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On the Gcd-Sum Function

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Abstract

In this paper we give a unified asymptotic formula for the partial gcd-sum function. We also study the mean-square of the error in the asymptotic formula.

1 Introduction

Pillai [6] first defined the gcd-sum (Pillai's function) by the relation

$$g(n) := \sum_{j=1}^{n} \gcd(j, n), \tag{1}$$

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where $\gcd(a, b)$ denotes the greatest common divisor of a and b. This is Sequence A018804 in Sloane's *Online Encyclopedia of Integer Sequences*. Pillai [6] proved that

$$g(n) = n \sum_{d|n} \frac{\varphi(n)}{d},$$

where $\varphi(n)$ is Euler's function. This fact was proved again by Broughan [2]. He also obtained the asymptotic formula of the partial sum function

$$G_\alpha(x) := \sum_{n \le x} g(n) n^{-\alpha}$$

for any $\alpha \in \mathbb{R}$, which was further improved in Bordellès [1] (the case $\alpha = 0$) and Broughan [3] (the general case) respectively.

The estimate of $G_\alpha(x)$ is closely related to the well-known Dirichlet divisor problem. For any $x > 0$, define

$$\Delta(x) := \sum_{n \le x} d(n) - x(\log x + 2\gamma - 1),$$

where $d(n)$ denotes the Dirichlet divisor function and γ is Euler's constant. Dirichlet first proved that $\Delta(x) = O(x^{1/2})$. The exponent $1/2$ was improved by many authors. The latest result reads

$$\Delta(x) \ll x^{131/416} (\log x)^{26947/8320}, \tag{2}$$

due to Huxley [4]. It is conjectured that

$$\Delta(x) = O(x^{1/4+\varepsilon}), \tag{3}$$

which is supported by the classical mean square result

$$\int_1^T \Delta^2(x) dx = \frac{\zeta^4(3/2)}{6\pi^2 \zeta(3)} T^{3/2} + O(T \log^5 T) \tag{4}$$

proved by Tong [7].

In the sequel of this paper, θ denotes the number defined by

$$\theta := \inf\{a \,|\, \Delta(x) \ll x^a\}. \tag{5}$$

Broughan [3] proved that

(1) If $\alpha \le 1 + \theta$, then

$$\begin{aligned} G_\alpha(x) =& \frac{x^{2-\alpha} \log x}{(2-\alpha)\zeta(2)} + \frac{x^{2-\alpha}}{(2-\alpha)\zeta(2)} \left(2\gamma - \frac{1}{2-\alpha} - \frac{\zeta'(2)}{\zeta(2)}\right) \\ &+ O(x^{\theta+1-\alpha+\varepsilon}); \end{aligned} \tag{6}$$

(2) If $1 + \theta < \alpha < 2$, then

$$\begin{aligned} G_\alpha(x) =& \frac{x^{2-\alpha} \log x}{(2-\alpha)\zeta(2)} + \frac{x^{2-\alpha}}{(2-\alpha)\zeta(2)} \left(2\gamma - \frac{1}{2-\alpha} - \frac{\zeta'(2)}{\zeta(2)}\right) \\ &+ O(1); \end{aligned} \tag{7}$$

(3) If $\alpha = 2$, then

$$G_\alpha(x) = \frac{\log^2 x}{2\zeta(2)} + \frac{\log x}{\zeta(2)}\left(2\gamma - \frac{\zeta'(2)}{\zeta(2)}\right) + O(1); \tag{8}$$

(4) If $\alpha > 2$, then

$$\begin{aligned} G_\alpha(x) =& \frac{x^{2-\alpha}\log x}{(2-\alpha)\zeta(2)} + \frac{x^{2-\alpha}}{(2-\alpha)\zeta(2)}\left(2\gamma - \frac{1}{2-\alpha} - \frac{\zeta'(2)}{\zeta(2)}\right) \\ &+ \frac{\zeta^2(\alpha-1)}{\zeta(\alpha)} + O(x^{\theta+1-\alpha+\varepsilon}). \end{aligned} \tag{9}$$

