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A Combinatorial Interpretation for Certain Relatives of the Conolly Sequence

B. Balamohan, Zhiqiang Li, and Stephen Tanny[1]
Department of Mathematics
University of Toronto
Toronto, Ontario M5S 2E4
Canada
bbalamoh@math.utoronto.ca
zhiqiang.li@utoronto.ca
tanny@math.utoronto.ca

Abstract

For any integer $s \geq 0$, we derive a combinatorial interpretation for the family of sequences generated by the recursion (parameterized by s) $h_s(n) = h_s(n - s - h_s(n - 1)) + h_s(n - 2 - s - h_s(n - 3)), n > s + 3$, with the initial conditions $h_s(1) = h_s(2) = \cdots = h_s(s+2) = 1$ and $h_s(s+3) = 2$. We show how these sequences count the number of leaves of a certain infinite tree structure. Using this interpretation we prove that h_s sequences are "slowly growing", that is, h_s sequences are monotone nondecreasing, with successive terms increasing by 0 or 1, so each sequence hits every positive integer. Further, for fixed s the sequence $h_s(n)$ hits every positive integer twice except for powers of 2, all of which are hit $s+2$ times. Our combinatorial interpretation provides a simple approach for deriving the ordinary generating functions for these sequences.

1 Introduction

Conolly [2] introduced the sequence defined by the meta-Fibonacci (self-referencing) recursion

$$F(n) = F(n - F(n - 1)) + F(n - 1 - F(n - 2)), \qquad n > 2 \tag{1.1}$$

with initial conditions $(F(1) = 0, F(2) = 1)$ or $(F(1) = 1, F(2) = 1)$. He notes that for $n > 2$ the recursion yields the same sequence whether $F(1) = 0$ or $F(1) = 1$. The resulting

[1] Corresponding author.

sequence behaves in a very simple fashion: it is monotonic non-decreasing, with successive terms differing by either 0 or 1, so the sequence hits every positive integer. Following Ruskey [8] we term such a sequence "slowly growing".

Variants and generalizations of the recursion (1.1) and its initial conditions have been studied (see, for example, [1, 4, 6, 10]). The most comprehensive of these is the following k-term generalization, where s and $k \geq 2$ are nonnegative integers:

$$T_{s,k}(n) = \sum_{i=0}^{k-1} T_{s,k}(n-i-s-T_{s,k}(n-i-1)), \qquad n > s+k. \tag{1.2}$$

The special case of (1.2) for $s = 1$ and initial conditions $T_{1,k}(1) = T_{1,k}(2) = 1$, and $T_{1,k}(j) = j-1$ for $3 \leq j \leq k+1$ is first discussed by Higham and Tanny [6]. It is shown there that the resulting sequence is slowly growing.

Recently Jackson and Ruskey [7] (and see [3] for a generalization) derived a beautiful combinatorial interpretation for the sequences generated by (1.2) with initial conditions $T_{s,k}(j) = 1$ for $1 \leq j \leq s+1$, and $T_{s,k}(j) = j-s$ for $s+2 \leq j \leq s+k$. These initial conditions are natural extensions for arbitrary s of those appearing in [6]. Using their interpretation they show that all of these sequences are slowly growing.

Even further, Jackson and Ruskey's interpretation provides a highly intuitive understanding for the similarity of these sequences for fixed k and varying values of the "shift" parameter s, a fact that had been noted by others but never proved. For fixed k the only differences between the sequences $T_{s,k}(n)$ and $T_{0,k}(n)$ occur in the frequencies with which each sequence hits the powers of k. More precisely, for fixed k and any positive integer r, the value k^r occurs s more times in the sequence $T_{s,k}(n)$ than in the sequence $T_{0,k}(n)$.

The present work is inspired by their approach. We begin with a meta-Fibonacci recursion closely related to (1.2), namely,

$$h_{s,k}(n) = \sum_{j=0}^{k-1} h_{s,k}(n-2j-s-h_{s,k}(n-(2j+1))), \qquad n > s+2k+1. \tag{1.3}$$

The right hand side of (1.3) includes precisely those terms in the sum on the right hand side of (1.2) that derive from the even values of the index. For $k > 2$ there does not appear to be any set of initial conditions for which (1.3) generates an interesting infinite sequence. For $k = 2$, however, (1.3) reduces to the two term recurrence

$$h_s(n) = h_s(n-s-h_s(n-1)) + h_s(n-2-s-h_s(n-3)), \qquad n > s+3. \tag{1.4}$$

For any integer $s \geq 0$ we derive a combinatorial interpretation for the sequence generated by the recursion (1.4) with the initial values $h_s(1) = h_s(2) = \cdots = h_s(s+2) = 1$ and $h_s(s+3) = 2$. These initial conditions are the natural analogue of the ones described above by Deugau and Ruskey [3]. It turns out that this sequence is slowly growing. Further, the role played by the parameter s in this family of sequences appears to be the same as in the generalized Conolly recursion analyzed by Deugau and Ruskey [3]. For given s each sequence $h_s(n)$ hits every positive integer twice except for the powers of 2, all of which are hit $(s+2)$ times in the sequence. See Table 1.1 for the first 50 terms of each sequence for $0 \leq s \leq 6$.

Table 1.1: First 50 entries of h_s for $0 \leq s \leq 6$.

