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Unimodal Rays in the Ordinary and Generalized Pascal Triangles

Hacène Belbachir[1]
USTHB, Faculty of Mathematics
P.O. Box 32
El Alia, Algiers
Algeria
hbelbachir@usthb.dz
hacenebelbachir@gmail.com

László Szalay[2]
Institute of Mathematics and Statistics
University of West Hungary
Erzsébet utca 9
H-9400 Sopron
Hungary
laszalay@ktk.nyme.hu

Abstract

The present paper provides the solution of two problems recently posed by Bencherif, Belbachir and Szalay. For example, they conjectured that any sequence of binomial coefficients lying along a ray in Pascal's triangle is unimodal.

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1 Introduction

Let ω denote a positive integer or infinity. A real sequence $\{a_k\}_{k=0}^{\omega}$ is *unimodal* if there exists a non-negative integer λ such that the subsequence $\{a_k\}_{k=0}^{\lambda}$ increases, while $\{a_k\}_{k=\lambda}^{\omega}$ decreases.

If $\lambda = 0$ then the sequence is monotone decreasing. Therefore it is also natural to consider a monotone increasing sequence as unimodal with $\lambda = \omega$, even if $\omega = \infty$.

If $a_0 \leq a_1 \leq \cdots \leq a_{m-1} < a_m = \cdots = a_M > a_{M+1} \geq a_{M+2} \geq \cdots$ then the integers $m, \ldots, M$ are called the *modes* of the sequence. In case of $m = M$, we talk about a *peak*; otherwise the set of modes is called a *plateau*.

A non-negative real sequence $\{a_k\}$ is *logarithmically concave* (*log-concave* or *LC* for short) if

$$a_k^2 \geq a_{k-1}a_{k+1}$$

for all $k \geq 1$.

Unimodal and log-concave sequences occur in several branches of mathematics (see, for example [5, 6]).

Our main interest is to examine combinatorial sequences connected to Pascal's triangle and its generalizations. The first result dealing with unimodality of the elements of the Pascal triangle is due to Tanny and Zuker [7], who showed that the sequence of the terms $\binom{n-k}{k}$ ($k = 0, \ldots, \lfloor \frac{n}{2} \rfloor$) is unimodal. In fact, the LC property of the sequence formed by the binomial coefficients $\binom{n-k}{k}$ was proved, and this implies unimodality by Theorem 1. They also investigated the unimodality of $\left\{\binom{n-\alpha k}{k}\right\}$ in [8, 9].

Benoumhani [4] justified the unimodality of the sequence $\left\{\frac{n}{n-k}\binom{n-k}{k}\right\}$ connected to Lucas numbers. Recently, Belbachir and Bencherif [1] proved that the elements $2^{n-2k}\binom{n-k}{k}$ and $2^{n-2k}\frac{n}{n-k}\binom{n-k}{k}$ linked to the Pell sequence and its companion sequence, respectively, provide unimodal sequences. In all the aforesaid cases the authors descibe the peaks and the plateaus with two elements. Incidentally, the paper [2] generalizes certain results on unimodality of sequences mentioned above.

One of the purposes of this work is to prove that any ray crossing Pascal's triangle hits elements of an unimodal sequence (Theorem 4 and Corollary 5). Further, we will show that unimodality appears when we consider the generalized Pascal triangle linked to the homogeneous linear recurrence $\{T_n\}$ of order s given by the recurrence relation

$$T_n = T_{n-1} + \cdots + T_{n-s}, \quad n \geq s \tag{1}$$

and by the initial values

$$T_0 = \cdots = T_{s-2} = 0, \; T_{s-1} = 1 \tag{2}$$

(Theorem 7 and Corollary 8). Questions of a similar kind were posed in the paper of Belbachir, Bencherif and Szalay [2].

For the proofs we will need the following two theorems.

Theorem 1. *A log-concave sequence $\{a_k\}$ with no internal zeros is also unimodal.*

See, for instance, [5].

Theorem 2. *The ordinary convolution of two log-concave sequences preserves the LC property. More precisely, if both $\{a_n\}$ and $\{b_n\}$ are log-concave, then so is $\{c_n\}$ with*

$$c_n = \sum_{i=0}^{n} a_i b_{n-i} \qquad (n = 0, 1, \dots).$$

See, for instance, [10].

2 Rays in Pascal's triangle

Let $u \geq 0$ and v denote integers, and as usual let

$$\binom{u}{v} = \begin{cases} \frac{u!}{v!(u-v)!}, & \text{if } 0 \leq v \leq u; \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

The main advantage of such an interpretation of binomial coefficients is, for example, that one can omit the use of exact limits in sums like $\sum_{i=0}^{u} \binom{u}{i}$ by simply writing $\sum_i \binom{u}{i}$ instead. In the sequel, for the sake of convenience, we exploit this kind of allowance. As a demonstration, we state Vandermonde's identity here, without proof.