In this paper we first give a unified asymptotic formula of $G_\alpha(x)$. Before stating our result, we introduce the following definitions:

$$M_\alpha(x) := \begin{cases} \dfrac{x^{2-\alpha}\log x}{(2-\alpha)\zeta(2)} + \dfrac{x^{2-\alpha}}{(2-\alpha)\zeta(2)}\left(2\gamma - \dfrac{1}{2-\alpha} - \dfrac{\zeta'(2)}{\zeta(2)}\right), & \text{if } \alpha \neq 2; \\ \dfrac{\log^2 x}{2\zeta(2)} - \left(\dfrac{\zeta'(2)}{\zeta^2(2)} + \dfrac{2\gamma}{\zeta(2)}\right)\log x, & \text{if } \alpha = 2, \end{cases} \tag{10}$$

and

$$c(\beta) := \begin{cases} 0, & \text{if } \beta \le 0; \\ \beta\displaystyle\int_0^\infty \Delta(x)x^{-\beta-1}dx, & \text{if } 0 < \beta < 1; \\ 2\gamma - 1 + \displaystyle\int_1^\infty \Delta(x)x^{-2}dx, & \text{if } \beta = 1; \\ \zeta^2(\beta), & \text{if } \beta > 1. \end{cases} \tag{11}$$

We then have

Theorem 1. *Suppose $\alpha \in \mathbb{R}$ is fixed. Then*

$$G_\alpha(x) = M_\alpha(x) + C(\alpha) + O(x^{\theta+1-\alpha+\varepsilon}), \tag{12}$$

where

$$C(\alpha) := \begin{cases} 0, & \textit{if } \alpha \le 1; \\ c(\alpha-1)/\zeta(\alpha), & \textit{if } 1 < \alpha < 2; \\ \dfrac{c(1)}{\zeta(2)} + \dfrac{2\zeta'^2(2) - \zeta(2)\zeta''(2)}{2\zeta^3(2)} - \dfrac{2\gamma\zeta'(2)}{\zeta^2(2)}, & \textit{if } \alpha = 2; \\ \dfrac{\zeta^2(\alpha-1)}{\zeta(\alpha)}, & \textit{if } \alpha > 2. \end{cases}$$

Remark 1. Theorem 1 slightly improves Broughan's result in the case $1 + \theta < \alpha \le 2$.

Define the error term $E_\alpha(x)$ by

$$E_\alpha(x) := G_\alpha(x) - M_\alpha(x) - C(\alpha).$$

For this error term, we have the following mean square results.

Theorem 2. *For any fixed $\alpha \in \mathbb{R}$, we have the asymptotic formula*

$$\int_1^T x^{2\alpha-2} E_\alpha^2(x)dx = C_2 T^{3/2} + O(T^{5/4+\varepsilon}), \tag{13}$$

where

$$C_2 = \frac{1}{6\pi^2}\sum_{n=1}^{\infty} h_0^2(n) n^{-3/2}, \quad h_0(n) = \sum_{n=ml} \mu(m)d(l)m^{-1/2}.$$

Corollary 1. *If $\alpha < 7/4$, then*

$$\int_1^T E_\alpha^2(x)dx = \frac{3C_2}{7-4\alpha} T^{7/2-2\alpha} + O(T^{13/4-2\alpha+\varepsilon}). \tag{14}$$

For the upper bound of $E_\alpha(x)$, we propose the following conjecture.
Conjecture. The estimate

$$E_\alpha(x) \ll x^{5/4-\alpha+\varepsilon}$$

holds for any $\alpha \in \mathbb{R}$.

Throughout this paper, ε denotes an arbitrary small positive number which does not need to be the same at each occurrence. When the summation conditions of a sum are complicated, we write the conditions separately like $SC(\Sigma)$.

2 Proof of Theorem 1

In this section we prove Theorem 1. First we prove the following

Lemma 2.1. *Suppose $\beta \in \mathbb{R} \setminus \{0\}$ and define $D_\beta(x) := \sum_{n \le x} d(n) n^{-\beta}$. Then we have*

$$\begin{aligned} D_\beta(x) = {} & \frac{x^{1-\beta}\log x}{1-\beta} + \frac{x^{1-\beta}}{1-\beta}\left(2\gamma - \frac{1}{1-\beta}\right) \\ & + c(\beta) + x^{-\beta}\Delta(x) + O(x^{-\beta}) \qquad \textit{for } \beta \neq 1 \end{aligned} \tag{15}$$

and

$$D_1(x) = \frac{\log^2 x}{2} + 2\gamma \log x + c(1) + \Delta(x)x^{-1} + O(x^{-1}), \tag{16}$$

where $c(\beta)$ is defined in (1.10).