$n \backslash s$	0	1	2	3	4	5	6
1	1	1	1	1	1	1	1
2	1	1	1	1	1	1	1
3	2	1	1	1	1	1	1
4	2	2	1	1	1	1	1
5	3	2	2	1	1	1	1
6	3	2	2	2	1	1	1
7	4	3	2	2	2	1	1
8	4	3	2	2	2	2	1
9	5	4	3	2	2	2	2
10	5	4	3	2	2	2	2
11	6	4	4	3	2	2	2
12	6	5	4	3	2	2	2
13	7	5	4	4	3	2	2
14	7	6	4	4	3	2	2
15	8	6	5	4	4	3	2
16	8	7	5	4	4	3	2
17	9	7	6	4	4	4	3
18	9	8	6	5	4	4	3
19	10	8	7	5	4	4	4
20	10	8	7	6	4	4	4
21	11	9	8	6	5	4	4
22	11	9	8	7	5	4	4
23	12	10	8	7	6	4	4
24	12	10	8	8	6	5	4
25	13	11	9	8	7	5	4
26	13	11	9	8	7	6	4
27	14	12	10	8	8	6	5
28	14	12	10	8	8	7	5
29	15	13	11	9	8	7	6
30	15	13	11	9	8	8	6
31	16	14	12	10	8	8	7
32	16	14	12	10	8	8	7
33	17	15	13	11	9	8	8
34	17	15	13	11	9	8	8
35	18	16	14	12	10	8	8
36	18	16	14	12	10	8	8
37	19	16	15	13	11	9	8
38	19	17	15	13	11	9	8
39	20	17	16	14	12	10	8
40	20	18	16	14	12	10	8
41	21	18	16	15	13	11	9
42	21	19	16	15	13	11	9
43	22	19	17	16	14	12	10
44	22	20	17	16	14	12	10
45	23	20	18	16	15	13	11
46	23	21	18	16	15	13	11
47	24	21	19	16	16	14	12
48	24	22	19	17	16	14	12
49	25	22	20	17	16	15	13
50	25	23	20	18	17	15	13

In Section 2 we show how the sequence $\{h_s(n)\}_{n=1}^{\infty}$ can be interpreted in terms of the leaves in an infinite tree structure, from which the properties stated above follow readily. This interpretation also allows us to derive the generating function for this sequence in Section 3. We provide some final thoughts in Section 4.

2 Tree Structure

We begin this section by constructing a tree structure $\mathcal{T}_s$ that consists of an infinite number of rooted trees joined together at their respective roots plus a single initial isolated node. Subsequently we show how to label the vertices in $\mathcal{T}_s$ so that for any $s \geq 0$ the number of leaves up to the vertex with label n in this infinite tree structure is equal to $h_s(n)$.

Figure 2.1 shows the initial portion of $\mathcal{T}_s$. To create $\mathcal{T}_s$ first we join an infinite chain of nodes $\{s_i\}_{i=1}^{\infty}$, with node s_{i+1} connected to node s_i for $i = 1, 2, \ldots$. We say that these nodes, which we distinguish by the term *super-nodes*, are at level 0 in the graph. Each super-node s_i is the root of a subtree of $\mathcal{T}_s$ and has 2^{i-1} children, or level 1 nodes. Each level 1 node has one child, so a level 2 node. Level 2 nodes have no children, hence these nodes are called *leaves*. Denote by $\mathcal{S}_i$, $i = 1, 2, \ldots$ the rooted tree consisting of the root s_i and all its descendants (level 1 and level 2 nodes). Finally we complete $\mathcal{T}_s$ by adding an isolated node I. Note that I is itself a leaf of a trivial tree, but is not of level 0, 1 or 2. As usual, for a graph $\mathcal{G}$, let $V(\mathcal{G})$ denote the set of nodes (vertices) of $\mathcal{G}$. Call the subgraph of $\mathcal{T}_s$ induced by $\{I\} \cup \bigcup_{k=1}^{i} V(\mathcal{S}_k)$ the ith *complete partial graph*.

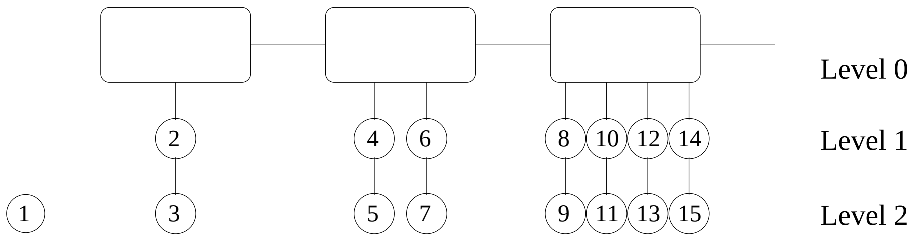

Figure 2.1: The initial portion of $\mathcal{T}_s$ with $s = 0$.

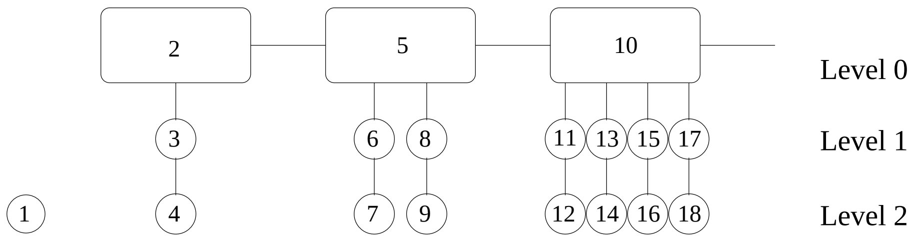

Figure 2.2: The initial portion of $\mathcal{T}_s$ with $s = 1$.