Lemma 3. *For arbitrary integers $u \geq 0$, $k \geq 0$ and v we have*

$$\binom{u+k}{v} = \sum_i \binom{k}{i} \binom{u}{v-i}.$$

For the set of non-zero binomial coefficients we use the phrase *Pascal's triangle*. Since any row $R^{(u)}$ ($u \geq 0$) of the Pascal triangle is the convolution powers of the log-concave constant sequences $\{1\}$ and $\{1\}$, it follows that $R^{(u)}$ is a log-concave sequence by Theorem 2, even if we extend $R^{(u)}$ by (3). (The log-concavity of a row in the Pascal triangle also follows from the direct application of the definition of LC property.)

Now we are ready to prove

Theorem 4. *The sequence of binomial coefficients located along a ray is log-concave.*

By Theorem 1, we immediately obtain

Corollary 5. *The sequence of binomial coefficients located along a ray is unimodal.*

Proof of Theorem 4. Since the sequence of zeros is trivially log-concave, it is sufficient to prove the case when the ray passes through at least one binomial coefficient in the Pascal triangle. Let $\binom{u_0}{v_0}$ denote such a coefficient. Thus $0 \leq v_0 \leq u_0$, and with the integers α and β satisfying $\alpha^2 + \beta^2 > 0$ let us consider the sequence $\{x_k\}$ given by

$$x_k = \binom{u_0 + \alpha k}{v_0 + \beta k}, \tag{4}$$

where the integer k varies over all cases when $u_0 + \alpha k \geq v_0 + \beta k \geq 0$. Since the choice $k = 0$ is feasible, the sequence $\{x_k\}$ contains at least one positive element. Without loss of generality we may suppose that $\gcd(\alpha, \beta) = 1$. If $\alpha = 0$ then $\{x_k\} = \left\{\binom{u_0}{v_0 \pm k}\right\}$ is the log-concave row $R^{(u_0)}$ of binomial coefficients.

Similarly, if $\beta = 0$ then one can readily show that the terms $x_k = \binom{u_0 \pm k}{v_0}$ provide log-concave sequence. To deduce this statement, it suffices to show by recalling the definition of log-concavity, that

$$\binom{u_0+1}{v_0}\binom{u_0-1}{v_0} \leq \binom{u_0}{v_0}^2, \qquad \text{if } 0 \leq v_0 \leq u_0 - 1, \tag{5}$$

since the direction of a ray is reversible. By simply extracting the binomial coefficients, we can check that the above inequality is equivalent to $u_0 - v_0 \leq u_0$.

Moreover, there is no restriction in assuming that α and β are positive because of the symmetry $\binom{n}{k} = \binom{n}{n-k}$ of Pascal's triangle (or generally, by (3), the symmetry of binomial coefficients).

Suppose now, that $\binom{u}{v}$ (where $0 \leq v \leq u$) is the element of $\{x_k\}$ for which $0 \leq u = u_0 + \alpha k$ is minimum. Then the sequence $\{\bar{x}_k\} = \left\{\binom{u+\alpha k}{v+\beta k}\right\}_{k\geq 0}$ contains $\{x_k\}$ as a subsequence.

By Lemma 3, it is clear that

$$\bar{x}_k = \binom{u+\alpha k}{v+\beta k} = \sum_i \binom{k}{i}\binom{u+(\alpha-1)k}{v+\beta k - i}. \tag{6}$$

Since the terms $a_i = \binom{k}{i}$, and $b_i = \binom{u+(\alpha-1)k}{v+\beta k-i}$ are from the row $R^{(k)}$ and $R^{(u+(\alpha-1)k)}$ of binomial coefficients, respectively, they clearly form log-concave sequences. Thus $\{\bar{x}_k\}$ is also log-concave. Indeed, $k \geq 0$, and by $u \geq 0$ and $\alpha \geq 1$ we have $u + (\alpha - 1)k \geq 0$, so the two sequences framing the convolution really exist. Then the proof of Theorem 4 is complete. $\diamond$

3 Generalized Pascal triangle and sequences T_n

The elements $\binom{n}{k}_s$ of the generalized Pascal triangle have the following combinatorial interpretation. The term $\binom{n}{k}_s$ assigns the number of different ways of distributing k uniform objects among n boxes, where each box may contain at most s objects. Clearly, $0 \leq k \leq sn$. In other words, $\binom{n}{k}_s = \left|\left\{f : \{0,\ldots,n-1\} \to \{0,\ldots,s\} \mid \sum_{i=0}^{n-1} f(i) = k\right\}\right|$.