Proof. First consider the case $\beta < 1$. By integration by parts we have

$$\begin{aligned}
D_\beta(x) &= \sum_{0<n\le x} d(n)n^{-\beta} = \int_0^x t^{-\beta} dD(t) \\
&= \int_0^x t^{-\beta} dH(t) + \int_0^x t^{-\beta} d\Delta(t) \\
&= \int_0^x t^{-\beta}(\log t + 2\gamma)dt + t^{-\beta}\Delta(t)\Big|_0^x + \beta \int_0^x \Delta(t)t^{-\beta-1}dt \\
&= \frac{x^{1-\beta}\log x}{1-\beta} + \frac{x^{1-\beta}}{1-\beta}\left(2\gamma - \frac{1}{1-\beta}\right) + x^{-\beta}\Delta(x) \\
&\quad + \beta \int_0^x \Delta(t)t^{-\beta-1}dt,
\end{aligned} \tag{17}$$

where

$$D(t) = \sum_{n\le t} d(n)$$

and

$$H(t) = t\log t + (2\gamma - 1)t.$$

Note that, from the definition of $\Delta(x)$,

$$\Delta(x) = -x\log x - (2\gamma - 1)x$$

for $0 < x < 1$, which implies that the integral $\int_0^1 \Delta(t)t^{-\beta-1}dt$ is convergent. To treat the last integral in (2.3) we recall the well-known formula (see Voronoï [8])

$$\int_0^T \Delta(x)dx = T/4 + O(T^{3/4}), \tag{18}$$

which combined with integration by parts gives

$$\beta \int_0^x \Delta(t)t^{-\beta-1}dt \ll x^{-\beta} \qquad (\beta < 0) \tag{19}$$

and

$$\beta \int_x^\infty \Delta(t)t^{-\beta-1}dt \ll x^{-\beta} \qquad (\beta > 0). \tag{20}$$

Especially, (2.6) shows that the infinite integral $\int_0^\infty \Delta(t)t^{-\beta-1}dt$ converges in the case $\beta > 0$. The assertion of Lemma 2.1 for the case $\beta < 1$ follows from (2.3), (2.5) and (2.6).

Next consider the case $\beta > 1$. Since the infinite series $\sum_{n=1}^\infty d(n)n^{-\beta}$ converges to $\zeta^2(\beta)$, we may write

$$D_\beta(x) = \zeta^2(\beta) - \sum_{n>x} d(n)n^{-\beta}. \tag{21}$$

By integration by parts and (2.6) we get

$$\begin{aligned}\sum_{n>x} d(n)n^{-\beta} &= \int_x^\infty t^{-\beta} dH(t) + \int_x^\infty t^{-\beta} d\Delta(t) \\ &= \int_x^\infty t^{-\beta}(\log t + 2\gamma)dt + t^{-\beta}\Delta(t)|_x^\infty + \beta \int_x^\infty \Delta(t)t^{-\beta-1}dt \\ &= -\frac{x^{1-\beta}\log x}{1-\beta} - \frac{x^{1-\beta}}{1-\beta}\left(2\gamma - \frac{1}{1-\beta}\right) - x^{-\beta}\Delta(x) + O(x^{-\beta}). \qquad (22)\end{aligned}$$

The assertion of Lemma 2.1 for the case $\beta > 1$ follows from (2.7) and (2.8).

Finally consider the case $\beta = 1$. We have

$$\begin{aligned}D_1(x) &= 1 + \sum_{1<n\le x} d(n)n^{-1} \\ &= 1 + \int_1^x t^{-1}dH(t) + \int_1^x t^{-1}d\Delta(t) \\ &= 1 + \int_1^x t^{-1}(\log t + 2\gamma)dt + \Delta(t)t^{-1}\Big|_1^x + \int_1^x \Delta(t)t^{-2}dt \\ &= \frac{\log^2 x}{2} + 2\gamma\log x + 1 + \Delta(x)x^{-1} - \Delta(1) + \int_1^\infty \Delta(t)t^{-2}dt + O(x^{-1}) \\ &= \frac{\log^2 x}{2} + 2\gamma\log x + 2\gamma - 1 + \int_1^\infty \Delta(t)t^{-2}dt + \Delta(x)x^{-1} + O(x^{-1}), \qquad (23)\end{aligned}$$

where we used (2.6) and the fact $\Delta(1) = 2 - 2\gamma$ which follows from the definition of $\Delta(x)$.