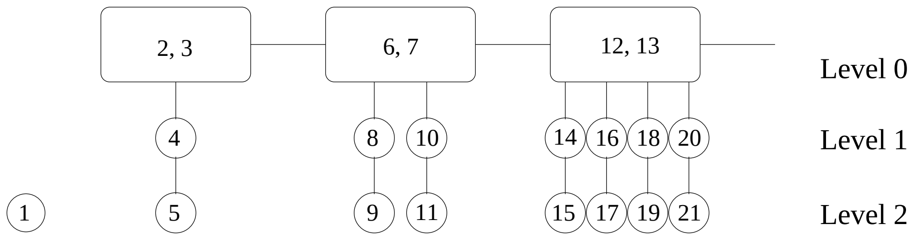

Figure 2.3: The initial portion of $\mathcal{T}_s$ with $s = 2$.

We now label each node with the integers $1, 2, 3, \ldots$ in the following way. Label I, the isolated node, 1. All super-nodes, starting with s_1, have the smallest s consecutive labels not yet used, where s is the parameter of the sequence $h_s(n)$ that we are considering. Note that if $s = 0$ the super-nodes have no labels. So for $s \geq 1$ the first super-node s_1 is labelled

$2, 3, \ldots, s+1$, which are the next s consecutive integers. Label each of the level 1 and level 2 nodes of the subtree $\mathcal{S}_1$ with a single integer, in increasing order first from level 1 to level 2 nodes and then from left to right. Do the same thing for $\mathcal{S}_2$, $\mathcal{S}_3$ and so on. See Figures 2.1, 2.2 and 2.3 where we show the initial portion of $\mathcal{T}_s$ for $s = 0, 1, 2$, respectively.

Denote by $v_s(n)$ the node with label n. Let $\mathcal{T}_s(n)$ be the subgraph of $\mathcal{T}_s$ induced by $\{v_s(i) : 1 \leq i \leq n\}$. Observe that if $v_s(n) = v_s(n-1) = s_i$ for some i, we have $\mathcal{T}_s(n) = \mathcal{T}_s(n-1)$. Note that the isolated node I is a leaf of $\mathcal{T}_s(n)$ for all $n = 1, 2, \ldots$. Let $a_s(n)$ be the number of leaves of $\mathcal{T}$ in $\mathcal{T}_s(n)$. That is, $a_s(n)$ equals the number of level 2 nodes in $\mathcal{T}_s(n)$ plus 1 (for the isolated node I). Finally, define $d_s(n)$ to be 1 if $v_s(n)$ is a leaf of $\mathcal{T}_s$ and to be 0 otherwise. Then it follows directly from these definitions that

$$a_s(n) = \sum_{k=1}^{n} d_s(k). \tag{2.1}$$

It is immediate from (2.1) that the sequence $\{a_s(n)\}_{n=1}^{\infty}$ is slowly growing. In addition

Lemma 2.1. *The sequence $\{a_s(n)\}_{n=1}^{\infty}$ satisfies the following properties:*
(a) *Let $s > 0$ and $r \geq 1$. Suppose n is the rth label in a super-node. Then for $-r \leq i \leq s+1-r$, $a_s(n-r-1)+1 = a_s(n+i) = a_s(n-r+s+2)-1$.*
(b) *If $v_s(n)$ is a level 1 node, then $a_s(n-1) = a_s(n) = a_s(n+1)-1$.*
(c) *If $v_s(n)$ is a level 2 node, then $a_s(n-1)+1 = a_s(n) = a_s(n+1)$.*

Proof. (a) Observe that if n is the r-th label in a super-node, then $n-r+1$ and $n-r+s$ are the first and last labels respectively in this super-node. Further, $v_s(n-r)$ is the last node in the immediately preceding complete partial graph, so is a leaf. Thus, $v_s(n-r-1)$ is the parent of this leaf and is a child of a super-node. It follows from (2.1) and our definitions that $a_s(n-r-1)+1$ counts the number of leaves in that preceding complete subtree.

Since $-r \leq i \leq s+1-r$, then $n-r \leq n+i \leq n-r+s+1$. All the values of $n+i$ lie between the label of the last leaf in the immediately preceding complete subtree and the label of the first child of this super-node. Thus,

$$a_s(n-r-1)+1 = a_s(n+i) = a_s(n-r+s+2)-1.$$

Both (b) and (c) are immediate from the definitions of level 1 and level 2 and (2.1). $\square$

Lemma 2.2. *For $i \geq 1$, $1 \leq r \leq 2^{i-1}$,*
(a) *If $v_s(n)$ is the super-node s_i (written compactly $v_s(n) = s_i$), then $a_s(n) = 2^{i-1}$.*
(b) *If $v_s(n)$ is a level 1 node, the rth child of s_i, then $a_s(n) = 2^{i-1}+r-1$.*
(c) *If $v_s(n)$ is a level 2 node, a leaf whose parent is the rth child of s_i, then $a_s(n) = 2^{i-1}+r$.*