For instance, if $s = 2$ we obtain the triangle

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 1 & 1 & 1 & & \\
 & 1 & 2 & 3 & 2 & 1 & \\
1 & 3 & 6 & 7 & 6 & 3 & 1 \\
 & & & \vdots & & &
\end{array},$$

where $\binom{n}{k}_2 = \binom{n-1}{k-2}_2 + \binom{n-1}{k-1}_2 + \binom{n-1}{k}_2$, supposing non-negative u in $\binom{u}{v}_2$ having the value zero if $v < 0$ or $2u < v$. Note that we omit the subscript 1 and write only $\binom{n}{k}$ for the usual binomial coefficients if $s = 1$. Now we formulate a lemma on generalized binomial coefficients, which will be useful in the proof of Theorem 7.

Lemma 6. *If $s \geq 2$ then we have*

$$\binom{n}{k}_s = \sum_{k_1=\lceil \frac{k}{s} \rceil}^{\min\{k,n\}} \binom{n}{k_1} \binom{k_1}{k-k_1}_{s-1}. \tag{7}$$

Proof of Lemma 6. Clearly, if we want to distribute k elements, first we choose k_1 boxes by putting one object into each of them, and then we distribute the remaining $k-k_1$ elements among the k_1 boxes that have been chosen, with at most $s-1$ elements per box. $\diamond$

Notice that we can ignore the indication of limits in the sum (7) by recalling that for non-negative u the coefficent $\binom{u}{v}_s = 0$ if the integer v is out of the range $0, \ldots, su$.

The generalized Pascal triangle $\binom{n}{k}_s$, $n \in \mathbb{N}$; $0 \leq k \leq sn$ is linked to the linear recurrence $\{T_n\}$ given by (1) and (2) via the diagonal sum

$$\sum_{k=0}^{\lfloor \frac{sn}{s+1} \rfloor} \binom{n-k}{k}_s = T_{n+s}. \tag{8}$$

(For reference, see, for instance [3].) The case $s = 1$ returns the nice identity

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} = F_{n+1}$$

for Fibonacci numbers A000045, while $s = 2$ is connected to the Tribonacci numbers A000073. In the sequel, to extend Theorem 4, we investigate not only the diagonal elements $\binom{n-k}{k}_s$, but any sequence of elements locating along a ray. Hence, in the generalized Pascal triangle, we consider the log-concavity of a more general sequence $\{w_k\}$ given by $w_k(s; u, v; \alpha, \beta) = \binom{u+\alpha k}{v+\beta k}_s$. We will prove that the sequence $\{w_k\}$ is log-concave, and consequently unimodal. We have

Theorem 7. *The sequence containing the terms $w_k = \binom{u+\alpha k}{v+\beta k}_s$ is log-concave if $s \geq 2$.*

Corollary 8. *The sequence $\{w_k\} = \left\{\binom{u+\alpha k}{v+\beta k}_s\right\}$ is unimodal for any positive integer s.*

Proof of Corollary 8. It comes immediately from Corollary 5 and Theorem 7. $\diamond$

For the proof of Theorem 7 we need the following

Lemma 9. *Given positive integers n and s, the sequence $\{y_k\} = \left\{\binom{k}{n-k}_s\right\}$ is log-concave.*

Proof of Lemma 9. In case of $s = 1$ the possible values for k are

$$\left\lceil \frac{n}{2} \right\rceil, \left\lceil \frac{n}{2} \right\rceil + 1, \ldots, n. \tag{9}$$

Put $t = n - k$, which goes through on the range $\left\lfloor \frac{n}{2} \right\rfloor, \left\lfloor \frac{n}{2} \right\rfloor - 1, \ldots, 0$ when k takes its values from the range (9). From the reversibility of the order, the sequence of the terms $\binom{k}{n-k} = \binom{n-t}{t}$ is log-concave by [7] (or see Introduction of the present paper).

Assume now that Lemma 9 holds for $s-1 \geq 1$ with all non-negative integer n. We have

$$y_k = \binom{k}{n-k}_s = \sum_m \binom{k}{m} \binom{m}{n-k-m}_{s-1}. \tag{10}$$

As remarked above, the Pascal triangle row $R^{(k)} = \left\{ \binom{k}{m} \right\}_m$ is log-concave. By the assertion of the induction, $\left\{ \binom{m}{n-k-m}_{s-1} \right\}_m$ is also log-concave. Applying Theorem 2, we conclude that the terms $y_k = \binom{k}{n-k}_s$ form a log-concave sequence. ⋄

Now we turn our attention to the

Proof of Theorem 7.

If $s \geq 2$, by Lemma 6 we have

$$w_k = \binom{u+\alpha k}{v+\beta k}_s = \sum_m \binom{u+\alpha k}{m} \binom{m}{v+\beta k - m}_{s-1}. \tag{11}$$

As we have already noted, the row $R^{(u+\alpha k)} = \left\{ \binom{u+\alpha k}{m} \right\}_m$ of the Pascal triangle is log-concave. On the other hand, the log-concavity of the sequence $\left\{ \binom{m}{v+\beta k-m}_{s-1} \right\}_m$ is provided by Lemma 9. Now it follows from Theorem 2 that the ordinary convolution (11) is also log-concave. The proof of Theorem 7 is complete. ⋄

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