal{O}_K$

Given a positive integer N we write ω for the primitive 2^N-th root of unity $e^{2\pi\imath/2^N}$. We write θ for the element $\omega + \omega^{-1}$. We write θ_j ($j = 0, 1, \dots,$) for the element $\omega^j + \omega^{-j}$, so that $\theta_1 = \theta$ and $\theta_0 = 2$. We write L for the field $\mathbb{Q}(\omega)$, and K for the real subfield $\mathbb{Q}(\theta)$. Note that $[L : \mathbb{Q}] = 2^{N-1}$ and $[K : \mathbb{Q}] = 2^{N-2}$. We write m for 2^{N-2}.

We begin by computing traces of the elements θ^i as well as of products $\theta_i\theta_j$. (Lemmas 2 and 3 also appear in [9], and are implicit in [2, Prop. 4.3].)

Lemma 2. *For $1 \le s < 2 \cdot m$,*

$$\mathrm{Tr}_{K/\mathbb{Q}}(\theta^s) = \begin{cases} 0, & \text{if } s \text{ is odd;} \\ m\binom{s}{s/2}, & \text{if } s \text{ is even.} \end{cases}$$

Proof. Observe that $\mathrm{Tr}_{K/\mathbb{Q}}(\theta^s) = \mathrm{Tr}_{L/\mathbb{Q}(\imath)}(\theta^s)$. Now expanding $\theta^s = (\omega + \omega^{-1})^s$ binomially, we find

$$\theta^s = \sum_{j=0}^{s} \binom{s}{j} \omega^j \omega^{-(s-j)} = \sum_{j=0}^{s} \binom{s}{j} \omega^{2j-s}$$

Notice that when s is odd, only odd powers of ω appear in this expansion. Since ω raised to any odd power is also a primitive 2^N-th root of unity, it has minimal polynomial $x^m \pm \imath$ over $\mathbb{Q}(\imath)$, and consequently, any odd power of ω has trace zero from L to $\mathbb{Q}(\imath)$. It follows that

$\mathrm{Tr}_{L/\mathbb{Q}(\imath)}(\theta^s) = 0$ when s is odd. (Notice that this is true for all odd s and not just those in the range of the statement of the lemma.)

When s is even, we first assume that $s < m$. Then, the terms in the expansion of θ^s above have even powers of ω that run through $s, s-2, \ldots, 2, 0, -2, \ldots, -(s-2), -s$. Given any nonzero even integer $2l$ in this set, we write it as $2^e a$ for some e and odd integer a. Then ω^{2l} is a primitive 2^{N-e}-root of unity, and $[L : \mathbb{Q}(\omega^{2l})] = 2^e$. Since, by assumption, $e < N-2$, $\mathbb{Q}(\omega^{2l})$ strictly contains $\mathbb{Q}(\imath)$. Now, $\mathrm{Tr}_{L/\mathbb{Q}(\imath)}(\omega^{2l}) = \mathrm{Tr}_{\mathbb{Q}(\omega^{2l})/\mathbb{Q}(\imath)}\mathrm{Tr}_{L/\mathbb{Q}(\omega^{2l})}(\omega^{2l}) = 2^e\mathrm{Tr}_{\mathbb{Q}(\omega^{2l})/\mathbb{Q}(\imath)}(\omega^{2l})$. Just as in the previous paragraph, $\mathrm{Tr}_{\mathbb{Q}(\omega^{2l})/\mathbb{Q}(\imath)}(\omega^{2l})$ is zero since the minimal polynomial of ω^{2l} is $x^{2^{N-e-2}} \pm \imath$. Hence, all nonzero powers of ω contribute nothing to the trace, so $\mathrm{Tr}_{L/\mathbb{Q}(\imath)}(\theta^s)$ is m times the coefficient of the term ω^0 which is $\binom{s}{s/2}$.

When $2 \cdot m > s \geq m$, we need a small modification. The expansion of θ^s will have only even powers of ω as before, but continuing to write these powers as $2l$, we will now also have powers where $2 \cdot m > 2l \geq m$. We first consider the powers $2l > m$: we factor ω^m out to find $\omega^{2l} = \imath\omega^{2l-m}$. Thus, $\mathrm{Tr}_{L/\mathbb{Q}(\imath)}(\omega^{2l}) = \imath\mathrm{Tr}_{L/\mathbb{Q}(\imath)}(\omega^{2l-m})$. From our assumptions we find that $2l - m$ is a positive even integer and that $m > 2l - m$, so the arguments of the previous paragraph show that this trace is zero. Thus, we are left with the terms ω^m, ω^0, and ω^{-m}. But $\mathrm{Tr}_{L/\mathbb{Q}(\imath)}(\omega^m) = \mathrm{Tr}_{L/\mathbb{Q}(\imath)}(\imath) = m\imath$, while $\mathrm{Tr}_{L/\mathbb{Q}(\imath)}(\omega^{-m}) = \mathrm{Tr}_{L/\mathbb{Q}(\imath)}(-\imath) = -m\imath$, so these two terms cancel each other out. Once again, we are left with the term ω^0 whose trace is $m\binom{s}{s/2}$. □