Proof. If $v_s(n) = s_i$, $a_s(n)$ counts the number of leaves in the $(i-1)$st complete partial graph. So

$$a_s(n) = 1 + \sum_{k=1}^{i-1} 2^{k-1} = 2^{i-1}. \tag{2.2}$$

The other two cases follow readily. $\square$

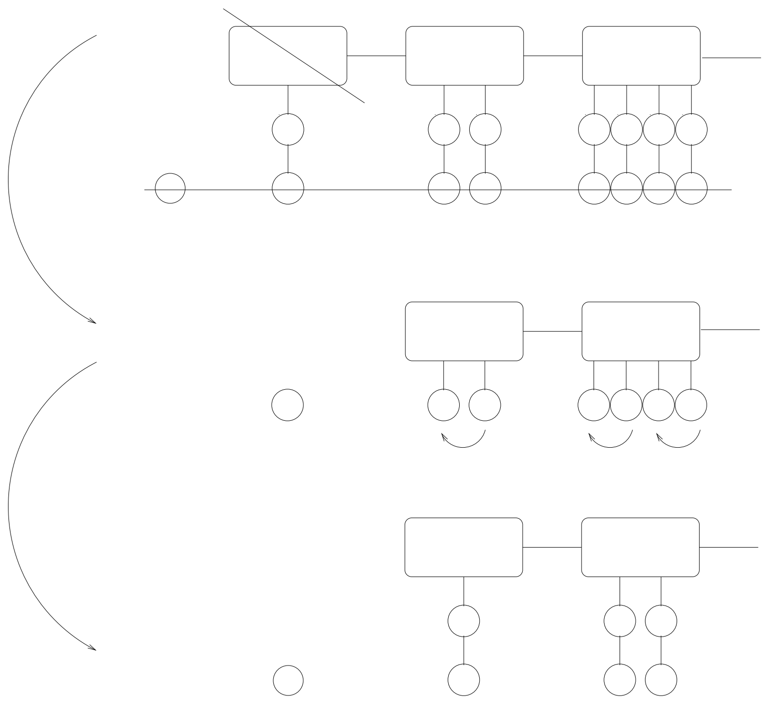

Figure 2.4: The chopping process P

For any $\mathcal{T}_s(n)$, we define a chopping process P that removes and rearranges nodes in the following way (see Figure 2.4): begin by removing the first super-node s_1 and all leaves of $\mathcal{T}_s$ (including the isolated node I). Then let the only child of s_1 (which is not removed) be the new isolated node I, and every second level 1 node in $\mathcal{S}_i$ ($i > 1$) be the child of its neighboring sibling on the left. Finally, relabel the tree so obtained in the same way as we label $\mathcal{T}_s$.

We observe several properties of the chopping process P:

- **P1.** The tree $\mathcal{T}_s$ is invariant under P. One still gets $\mathcal{T}_s$ after applying P on $\mathcal{T}_s$. See Figure 2.4.
- **P2.** Apply P on the ith complete partial graph, $i > 1$. We get the $(i-1)$st complete partial graph.
- **P3.** For any $n \geq s+4$, apply P on $\mathcal{T}_s(n)$ (the subgraph $\mathcal{T}_s(n)$ strictly contains the first complete subgraph). The resulting reduced graph is $\mathcal{T}_s(n - s - a_s(n))$, which has $s + a_s(n)$ fewer labels (note that $n - s - a_s(n) > 0$ by construction). For $i > 1$, $\mathcal{T}_s(n)$ contains $\mathcal{S}_i$ if and only if $\mathcal{T}_s(n - s - a_s(n))$ contains $\mathcal{S}_{i-1}$. See Figure 2.5 where we apply P to $\mathcal{T}_3(19)$ and get $\mathcal{T}_3(19 - 3 - a_3(19)) = \mathcal{T}_3(11)$.

Now we state our main theorem:

Theorem 2.3. *The sequence $h_s(n)$ defined by (1.4) and the given initial conditions counts the number of leaves in $\mathcal{T}_s(n)$. More precisely, for every positive integer n, $h_s(n) = a_s(n)$.*

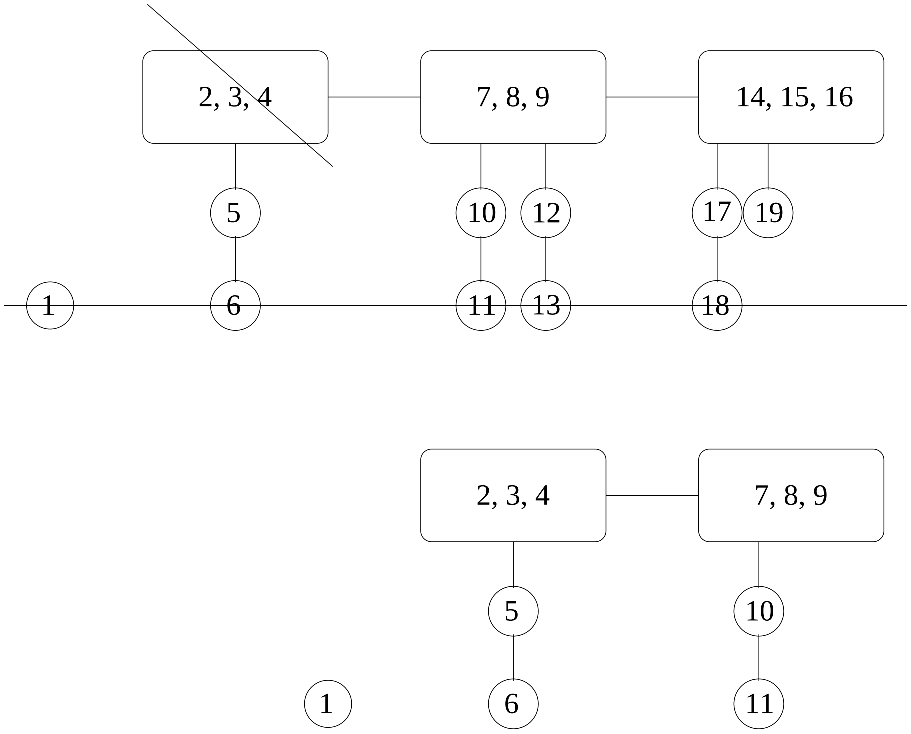

Figure 2.5: The chopping process P reduces $\mathcal{T}_3(19)$ to $\mathcal{T}_3(11)$.

