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Binary Relations on the Power Set of an n-Element Set

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Abstract

We define six binary relations on the power set of an n-element set and describe their basic structure and interrelationships. An auxiliary relation is noted that will assist in determining the cardinalities of each. We also indicate an eighth relation that may be of interest. We conclude the second section by computing several quantities related to walks in the graph of the sixth relation. In the third section we turn our attention to the basic structure and cardinalities of the auxiliary relation noted in section two and several additional relations. We also compute seven sums associated with these relations and indicate connections four relations have with Wieder's *conjoint* and *disjoint k-combinations*.

1 Introduction

Combinatorial interest has generally focused on the enumeration of various *types* of relations or associated structures such as topologies and graphs. In this paper we will be interested in the generally more tractable problem of defining and enumerating several binary relations on the power set of an n-element set and computing various quantities associated with them. We will conclude by indicating connections four relations have with Wieder's *conjoint* and *disjoint k-combinations*. The motivation here is to suggest some ways in which sets of k-subsets of some set may be studied under the rubric of asymmetric k-ary relations on the same. To the author's knowledge, the results of this paper or related results do not appear in the existing literature, except when so noted.

2 The six initial relations

2.1 Definitions and structure

Let $P(A)$ be the power set of an n-element set A[1]. Then $B = P(A) \times P(A)$ is just the Cartesian product of $P(A)$ with itself and the following binary relations on $P(A)$ are, of course, just subsets of B:

$$R_0 = \{(x, y) : x, y \in P(A), x \subseteq y \vee y \subseteq x\},$$

$$R_1 = \{(x, y) : x, y \in P(A), x \subset y \vee y \subset x\},$$

$$R_2 = \{(x, y) : x, y \in P(A), x \prec y \vee y \prec x\},$$

$$R_3 = \{(x, y) : x, y \in P(A), x \subseteq y\},$$

$$R_4 = \{(x, y) : x, y \in P(A), x \subset y\}, \text{ and}$$

$$R_5 = \{(x, y) : x, y \in P(A), x \prec y\}.$$

Note that $x \prec y$ denotes that $x \subset y$ and $|y| - |x| = 1$ [2]. As R_3 has been defined to be a *partial order* relation on $P(A)$, this just means that R_5 is the corresponding *covering* relation [2]. R_4 is the corresponding *strict order* [2]. Also, observe that B is a reflexive, symmetric, and transitive relation on $P(A)$ and so is an *equivalence* relation on the same.

	R_0	R_1	R_2	R_3	R_4	R_5
reflexive	yes	no	no	yes	no	no
irreflexive	no	yes	yes	no	yes	yes
symmetric	yes	yes	yes	no	no	no
asymmetric	no	no	no	no	yes	yes
antisymmetric	no	no	no	yes	yes	yes
transitive	no	no	no	yes	yes	no
intransitive	no	no	yes	no	no	yes

Table 1: Basic properties of R_0 – R_5

Tables 1, 2, and 3 list some basic information with regard to R_0 – R_5. While entries $=, \subseteq$, and $\supseteq$ in Table 3 are easily enough understood, we explain an entry of $\cap$ as follows. Given a row relation R_r and a column relation R_c in Table 3, an entry of $\cap$ indicates that, while $R_r \not\subseteq R_c$ and $R_c \not\subseteq R_r$, we still have that $R_r \cap R_c \neq \emptyset$.

Examples. With $A = \{1, 2\}$, $P(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$, and

$$R_4 = \{(\emptyset, \{1\}), (\emptyset, \{2\}), (\emptyset, \{1, 2\}), (\{1\}, \{1, 2\}), (\{2\}, \{1, 2\})\},$$

$$R_5 = \{(\emptyset, \{1\}), (\emptyset, \{2\}), (\{1\}, \{1, 2\}), (\{2\}, \{1, 2\})\}.$$

[1]For convenience we may assume that $A = \{1, 2, 3, \ldots, n\}$.