This completes the proof of Lemma 2.1. □

Proof of Theorem 1. Broughan [3] proved that

$$G_\alpha(x) = \sum_{m\le x} \mu(m)m^{-\alpha} \sum_{n\le x/m} d(n)n^{1-\alpha}. \qquad (24)$$

From (2.10), Lemma 2.1 and some easy calculations, we get

$$G_\alpha(x) = M_\alpha(x) + C(\alpha) + E_\alpha(x), \qquad (25)$$

where

$$E_\alpha(x) = x^{1-\alpha}\sum_{m\le x}\frac{\mu(m)}{m}\Delta(\frac{x}{m}) + O(x^{1-\alpha}\log x), \qquad (26)$$

which is $O(x^{\theta+1-\alpha+\varepsilon})$, completing the proof of Theorem 1.

Remark 2. Voronoï [8] actually proved

$$\int_0^T \Delta(x)dx = \frac{T}{4} + \frac{T^{3/4}}{2\sqrt{2}\pi^2}\sum_{n=1}^\infty \frac{d(n)}{n^{5/4}}\sin(4\pi\sqrt{nT} - \pi/4) + O(T^{1/4}). \qquad (27)$$

Replacing the formula (2.4) in the proof of Lemma 2.1 it is easy to check that $\log x$ in the error term of (2.12) can be removed.

3 Proof of Theorem 2

In order to prove Theorem 2 we need the following well-known Voronoï formula (see, e.g., Ivić [5]).

Lemma 3.1. *Suppose $A > 0$ is any fixed constant. If $1 \ll N \ll x^A$, then*

$$\Delta(x) = \frac{x^{1/4}}{\pi\sqrt{2}} \sum_{n\le N} \frac{d(n)}{n^{3/4}} \cos\left(4\pi\sqrt{nx} - \frac{\pi}{4}\right) + O(x^{\varepsilon} + x^{1/2+\varepsilon}N^{-1/2}).$$

It suffices to evaluate the integral $\int_T^{2T} x^{2\alpha-2}E_\alpha^2(x)dx$. Let

$$y = T^{1-\varepsilon}.$$

By (2.12) we have

$$\begin{aligned} x^{\alpha-1}E_\alpha(x) &= \sum_{m\le y} \frac{\mu(m)}{m}\Delta\left(\frac{x}{m}\right) + \sum_{y\le m\le x} \frac{\mu(m)}{m}\Delta\left(\frac{x}{m}\right) + O(\log x) \\ &=: F_1(x) + F_2(x) + O(\log x), \end{aligned}$$

say. For $F_2(x)$, we have $\Delta(x/m) \ll (x/y)^{1/3} \ll T^{\varepsilon}$, therefore

$$F_2(x) \ll \sum_{y<m\le x} \frac{1}{m}\left|\Delta\left(\frac{x}{m}\right)\right| \ll T^{\varepsilon}.$$

For $F_1(x)$, we can take $N = y$ in Lemma 3.1 with a suitable A (e.g., $A = (1+\varepsilon)/\varepsilon$), hence we get

$$F_1(x) = E^*(x) + O(T^{\varepsilon}),$$

where

$$E^*(x) = \frac{x^{1/4}}{\sqrt{2}\pi} \sum_{m\le y} \frac{\mu(m)}{m^{5/4}} \sum_{n\le y} \frac{d(n)}{n^{3/4}} \cos\left(4\pi\sqrt{\frac{nx}{m}} - \frac{\pi}{4}\right).$$

As a result we get an expression of $x^{\alpha-1}E_\alpha(x)$:

$$x^{\alpha-1}E_\alpha(x) = E^*(x) + O(T^{\varepsilon}). \tag{28}$$

Now we consider the mean square of $E^*(x)$. By the elementary formula

$$\cos u \cos v = \frac{1}{2}(\cos(u-v) + \cos(u+v))$$

we may write

$$\begin{aligned} |E^*(x)|^2 = \frac{x^{1/2}}{2\pi^2} &\sum_{m_1,m_2\le y} \frac{\mu(m_1)\mu(m_2)}{(m_1m_2)^{5/4}} \sum_{n_1,n_2\le y} \frac{d(n_1)d(n_2)}{(n_1n_2)^{3/4}} \\ &\times \cos\left(4\pi\sqrt{\frac{n_1x}{m_1}} - \frac{\pi}{4}\right)\cos\left(4\pi\sqrt{\frac{n_2x}{m_2}} - \frac{\pi}{4}\right) \\ &= S_1(x) + S_2(x) + S_3(x), \end{aligned} \tag{29}$$