Proof. We proceed by induction on n for fixed s. We first show that the assertion holds for the initial values $n = 1, 2, \ldots, 2s+7$. These comprise all the labels in the first two rooted trees $\mathcal{S}_1$ and $\mathcal{S}_2$ (together with I).

For $n = 1, 2, \ldots, s+2$, $h_s(n) = 1$ by the initial conditions assumed for (1.4), while $a_s(n) = 1$ because the isolated node I has label 1 and counts as a leaf, the super-node s_1 has s labels and its only child is labeled $s+2$. For $n = s+3$ $a_s(n) = 2$ because the node labeled $s+3$ is the first (and only) leaf in $\mathcal{S}_1$. Further, for $n = s+4, \ldots, 2s+4$, $a_s(n) = 2$ because the super-node s_2 contains s labels starting with $s+4$ and the node labeled $2s+4$ is the first child of s_2. But for the range of values $n = s+3, \ldots, 2s+4$ we can apply induction to the recursion for $h_s(n)$ to get $h_s(n) = h_s(n-s-h_s(n-1)) + h_s(n-2-s-h_s(n-3)) = 1+1 = 2 = a_s(n)$, as desired. The next two labels $n = 2s+5$ and $2s+6$ are assigned to the first leaf in $\mathcal{S}_2$ and the second child of s_2, respectively, so it follows that $a_s(2s+5) = a_s(2s+6) = 3$. But for these values of n, $h_s(n) = 3$ by (1.4). Finally, the label $2s+7$ is assigned to the second (and last) leaf in $\mathcal{S}_2$, so once again applying the above recursion we get $h_s(2s+7) = 4 = a_s(2s+7)$. See Figure 2.3.

We prove the assertion for n assuming the assertion is true for the first $n-1$ entries, where $v_s(n)$ is not a node of the 2nd complete partial graph (i.e., $n > 2s+7$).

Case 1: Node $v_s(n)$ is a level 0 node, i.e., a super-node. Assume $v_s(n) = s_i, i \geq 3$. By Lemma 2.2(a), we have $a_s(n) = 2^{i-1}$. We will prove that $a_s(n-s-a_s(n-1)) = a_s(n-2-s-a_s(n-3)) = 2^{i-2}$, from which it follows that $a_s(n) = 2^{i-1} = a_s(n-s-a_s(n-

$1)) + a_s(n-2-s-a_s(n-3))$.

By the induction assumption $a_s(n-1) = h_s(n-1)$ and $a_s(n-3) = h_s(n-3)$. Thus, $a_s(n-s-a_s(n-1)) = a_s(n-s-h_s(n-1))$ and $a_s(n-2-s-a_s(n-3)) = a_s(n-2-s-h_s(n-3))$. Apply the induction assumption once again to conclude that $a_s(n-s-h_s(n-1)) = h_s(n-s-h_s(n-1))$ and $a_s(n-2-s-h_s(n-3)) = h_s(n-2-s-h_s(n-3))$. But by (1.4) $h_s(n-s-h_s(n-1)) + h_s(n-2-s-h_s(n-3)) = h_s(n)$, so we get $a_s(n) = h_s(n)$, as desired.

To complete Case 1, we need to show that $a_s(n-s-a_s(n-1)) = a_s(n-2-s-a_s(n-3)) = 2^{i-2}$. Since $v_s(n) = s_i$, we have $a_s(n-1) = a_s(n)$ by Lemma 2.1(a). After the chopping process P on $\mathcal{T}_s(n)$, we get by P3 a new reduced graph with the last node $v_s(n-s-a_s(n)) = s_{i-1}$. By Lemma 2.2, $a_s(n-s-a_s(n)) = 2^{i-2}$, so $a_s(n-s-a_s(n-1)) = 2^{i-2}$.

If $v_s(n-2) = s_i$, let $n' := n-2$. Then $a_s(n-2-s-a_s(n-3)) = a_s(n'-s-a_s(n'-1)) = 2^{i-2}$. Otherwise, $v_s(n-2)$ is a level 2 or level 1 node, so by Lemma 2.1, $a_s(n-3) = a_s(n-1)-1$. Thus $a_s(n-2-s-a_s(n-3)) = a_s(n-2-s-a_s(n-1)+1) = a_s(n-1-s-a_s(n-1))$. But we have already shown that $v_s(n-s-a_s(n)) = v_s(n-s-a_s(n-1)) = s_{i-1}$, so by Lemmas 2.1 and 2.2, $a_s(n-1-s-a_s(n-1)) = a_s(n-s-a_s(n-1)) = 2^{i-2}$. This completes Case 1.