Lemma 3. *For $1 \leq j < 2m$,*

$$\mathrm{Tr}_{K/\mathbb{Q}}(\theta_j) = 0 \tag{6}$$

and for $1 \leq i, j < m$

$$\mathrm{Tr}_{K/\mathbb{Q}}(\theta_i\theta_j) = \begin{cases} 0, & \text{if } i \neq j; \\ 2m, & \text{if } i = j. \end{cases} \tag{7}$$

Proof. The proof of the first part is essentially contained in the proof of Lemma 2 above. We have $\mathrm{Tr}_{K/\mathbb{Q}}(\theta_j) = \mathrm{Tr}_{L/\mathbb{Q}(\imath)}(\theta_j) = \mathrm{Tr}_{L/\mathbb{Q}(\imath)}(\omega^j + \omega^{-j})$. We saw in that proof that $\mathrm{Tr}_{L/\mathbb{Q}(\imath)}(\omega^j) = 0$ for all $1 \leq j < 2m$ except when $j = m$, so $\mathrm{Tr}_{L/\mathbb{Q}(\imath)}(\theta_j) = 0$ for all such j. When $j = m$, we have $\mathrm{Tr}_{L/\mathbb{Q}(\imath)}(\omega^m) = m\imath$ and $\mathrm{Tr}_{L/\mathbb{Q}(\imath)}(\omega^{-m}) = -m\imath$. Hence $\mathrm{Tr}_{L/\mathbb{Q}(\imath)}(\theta_m) = 0$ as well.

For the second assertion, note that $\theta_i\theta_j = \theta_{i+j} + \theta_{j-i}$ where we can assume without loss of generality that $j - i \geq 0$. The result immediately follows from the calculations of $\mathrm{Tr}_{K/\mathbb{Q}}(\theta_j)$ above, noting that $i + j < 2m$, and $\theta_0 = 2$.

□

Note that $\mathcal{O}_K = \mathbb{Z}[\theta]$ (see [8, Exer. 35, Chap. 2] for instance). Expanding each power θ^s binomially and collecting terms we find

$$\theta^s = \begin{cases} \displaystyle\sum_{j=0}^{\lfloor s/2 \rfloor} \binom{s}{j}\theta_{s-2j}, & \text{if } s \text{ is odd}; \\ \displaystyle\sum_{j=0}^{(s/2)-1} \binom{s}{j}\theta_{s-2j} + \binom{s}{s/2}, & \text{if } s \text{ is even}. \end{cases} \tag{8}$$

For any positive integer n, let B_n be as in (2), and C_n as in (5). Let

$$\begin{aligned} V_e &= (1, \theta^2, \theta^4, \dots, \theta^{m-2})^T \\ V_o &= (\theta, \theta^3, \theta^5, \dots, \theta^{m-1})^T \\ W_e &= (1, \theta_2, \theta_4, \dots, \theta_{m-2})^T \\ W_o &= (\theta_1 = \theta, \theta_3, \theta_5, \dots, \theta_{m-1})^T \end{aligned}$$

Then Eq. (8) splits as two matrix relations:

$$V_e = B_{m/2} W_e; \tag{9}$$

$$V_o = C_{m/2} W_o. \tag{10}$$

Since 1, θ, θ^2, ..., θ^{m-1} is a $\mathbb{Z}$-basis for $\mathcal{O}_K$, and since B_m and C_m are integer matrices with determinant 1, these relations show that 1, $\theta_1 = \theta$, θ_2, ..., θ_{m-1} is also a $\mathbb{Z}$-basis for $\mathcal{O}_K$. But these relations show us even more: if we define

$$M_e = \mathbb{Z} \oplus \mathbb{Z}\theta^2 \oplus \mathbb{Z}\theta^4 \oplus \cdots \oplus \mathbb{Z}\theta^{m-2} \quad \text{and} \tag{11}$$

$$M_o = \mathbb{Z} \oplus \mathbb{Z}\theta \oplus \mathbb{Z}\theta^3 \oplus \cdots \oplus \mathbb{Z}\theta^{m-1} \tag{12}$$

then 1, θ_2, θ_4, ..., θ_{m-2} is also a $\mathbb{Z}$ basis for M_e, and θ, θ_3, ..., θ_{m-1} is also a $\mathbb{Z}$ basis for M_o.