where

$$\begin{aligned}
S_1(x) = &\ \frac{x^{1/2}}{4\pi^2}\sum\nolimits_1 \frac{\mu(m_1)\mu(m_2)}{(m_1m_2)^{5/4}}\frac{d(n_1)d(n_2)}{(n_1n_2)^{3/4}},\\
S_2(x) = &\ \frac{x^{1/2}}{4\pi^2}\sum\nolimits_2 \frac{\mu(m_1)\mu(m_2)}{(m_1m_2)^{5/4}}\frac{d(n_1)d(n_2)}{(n_1n_2)^{3/4}}\cos\left(4\pi\sqrt{x}\left(\sqrt{\frac{n_1}{m_1}}-\sqrt{\frac{n_2}{m_2}}\right)\right),\\
S_3(x) = &\ \frac{x^{1/2}}{4\pi^2}\sum\nolimits_3 \frac{\mu(m_1)\mu(m_2)}{(m_1m_2)^{5/4}}\frac{d(n_1)d(n_2)}{(n_1n_2)^{3/4}}\sin\left(4\pi\sqrt{x}\left(\sqrt{\frac{n_1}{m_1}}+\sqrt{\frac{n_2}{m_2}}\right)\right),
\end{aligned}$$

with summation conditions

$$\begin{aligned}
&SC(\Sigma_1): m_1, m_2, n_1, n_2 \le y, \quad n_1m_2 = n_2m_1,\\
&SC(\Sigma_2): m_1, m_2, n_1, n_2 \le y, \quad n_1m_2 \ne n_2m_1,\\
&SC(\Sigma_3): m_1, m_2, n_1, n_2 \le y,
\end{aligned}$$

respectively.

We have

$$\int_T^{2T} S_1(x)dx = \frac{B(T)}{4\pi^2}\int_T^{2T} x^{1/2}dx, \tag{30}$$

where

$$B(T) = \sum\nolimits_1 \frac{\mu(m_1)\mu(m_2)}{(m_1m_2)^{5/4}}\frac{d(n_1)d(n_2)}{(n_1n_2)^{3/4}}.$$

We evaluate $B(T)$. It is written as

$$\begin{aligned}
B(T) &= \sum\nolimits_1 \frac{\mu(m_1)\mu(m_2)d(n_1)d(n_2)(m_1m_2)^{-1/2}}{(n_1m_2n_2m_1)^{3/4}}\\
&= \sum_{n\le y^2} h^2(n;y)n^{-3/2},
\end{aligned}$$

where

$$h(n;y) = \sum_{\substack{n=ml\\ m,l\le y}} \mu(m)d(l)m^{-1/2}.$$

Let

$$h_0(n) = \sum_{n=ml} \mu(m)d(l)m^{-1/2},\ \ h_1(n) = \sum_{n=ml} d(l)m^{-1/2}.$$

Obviously,

$$\begin{aligned}
&h(n;y) = h_0(n),\ \ n \le y,\\
&|h(n;y)| \le h_1(n), \quad |h_0(n)| \le h_1(n), n \ge 1.
\end{aligned}$$

Since $h_1(n)$ is a multiplicative function, by using Euler's product, it is easy to show that

$$\sum_{n=1}^{\infty} h_1^2(n)n^{-s} = \zeta^4(s)M(s), \quad \Re s > 1,$$

where $M(s)$ is regular for $\Re s > 1/2$. Thus

$$\sum_{n\le x} h_1^2(n) \ll x\log^3 x.$$

From the above estimates we get

$$\begin{aligned}
B(T) &= \sum_{n\le y} h_0^2(n)n^{-3/2} + O\left(\sum_{y<n\le y^2} h_1^2(n)n^{-3/2}\right)\\
&= \sum_{n=1}^{\infty} h_0^2(n)n^{-3/2} + O\left(\sum_{n>y} h_1^2(n)n^{-3/2}\right)\\
&= \sum_{n=1}^{\infty} h_0^2(n)n^{-3/2} + O(y^{-1/2}\log^3 y). \qquad (31)
\end{aligned}$$