Case 2: Node $v_s(n)$ is a level 1 node. Assume $v_s(n)$ is a node of subtree $\mathcal{S}_i$ and the rth child of s_i, $i \geq 3$. By Lemma 2.2(b), we have $a_s(n) = 2^{i-1} + r - 1$. For $x \in \mathbb{R}$, let $\lfloor x \rfloor$ be the floor function of x. We will prove that $a_s(n-s-a_s(n-1)) = 2^{i-2} + \lfloor \frac{r}{2} \rfloor$ and $a_s(n-2-s-a_s(n-3)) = 2^{i-2} + \lfloor \frac{r-1}{2} \rfloor$, from which it follows that $a_s(n) = 2^{i-1} + r - 1 = a_s(n-s-a_s(n-1)) + a_s(n-2-s-a_s(n-3)) = h_s(n-s-h_s(n-1)) + h_s(n-2-s-h_s(n-3)) = h_s(n)$, from the induction assumption as in Case 1.

Since $v_s(n)$ is a level 1 node, we have $a_s(n-1) = a_s(n)$ by Lemma 2.1(b). After the chopping process P on $\mathcal{T}_s(n)$, we get by P3 a reduced graph with the last node $v_s(n-s-a_s(n))$, the $(\lfloor \frac{r}{2} \rfloor + 1)$st child ($r$ odd) of s_{i-1} or the leaf whose parent is the $\lfloor \frac{r}{2} \rfloor$th child ($r$ even) of s_{i-1}. By Lemma 2.2, $a_s(n-s-a_s(n)) = 2^{i-2} + \lfloor \frac{r}{2} \rfloor$, so $a_s(n-s-a_s(n-1)) = 2^{i-2} + \lfloor \frac{r}{2} \rfloor$.

If $v_s(n-2)$ is also a child of s_i, then it is the $(r-1)$st child. Let $n' := n-2$. Then $a_s(n-2-s-a_s(n-3)) = a_s(n'-s-a_s(n'-1)) = 2^{i-2} + \lfloor \frac{r-1}{2} \rfloor$. Otherwise, $r = 1$. We have two subcases:

(a) When $s > 1$, then $a_s(n) = a_s(n-3)$. After the chopping process P, by P3 the last node of the reduced graph is $v_s(n-s-a_s(n-3))$, the first child of s_{i-1}. So $v_s(n-2-s-a_s(n-3)) = s_{i-1}$. Thus by Lemma 2.2, $a_s(n-2-s-a_s(n-3)) = 2^{i-2} = 2^{i-2} + \lfloor \frac{r-1}{2} \rfloor$.

(b) When $s = 0$ or 1, then $a_s(n) = a_s(n-3)+1$. After the chopping process P, by P3 the last node of the reduced graph is $v_s(n-s-a_s(n))$, the first child of s_{i-1}. Thus by Lemma 2.1 and Lemma 2.2, $a_s(n-2-s-a_s(n-3)) = a_s(n-s-a_s(n)-1) = a_s(n-s-a_s(n)) = 2^{i-2} = 2^{i-2} + \lfloor \frac{r-1}{2} \rfloor$.

Case 3: $v_s(n)$ is a level 2 node, i.e., a leaf. Assume the parent of $v_s(n)$ is the rth child of s_i, $i \geq 3$. By Lemma 2.2(c), we have $a_s(n) = 2^{i-1} + r$. We will prove that $a_s(n-s-a_s(n-1)) = 2^{i-2} + \lfloor \frac{r+1}{2} \rfloor$ and $a_s(n-2-s-a_s(n-3)) = 2^{i-2} + \lfloor \frac{r}{2} \rfloor$, from which it follows that $a_s(n) = 2^{i-1} + r = a_s(n-s-a_s(n-1)) + a_s(n-2-s-a_s(n-3)) = h_s(n-s-h_s(n-1)) + h_s(n-2-s-h_s(n-3)) = h_s(n)$, from the induction assumption as in Case 1.

Since $v_s(n)$ is a level 2 node, we have $a_s(n-1) = a_s(n)-1$ by Lemma 2.1(c). After the chopping process P on $\mathcal{T}_s(n)$, we get by P3 a reduced graph with the last node $a_s(n-s-a_s(n-1))$ the $(\lfloor\frac{r}{2}\rfloor+1)$st child of s_{i-1} or the leaf whose parent is the $\lfloor\frac{r}{2}\rfloor$th child of s_{i-1}. By Lemma 2.2, $a_s(n-s-a_s(n)+1)=2^{i-2}+\lfloor\frac{r+1}{2}\rfloor$, so $a_s(n-s-a_s(n-1))=2^{i-2}+\lfloor\frac{r+1}{2}\rfloor$.

If $v_s(n-2)$ is also a node of $\mathcal{S}_i$, then it is the child of the $(r-1)$st child of s_i. Let $n':=n-2$. Then $a_s(n-2-s-a_s(n-3))=a_s(n'-s-a_s(n'-1))=2^{i-2}+\lfloor\frac{r}{2}\rfloor$. Otherwise, $r=1$. We have two subcases:

(a) When $s>0$, then $a_s(n)=a_s(n-3)$. After the chopping process P, by P3 the last node of the reduced graph is $v_s(n-s-a_s(n))$, the first child of s_{i-1}. So $v_s(n-s-a_s(n)-1)=s_{i-1}$. Thus by Lemma 2.1 and Lemma 2.2, $a_s(n-s-a_s(n-3)-2)=a_s(n-s-a_s(n)-2)=a_s(n-s-a_s(n)-1)=2^{i-2}=2^{i-2}+\lfloor\frac{r}{2}\rfloor$.