	R_0	R_1	R_2	R_3	R_4	R_5
reflexive closure		R_0			R_3	
reflexive reduction	R_1			R_4		
symmetric closure				R_0	R_1	R_2
symmetric reduction	R_3	R_4	R_5			
transitive closure			R_1			R_4
transitive reduction		R_2			R_5	

Table 2: Closures and reductions of R_0 – R_5

	R_0	R_1	R_2	R_3	R_4	R_5
R_0	$=$	$\supseteq$	$\supseteq$	$\supseteq$	$\supseteq$	$\supseteq$
R_1	$\subseteq$	$=$	$\supseteq$	$\cap$	$\supseteq$	$\supseteq$
R_2	$\subseteq$	$\subseteq$	$=$	$\cap$	$\cap$	$\supseteq$
R_3	$\subseteq$	$\cap$	$\cap$	$=$	$\supseteq$	$\supseteq$
R_4	$\subseteq$	$\subseteq$	$\cap$	$\subseteq$	$=$	$\supseteq$
R_5	$\subseteq$	$\subseteq$	$\subseteq$	$\subseteq$	$\subseteq$	$=$

Table 3: Inclusion and intersection relationships, R_0 – R_5

Define the partial order relation

$$E = \{(p, q) : p, q \in \{R_0, R_1, \ldots, R_5\}, p \subseteq q\}$$

so that G may be defined to be the (directed) graph of the covering relation corresponding to E. Then most of the information in Tables 1, 2, and 3 is illustrated in Figure 1. Observe that we have labelled the edges of G in the diagram to indicate the closure and reduction relationships that obtain between the vertices of this graph, i.e., R_0, R_1 etc. For example, an edge labelled s and directed from vertex u to vertex v indicates that u is the symmetric reduction of v (and v the symmetric closure of u); we can also deduce from this that v is a symmetric relation. Similar observations hold for the other two edge labels. Finally, note that if a walk exists from u to v, then $u \subseteq v$. However, if there is neither a walk from u to v nor a walk from v to u, then, while $u \not\subseteq v$ and $v \not\subseteq u$, it is still the case that $u \cap v \neq \emptyset$.

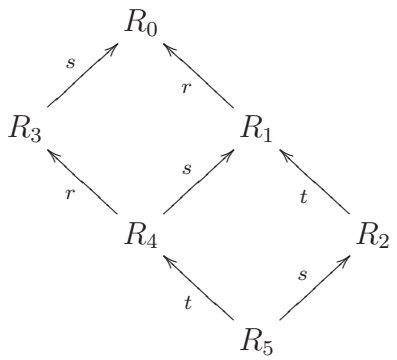

Figure 1: Hasse diagram of G

2.2 The cardinalities of $R_0 - R_5$

In determining the cardinalities of R_0 – R_5 it will be useful to recall [1] that

$$|P(A)| = 2^n. \tag{1}$$

To see this, consider that in enumerating a subset of A, i.e., an element of $P(A)$, we decide on a case by case basis which elements of A to include in the subset. As there are 2 outcomes possible for each of the elements of A – include in the subset or exclude from the subset – and n elements of A, there are 2^n subsets of A altogether.

Now recall that the $(x, y) \in R_3$ are such that $x \subseteq y$. It is not hard to see that enumerating each subset of each subset of A will enumerate the elements of R_3. We may do so by deciding on a case by case basis which elements of A to

1. include in both x and y,
2. exclude from both x and y, or
3. exclude from x and include in y.

As there are 3 outcomes possible for each of the n elements of A, we see that

$$|R_3| = 3^n. \tag{2}$$

The cardinalities of R_0, R_1, and R_4 are determined as follows. Let R_i^{-1} denote the *inverse* relation of R_i [3]. Then, define

$$R_{11} = \{(x, y) : x, y \in P(A), x = y\} = \{(x, x) : x \in P(A)\}.$$

It should be clear that $R_{11} \subset R_3$. Moreover, it is obvious that

$$|R_{11}| = 2^n. \tag{3}$$

It then follows that

$$|R_0| = 2 \cdot 3^n - 2^n, \tag{4}$$

as, by definition, $R_0 = R_3 \cup R_3^{-1}$, and R_{11} should only be counted once in this union.