Next we consider the integral of $S_2(x)$. By the first derivative test we get

$$\begin{aligned}
\int_T^{2T} S_2(x)dx &\ll T\sum\nolimits_2 \frac{d(n_1)d(n_2)}{(m_1m_2)^{5/4}(n_1n_2)^{3/4}}\frac{1}{\left|\sqrt{\frac{n_1}{m_1}}-\sqrt{\frac{n_2}{m_2}}\right|}\\
&\ll T\sum\nolimits_2 \frac{d(n_1)d(n_2)}{(n_2m_1n_1m_2)^{3/4}}\frac{1}{|\sqrt{n_1m_2}-\sqrt{n_2m_1}|}\\
&\ll T\sum_{\substack{n,m\le y^2\\ n\ne m}} \frac{d_3(m)d_3(n)}{n^{3/4}m^{3/4}}\frac{1}{|\sqrt{n}-\sqrt{m}|}\\
&\ll T\sum_{\substack{n,m\le y^2\\ |\sqrt{n}-\sqrt{m}|\ge\frac12(mn)^{1/4}}} \frac{d_3(m)d_3(n)}{n^{3/4}m^{3/4}}\frac{1}{|\sqrt{n}-\sqrt{m}|}\\
&\quad + T\sum_{\substack{n,m\le y^2\\ 0<|\sqrt{n}-\sqrt{m}|<\frac12(mn)^{1/4}}} \frac{d_3(m)d_3(n)}{n^{3/4}m^{3/4}}\frac{1}{|\sqrt{n}-\sqrt{m}|}\\
&\ll T\left(\sum_{n\le y^2} d_3(n)n^{-1}\right)^2 + T\sum_{\substack{n,m\le y^2\\ n\ne m}} \frac{d_3(m)d_3(n)}{n^{1/2}m^{1/2}}\frac{1}{|n-m|}\\
&\ll T\log^9 T + T\sum_{n\le y^2} d_3^2(n)n^{-1} \ll T\log^9 T, \qquad (32)
\end{aligned}$$

where we used the well-known Hilbert's inequality and the estimates

$$\sum_{n\le x} d_3(n) \ll x\log^2 x, \quad \sum_{n\le x} d_3^2(n) \ll x\log^8 x.$$

For the integral of $S_3(x)$, by the first derivative test again, we get

$$\begin{aligned}\int_T^{2T} S_3(x)dx &\ll T\sum\nolimits_3 \frac{d(n_1)d(n_2)}{(m_1m_2)^{5/4}(n_1n_2)^{3/4}}\frac{1}{\left|\sqrt{\frac{n_1}{m_1}}+\sqrt{\frac{n_2}{m_2}}\right|}\\
&\ll T\sum\nolimits_3 \frac{d(n_1)d(n_2)}{(m_1m_2)^{5/4}(n_1n_2)^{3/4}}\frac{1}{(\frac{n_1}{m_1})^{1/4}(\frac{n_2}{m_2})^{1/4}}\\
&\ll T\sum\nolimits_3 \frac{d(n_1)d(n_2)}{m_1m_2n_1n_2} \ll T\log^6 T. \qquad (33)\end{aligned}$$

From (3.2)-(3.6) we get

$$\begin{aligned}\int_T^{2T} |E^*(x)|^2dx &= \frac{1}{4\pi^2}\sum_{n=1}^{\infty} h_0^2(n)n^{-3/2}\int_T^{2T} x^{1/2}dx\\
&\quad + O(T^{3/2}y^{-1/2}\log^3 T + T\log^9 T)\\
&= \frac{1}{4\pi^2}\sum_{n=1}^{\infty} h_0^2(n)n^{-3/2}\int_T^{2T} x^{1/2}dx + O(T^{1+\varepsilon}). \qquad (34)\end{aligned}$$

From (3.1), (3.7) and Cauchy's inequality we get

$$\int_T^{2T} x^{2\alpha-2}|E_\alpha(x)|^2dx = \frac{1}{4\pi^2}\sum_{n=1}^{\infty} h_0^2(n)n^{-3/2}\int_T^{2T} x^{1/2}dx + O(T^{5/4+\varepsilon}), \qquad (35)$$

which implies Theorem 2 by a splitting argument.

Remark 3. The referee kindly indicated that the average order of $h_1^2(n)$ is also derived by Shiu's theorem. Indeed, since $h_1(n) \ll d_3(n) \ll n^\varepsilon$ we have

$$\begin{aligned}\sum_{n\le x} h_1^2(n) &\ll \frac{x}{\log x}\exp\left(\sum_{p\le x}\frac{h_1^2(p)}{p}\right)\\
&\ll \frac{x}{\log x}\exp\left(\sum_{p\le x}\left(\frac{4}{p}+\frac{4}{p^{3/2}}+\frac{1}{p^2}\right)\right)\\
&\ll x(\log x)^3.\end{aligned}$$

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