(b) When $s=0$, then $a_s(n)=a_s(n-3)+1$. After the chopping process P, by P3 the last node of the reduced tree is $v_s(n-s-a_s(n))$, the first child of s_{i-1}. Thus by Lemma 2.1 and Lemma 2.2, $a_s(n-2-s-a_s(n-3))=a_s(n-s-a_s(n)-1)=a_s(n-s-a_s(n))=2^{i-2}=2^{i-2}+\lfloor\frac{r}{2}\rfloor$.

This completes the proof.

□

From Theorem 2.3, Lemma 2.1 and the definition of $a_s(n)$ the following result is immediate .

Corollary 2.4. *For every $s\geq 0$, the sequence $h_s(n)$ is slowly growing. The sequence hits every positive integer twice except for the powers of 2; the number of repetitions of $2^r, r\geq 0$, in the sequence $h_s(n)$ is $s+2$.*

For $s=0$ it is evident from the definition of $a_0(n)$ and Theorem 2.3 that $h_0(n)=\lceil\frac{n}{2}\rceil$, which provides a simple closed form for $h_0(n)$. This formula is immediate from a casual inspection of Table 1.1, and also follows directly from Corollary 2.4.

Observe that Corollary 2.4 implies that 2^k+sk is the index of the last term in $h_s(n)$ that equals 2^{k-1}. The argument for this is as follows: there are k powers of 2 less than or equal to 2^{k-1}, each of which occurs $s+2$ times in $h_s(n)$ while each of the other $2^{k-1}-k$ positive integers less than or equal to 2^{k-1} occurs twice. But since $2(2^{k-1}-k)+(s+2)k=2^k+sk$ we can write $h_s(n)$ in terms of $h_0(n)$. Since we have a closed form for $h_0(n)$ this provides a "piecewise" closed formula for $h_s(n)$ where $s\geq 1$:

$$h_s(n)=\begin{cases} h_0(n-sk) & 2^k+sk\leq n<2^{k+1}+sk \\ 2^k & 2^{k+1}+sk\leq n<2^{k+1}+s(k+1)\end{cases} \tag{2.3}$$

In a surprising twist it turns out that the sequence $h_1(n)$ also has an explicit closed form. This sequence, with the additional initial term $h_1(0)=1$, appears in [9] as entry A109964 (Bottomley's sequence), though the latter is generated by a very different recursion that is not self-referencing. To prove that the two sequences are the same it is enough to show that for $n>0$ Bottomley's sequence satisfies Corollary 2.4. This follows from a straightforward induction; we omit the details. Thus, for $n>0$ $h_1(n)$ satisfies the recursion in [9] for A109964(n), namely, $h_1(n)=\lfloor\sqrt{\sum_{j=0}^{n-1}h_1(j)}\rfloor$.

Further, from [9] we have that for $n>0$ $A109964(n)=\lfloor\sqrt{b(n-1)}\rfloor$ where $b(n)$ is defined by the recursion $b(n)=b(n-1)+\lfloor\sqrt{b(n-1)}\rfloor$, with $b(0)=1$. But in [5, Chap. 3, Exer.

3.28], the following closed form for $b(n)$ is attributed to Carl Witty:

$$b(n-1) = 2^k + \lfloor (\frac{n-k}{2})^2 \rfloor, \qquad 2^k + k \leq n < 2^{k+1} + k + 1.$$

Thus we have the following closed form for $h_1(n)$:

$$h_1(n) = \lfloor \sqrt{2^k + \lfloor (\frac{n-k}{2})^2 \rfloor} \rfloor, \qquad 2^k + sk \leq n \leq 2^{k+1} + sk.$$

For $s > 1$ there does not seem to be any comparable closed form for $h_s(n)$.

Note that it is possible to generate the family of sequences $h_s(n)$ from a slightly different infinite tree graph and a modified numbering scheme for the nodes.

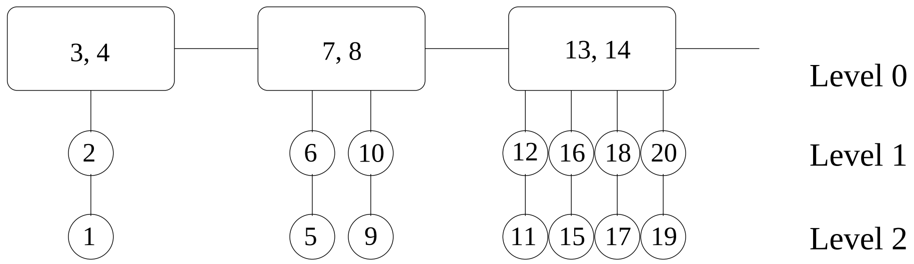

Figure 2.6: The initial portion of $\mathcal{T}'_s$ with $s = 2$.