Similarly, we see that

$$|R_4| = 3^n - 2^n, \tag{5}$$

as, by definition, $R_4 = R_3 \setminus R_{11}$.

Furthermore,

$$|R_1| = 2 \cdot (3^n - 2^n), \tag{6}$$

as, by definition, $R_1 = R_4 \cup R_4^{-1}$.

Recall that for each $(x, y) \in R_5$, $x \subset y$ and $|y| - |x| = 1$. This suggests the following enumeration of this relation. First, select 1 of the n elements of A to add to x (to get y). x is then constructed from the remaining elements of A, and this can be done in 2^{n-1} ways. As y is now completely determined, we have that

$$|R_5| = n \cdot 2^{n-1}. \tag{7}$$

It is then easy to see that

$$|R_2| = n \cdot 2^n, \tag{8}$$

as, by definition, $R_2 = R_5 \cup R_5^{-1}$.

Before concluding here we note one additional relation that may be of interest:

$$R_6 = R_4 \setminus R_5.$$

R_6 is an irreflexive, asymmetric, antisymmetric, and transitive relation and so a strict order. We also note that this relation is a proper subset of R_0, R_1, R_3, and R_4. Finally, that

$$|R_6| = 3^n - 2^n - n \cdot 2^{n-1} \tag{9}$$

is easily deduced from Eqns. (5) and (7).

Relation	Cardinality	Sequence	Sequence	Sequence
R_0	$2 \cdot 3^n - 2^n$	A027649(n)	A090888$(n, 3)$	
R_1	$2 \cdot (3^n - 2^n)$	A056182(n)		
R_2	$n \cdot 2^n$	A036289(n)		
R_3	3^n	A000244(n)	A090888$(n, 2)$	A112626$(n, 0)$
R_4	$3^n - 2^n$	A001047(n)	A090888$(n, 1)$	A112626$(n, 1)$
R_5	$n \cdot 2^{n-1}$	A001787(n)	A090802$(n, 1)$	
R_6	$3^n - 2^n - n \cdot 2^{n-1}$	A066810(n)		A112626$(n, 2)$

Table 4: The cardinalities of R_0 – R_6

Table 4 lists the cardinalities of R_0 – R_6 with references to the corresponding integer sequences in Sloane's *Encyclopedia* [5].

2.3 Walks and lengths of walks in the graph of R_5

Let H be the (directed) graph of R_5 and recall that $P(A)$ is the power set of an n-element set A. Also, let $C(n, k)$ denote the *binomial coefficient*, i.e., the number of ways to select k elements from an n-element set.

We begin by counting the number of k-length walks $w(n, k)$ in H [3]. As the vertices of H are just elements of $P(A)$, a little thought shows that a k-length walk in H begins at some $x \in P(A)$ and adds k elements of A not in x to x, where the order of addition is important. We may enumerate such a walk by first selecting the k elements to add to x. There are $C(n, k)$ ways to do this. This then leaves 2^{n-k} ways to construct x. Once constructed, we then just need to decide which of the $k!$ ways we will add the k selected elements in. Hence

$$w(n, k) = k! \cdot C(n, k) \cdot 2^{n-k}. \tag{10}$$

The total number of walks $W(n)$ in H is then just given by

$$W(n) = \sum_{0 \le k \le n} w(n, k) = \sum_{0 \le k \le n} k! \cdot C(n, k) \cdot 2^{n-k}. \tag{11}$$