Hence, since $\mathrm{Tr}_{K/\mathbb{Q}}$ is $\mathbb{Z}$ linear, for any $x \in M_e$, $x = b_0 + b_2\theta^2 + \cdots + b_{m-2}\theta^{m-2}$ and any $y \in M_e$, $y = \sum c_0 + c_2\theta^2 + \cdots + c_{m-2}\theta^{m-2}$, the value of $\mathrm{Tr}(x, y) = \mathrm{Tr}_{K/\mathbb{Q}}(xy)$ is determined by the values of $\mathrm{Tr}_{K/\mathbb{Q}}(\theta^i\theta^j)$, $i, j = 0, 2 \dots, m-2$. By a similar reasoning, writing x and y in terms of the basis 1, θ_2, θ_4, ..., θ_{m-2}, the values of $\mathrm{Tr}(x, y)$ on M_e is also determined by the values of $\mathrm{Tr}_{K/\mathbb{Q}}(\theta_i\theta_j)$, $i, j = 0, 2 \dots, m-2$. Lemmas 2 and 3 immediately give us the following result which connects our Hankel matrix to the trace form:

Corollary 4. *The matrix* $(\mathrm{Tr}_{K/\mathbb{Q}}(\theta^i\theta^j))$ *$(i, j = 0, 2 \dots, m-2)$ equals m times the Hankel matrix $H_{m/2}$ and* $(\mathrm{Tr}_{K/\mathbb{Q}}(\theta_i\theta_j)$ *$(i, j = 0, 2 \dots, m-2)$ equals m times the matrix $D_{m/2}$ defined in Equation (3).*

Similarly, by considering the values of $\mathrm{Tr}(x, y)$ on the $\mathbb{Z}$ module M_o in the two bases θ, θ^3, ..., θ^{m-1} and θ, θ_3, ..., θ_{m-1}, we have the following:

Corollary 5. *The matrix* $(\mathrm{Tr}_{K/\mathbb{Q}}(\theta^i\theta^j))$ *$(i, j = 1, 3 \dots, m-1)$ equals m times the Hankel matrix $H^{(1)}_{m/2}$ and* $(\mathrm{Tr}_{K/\mathbb{Q}}(\theta_i\theta_j)$ *$(i, j = 1, 3 \dots, m-1)$ equals $2m$ times the identity matrix.*

Now observe that the matrix $(\theta^i\theta^j)$ $(i, j = 0, 2 \dots, m-2)$ is just $V_e \cdot {V_e}^T$ (a product of $n \times 1$ and $1 \times n$ matrices), and that $(\theta_i\theta_j)$ $(i, j = 0, 2 \dots, m-2)$ equals $W_e \cdot {W_e}^T$. Similarly, $(\theta^i\theta^j)$ $(i, j = 1, 3 \dots, m-1)$ equals $V_o \cdot {V_o}^T$ and $(\theta_i\theta_j)$ $(i, j = 1, 3 \dots, m-1)$ equals $W_e \cdot {W_e}^T$.

Equations (9) and (10), the $\mathbb{Z}$ bilinearity of $\mathrm{Tr}_{K/\mathbb{Q}}$, and Corollaries 4 and 5 now give us the following (here, given a matrix M, $\mathrm{Tr}_{K/\mathbb{Q}}(M)$ stands for the matrix whose entries are the traces of the entries of M):

$$\begin{aligned}
H_{m/2} &= \frac{1}{m}(\mathrm{Tr}_{K/\mathbb{Q}}(\theta^i\theta^j))_{i,j=0,2\ldots,m-2} = \frac{1}{m}\mathrm{Tr}_{K/\mathbb{Q}}(V_e \cdot {V_e}^T) \\
&= \frac{1}{m}B_{m/2}\mathrm{Tr}_{K/\mathbb{Q}}(W_e \cdot {W_e}^T)B_{m/2}^T \\
&= \frac{1}{m}B_{m/2}(\mathrm{Tr}_{K/\mathbb{Q}}(\theta_i\theta_j)_{i,j=0,2\ldots,m-2}B_{m/2}^T \\
&= B_{m/2}D_{m/2}B_{m/2}^T. \\
H_{m/2}^{(1)} &= \frac{1}{m}(\mathrm{Tr}_{K/\mathbb{Q}}(\theta^i\theta^j))_{i,j=1,3\ldots,m-1} = \frac{1}{m}\mathrm{Tr}_{K/\mathbb{Q}}(V_o \cdot {V_o}^T) \\
&= \frac{1}{m}C_{m/2}\mathrm{Tr}_{K/\mathbb{Q}}(W_o \cdot {W_o}^T)C_{m/2}^T \\
&= \frac{1}{m}C_{m/2}(\mathrm{Tr}_{K/\mathbb{Q}}(\theta_i\theta_j)_{i,j=1,3\ldots,m-1}C_{m/2}^T \\
&= 2C_{m/2}C_{m/2}^T.
\end{aligned}$$

Let $G_n^{(k)}$ denote the Hankel matrices formed from the sequence of *odd* central binomial coefficients $\binom{2l+1}{l}$, $l = 0, 1, \ldots$. Note that since $\binom{2n}{n} = 2\binom{2n-1}{n-1}$, we have the relation $H_n^{(k+1)} = 2G_n^{(k)}$.