The new tree structure $\mathcal{T}'_s$ is our original $\mathcal{T}_s$ minus the isolated node I. We label each node with integers 1, 2, 3, $\cdots$ as follows: All subtrees, starting with $\mathcal{S}_1$, are labelled consecutively with the smallest labels not yet used. For each subtree $\mathcal{S}_i$, first label the leaf of the first child of super-node s_i and then label the first child of s_i, each node receiving one label. Next, label the super-node s_i with s consecutive labels not yet used, where s is the parameter of the sequence $h_s(s)$. Now label each of the remaining level 2 and level 1 nodes of the subtree $\mathcal{S}_i$ with a single label in increasing order first from level 2 to level 1 nodes and then from left to right. See Figure 2.6, where we show the initial portion of $\mathcal{T}'_s$ for $s = 2$. If we define the chopping process in the same way as before, with the obvious modification since we no longer have an isolated node, then it is easy to see that the sequence $h_s(n)$ can be interpreted in terms of the leaves in the this new graph $\mathcal{T}'_s$.

3 Generating Function

We compute a generating function for $\{a_s(n)\}_{n=1}^{\infty}$, which is also the generating function for $\{h_s(n)\}_{n=1}^{\infty}$ by Theorem 2.3. Define $\mathcal{A}_s(z) := \sum_{k=1}^{\infty} a_s(k) z^k$ and $\mathcal{D}_s(z) := \sum_{k=1}^{\infty} d_s(k) z^k$. By (2.1), we know that $\mathcal{A}_s(z) = \frac{\mathcal{D}_s(z)}{1-z}$. So it's enough to find $\mathcal{D}_s(z)$. From the definition of $d_s(n)$ it is clear that the subsequence of the sequence $d_s(n)$ corresponding to $\mathcal{S}_i$, $i > 0$, is of the form $0^s 01 \cdots 01$, with 2^{i-1} consecutive 01's, preceded by s consecutive 0's, denoted by 0^s. Thus $\{d_s(n)\}_{n=1}^{\infty}$ is $1 \vdots 0^s 01 \vdots 0^s 0101 \vdots 0^s 01010101 \vdots \cdots$

For $n > 1$, define the concatenation $C_n = C_{n-1}C_{n-1}$, with $C_0 = 1$ and $C_1 = 01$. Let $C_n(k)$ be the kth entry of C_n. It is obvious that $\{d_s(n)\}_{n=1}^{\infty}$ can be written as $C_0 0^s C_1 0^s C_2 0^s C_3 0^s \cdots$. Further we have

Lemma 3.1. *The length of C_n is $|C_n| = 2^n$ and for $n > 0$,*

$$\mathcal{C}_n(z) := \sum_{k=1}^{2^n} C_n(k) z^k = \sum_{i=1}^{2^{n-1}} z^{2i} = \frac{z^2}{1-z^2}(1 - z^{2^n}).$$

Proof. Evidently $|C_0| = 1$ and for $n > 0$, C_n has 2^{n-1} pairs of 01, so $|C_n| = 2^n$. Each 1 occurs as the second entry in every pair, so the result follows. □

Theorem 3.2. *The generating function $\mathcal{D}_s(z)$ is equal to*

$$z + \frac{z^2}{1-z^2} \sum_{i=1}^{\infty} z^{si+2^i-1}(1 - z^{2^i})$$

Proof.

$$\begin{aligned}
\mathcal{D}_s(z) &= z + \sum_{i=1}^{\infty} z^{si+\sum_{j=0}^{i-1}|C_j|}\mathcal{C}_i(z) \\
&= z + \sum_{i=1}^{\infty} z^{si+2^i-1} \frac{z^2}{1-z^2}(1 - z^{2^i}) \\
&= z + \frac{z^2}{1-z^2} \sum_{i=1}^{\infty} z^{si+2^i-1}(1 - z^{2^i})
\end{aligned}$$

□

Remark. We can also rewrite the expression for $\mathcal{D}_s(z)$

$$\begin{aligned}
\mathcal{D}_s(z) &= z + \frac{z^2}{1-z^2} \sum_{i=1}^{\infty} z^{si+2^i-1}(1 - z^{2^i}) \\
&= z + \frac{z^2}{1-z^2} z^{s+2^0}[1 - z^{2^1} + z^{s+2^1}[1 - z^{2^2} + z^{s+2^2}[1 - z^{2^3} + z^{s+2^3}[\cdots \\
&= \frac{z^2}{1-z^2}[\frac{1}{z} - z^{2^0} + z^{s+2^0}[1 - z^{2^1} + z^{s+2^1}[1 - z^{2^2} + z^{s+2^2}[1 - z^{2^3} + z^{s+2^3} \cdots
\end{aligned}$$

Corollary 3.3. *The generating function $\mathcal{A}_s(z)$ is equal to*

$$\frac{z}{1-z} + \frac{z^2}{(1-z^2)(1-z)} \sum_{i=1}^{\infty} z^{si+2^i-1}(1 - z^{2^i})$$

Proof. By the proposition and $\mathcal{A}_s(z) = \frac{\mathcal{D}_s(z)}{1-z}$. □

4 Concluding Remark

It is equally natural to investigate the meta-Fibonacci recursion that derives from the odd values of the index on the right hand side of (1.2), that is,

$$g_{s,k}(n) = \sum_{j=0}^{k-1} g_{s,k}(n-(2j+1)-s-g_{s,k}(n-(2j+2))), \qquad n > s+2k. \tag{4.1}$$

For general k and s, the recursion (4.1) produces a much richer collection of infinite sequences (that is, sequences that do not die) for various sets of initial conditions. In many instances these sequences exhibit considerably more complicated behavior than $h_s(n)$. To date we have not been able to discover a combinatorial interpretation for any of these sequences, whose behavior will be the subject of a future communication.

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