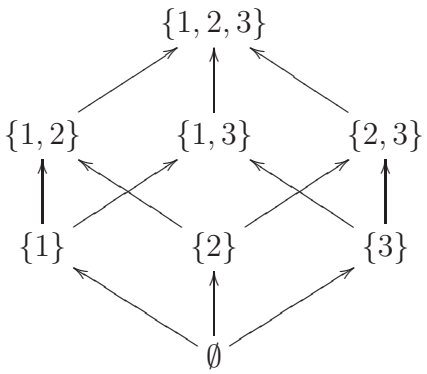

Figure 2: Hasse diagram of H with $A = \{1, 2, 3\}$

Now let $l(n, k)$ denote the total length of all k-length walks in H. It is not hard to see that

$$l(n, k) = \frac{k}{k!} \cdot w(n, k) = k \cdot C(n, k) \cdot 2^{n-k} \tag{12}$$

as in computing $l(n, k)$ we are interested in determining only *that* in fact a k-length walk exists in H and accounting for the length of this walk, and this is easily accomplished by dividing $w(n, k)$ by $k!$ and then, of course, just multiplying the result by k. The total length of all walks $L(n)$ in H is then just given by

$$L(n) = \sum_{0 \le k \le n} l(n, k) = \sum_{0 \le k \le n} k \cdot C(n, k) \cdot 2^{n-k}. \tag{13}$$

$w(n, k)$ is given by A090802(n, k) in Sloane's *Encyclopedia* [5], while $W(n)$ is given by A010842(n). There is no sequence listed that gives $l(n, k)$. However, $L(n)$ is given by A027471$(n + 1)$.

We conclude this section by observing that the number of 1-length walks in H is just $1! \cdot C(n, 1) \cdot 2^{n-1} = n \cdot 2^{n-1}$. It should be clear that the enumeration of these walks is fundamentally equivalent to the enumeration of R_5 given above (7).

3 The additional relations

3.1 Definitions and structure

As before, let $P(A)$ be the power set of an n-element set A. Here we will be chiefly interested in five pairwise disjoint relations on $P(A)$ whose union we define to be R_{35}. It should be clear that the cardinality of R_{35} is easily computed as the sum of the cardinalities of these five relations. Moreover, we will be able to easily define and determine the cardinalities of the $2^5 - 7 = 25$ remaining possible unions of the relations (the empty union is excluded, of course). So we define

$$R_7 = \{(x, y) : x, y \in P(A), x \subset y \wedge x \cap y = \emptyset\}, \text{ and}$$

$$R_{10} = \{(x, y) : x, y \in P(A), x \subset y \wedge x \cap y \neq \emptyset\},$$

and recall that

$$R_{11} = \{(x, y) : x, y \in P(A), x = y\}.$$

Then we let R_8 be an asymmetric binary relation on $P(A)$ whose (x, y) are such that $x \not\subseteq y$ and $y \not\subseteq x$ and $x \cap y = \emptyset$. Similarly, we define R_9 be an asymmetric binary relation on $P(A)$ whose (x, y) are such that $x \not\subseteq y$ and $y \not\subseteq x$ and $x \cap y \neq \emptyset$. It should be clear that R_7 and R_{10} are also asymmetric relations. We also observe that R_7 - R_{10} are irreflexive and vacuously antisymmetric, while R_{11} is reflexive, vacuously symmetric, and antisymmetric.

R_{35} is a reflexive, antisymmetric, and transitive relation, and so a partial order. Let Q be the *quotient set* [4] $B/\equiv$, i.e., an equivalence relation on B, where

$$(a, b) \equiv (c, d) \Leftrightarrow (a = d) \wedge (b = c)$$

for all $(a, b), (c, d) \in B$. Then R_{35} may also be thought of as a set of equivalence class representatives of Q. Furthermore, we note that

$$R_3 = R_7 \cup R_{10} \cup R_{11}, \text{ and}$$

$$R_4 = R_7 \cup R_{10}.$$

Examples. With $A = \{1, 2\}$, $P(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$, and

$$R_7 = \{(\emptyset, \{1\}), (\emptyset, \{2\}), (\emptyset, \{1, 2\})\},$$

$$R_8 = \text{either } \{(\{1\}, \{2\})\} \text{ or } \{(\{2\}, \{1\})\},$$

$$R_9 = \emptyset,$$

$$R_{10} = \{(\{1\}, \{1, 2\}), (\{2\}, \{1, 2\})\},$$

$$R_{11} = \{(\emptyset, \emptyset), (\{1\}, \{1\}), (\{2\}, \{2\}), (\{1, 2\}, \{1, 2\})\}.$$