The discussions above now immediately yield the following theorem:

Theorem 6. *For any $n \geq 1$ and any $k \geq 1$, we have the factorizations:*

i. $H_n = B_nD_nB_n^T$, *and*

ii. $H_n^{(1)} = 2C_nC_n^T$.

iii. $G_n := G_n^{(0)} = C_nC_n^T$.

iv. $H_n^{(k)} = B_{n+k,n}D_nB_n^T$, *where $B_{n+k,n}$ denotes the lower left $n \times n$ block of B_{n+k}.*

v. $G_n^{(k)} = \frac{1}{2}B_{n+k+1,n}D_nB_n^T$,

vi. $H_n^{(k)} = 2C_{n+k-1,n}C_n^T$. *(Of course, when $k = 1$, this is the same as Part (ii) above.)*

vii. $G_n^{(k)} = C_{n+k,n}C_n^T$.

Proof. We pick an N such that $2^N \geq 8n$, and work in the maximal real subfield K of the 2^N-th cyclotomic extension of $\mathbb{Q}$. The equations preceding the statement of the theorem yield the factorizations $H_{m/2} = B_{m/2}D_{m/2}B_{m/2}^T$ and $H_{m/2}^{(1)} = 2C_{m/2}C_{m/2}^T$. By the choice of N, we have $n \leq m/2$. Note that H_n, $H_n^{(1)}$, B_n, C_n and D_n are all just the upper left $n \times n$ blocks of the corresponding matrices $H_{m/2}$, $H_{m/2}^{(1)}$, $B_{m/2}$, $C_{m/2}$ and $D_{m/2}$. Studying the upper left $n \times n$ blocks of the products $B_{m/2}D_{m/2}B_{m/2}^T$ and $C_{m/2}C_{m/2}^T$, and noting the lower triangular nature of $B_{m/2}$ and $C_{m/2}$ and the diagonal nature of $D_{m/2}$, we get the first two factorizations of the theorem.

We now substitute $n+k$ for n throughout in the first two factorizations, and observe that $H_n^{(k)}$ is the lower left $n \times n$ block of H_{n+k}, as also the lower left $n \times n$ block of $H_{n+k-1}^{(1)}$. Studying the products $B_{n+k} D_{n+k} B_{n+k}^T$ and $C_{n+k-1} C_{n+k-1}^T$ yields the two factorizations in (iv) and (vi) as well.

The factorizations in (iii), (v), and (vii) are a direct consequence of the relation $H_n^{(k+1)} = 2G_n^{(k)}$. □

Note that Factorization (iii) of G_n was described in [1, Prop. 6] by showing that odd binomial coefficients could be regarded as *Catalan-like numbers*. (In the notation of [1, Prop. 6], the odd binomial coefficients are $C_n^{(3,2)}$, the matrix $\tilde{A}_n$ is our G_{n+1} and the matrix A_n is our C_{n+1}.)

Taking the determinants on both sides of Parts (i) and (ii) of Theorem 6 above yeilds Corollary 1.

Let $c_n^{(k)}$ denote the determinant of $G_n^{(k)}$, and note that the relation $H_n^{(k+1)} = 2G_n^{(k)}$ shows that $d_n^{(k+1)} = 2^n c_n^{(k)}$. We therefore also have the following:

Corollary 7. *(See [13, Eq. 1.5], also [1, Prop. 6].)* $c_n^{(0)} = 1$.

Example 8. Parts (iv) or (vi) of Theorem 6 show that the computation of $d_n^{(k)}$ for $k \geq 2$ can be accomplished by computing the determinants of $B_{n+k,n}$ or $C_{n+k,n}$. Since the primary goal of this note is to establish the connection between the $H_n^{(k)}$ and number theory we will not do this here, but we note that these can be computed using, for instance, the very general techniques described in [6, Thm. 26] (as can the determinants of the original $H_n^{(k)}$ themselves!). Computing these determinants shows that for $k \geq 2$

$$d_n^{(k)} = 2^n \prod_{1 \leq i \leq j \leq k-1} \frac{i+j-1+2n}{i+j-1} \tag{13}$$

(recovering, for example, [13, Eq. 1.5]).

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