3.2 The cardinalities of $R_7 - R_{11}$ and their possible unions

The *Stirling numbers of the second kind* $S(n, k)$ count the number of ways to partition an n-element set into k non-empty subsets, or, *parts*. They satisfy the recurrence

$$S(n + 1, k + 1) = S(n, k) + (k + 1) \cdot S(n, k + 1). \tag{14}$$

It is also the case that

$$S(n, 1) = S(n, n) = 1, \tag{15}$$

as there is clearly just 1 way to partition an n-element set into either 1 part or n parts.

We assume $S(i, j) = 0$ when $i < j$.

The main approach we will take in determining the cardinalities of R_7 – R_{11} will be to form two partitions of the n-element set A and then assign one part of each to x and one

or two of each to y for all the (x, y) of R_7 - R_{11}. For one of these partitions, it will be the case that $|x \cup y| = |A|$; for the other, $|x \cup y| < |A|$. In other words, we will enumerate the elements of R_7 - R_{11} based upon which of these results will obtain.

We begin with R_7. Observe that for the $(x, y) \in R_7$ it is the case that $x = \emptyset$ and $x \neq y$. So we may enumerate R_7 by fixing $x = \emptyset$, and then partitioning A into either 1 or 2 parts and assigning one part to y. There are clearly $1 = S(n, 1)$ ways to accomplish the 1-part enumeration and $C(2, 1) \cdot S(n, 2) = 2 \cdot S(n, 2)$ ways to accomplish the 2-part enumeration. With $k = 1$ in Eqn. (14), it then follows that

$$|R_7| = S(n + 1, 2). \tag{16}$$

With regard to the $(x, y) \in R_8$, we partition A into either 2 or 3 parts and assign one part to x and another to y. It is easy to see that there are $S(n, 2)$ ways to accomplish the 2-part enumeration, as we can only assign the parts in 1 way. The 3-part enumeration may be accomplished in $C(3, 2) \cdot S(n, 3) = 3 \cdot S(n, 3)$ ways, as we select 2 of the 3 parts and assign one to x and the other to y. With $k = 2$ in Eqn. (14), we then see that

$$|R_8| = S(n + 1, 3). \tag{17}$$

With regard to the $(x, y) \in R_9$, we partition A into either 3 or 4 parts and assign one part to x, another to y, and another to their intersection. There are $C(3, 2) \cdot S(n, 3) = 3 \cdot S(n, 3)$ ways to accomplish the 3-part enumeration, as selecting 2 of the 3 parts and assigning one to x and the other to y leaves only 1 way to assign the remaining part to $x \cap y$. The 4-part enumeration may be accomplished in $C(4, 2) \cdot C(2, 1) \cdot S(n, 4) = 12 \cdot S(n, 4)$ ways, as we first select 2 of the 4 parts and assign one to x and the other to y, and then select 1 of the 2 remaining parts to assign to $x \cap y$. With $k = 3$ in Eqn. (14), we then have that

$$|R_9| = 3 \cdot S(n + 1, 4). \tag{18}$$

With regard to the $(x, y) \in R_{10}$, we partition A into either 2 or 3 parts and assign one part to x and this part and another part to y. There are $C(2, 1) \cdot S(n, 2) = 2 \cdot S(n, 2)$ ways to accomplish the 2-part enumeration, as selecting 1 of the 2 parts to assign to x leaves just 1 part to assign to y (in 1 way). The 3-part enumeration may be accomplished in $C(3, 1) \cdot C(2, 1) \cdot S(n, 3) = 6 \cdot S(n, 3)$ ways, as we first select 1 of the 3 parts to assign to x and then combine this with 1 of the 2 remaining parts and assign the result to y. With $k = 2$ in Eqn. (14), it then follows that

$$|R_{10}| = 2 \cdot S(n + 1, 3). \tag{19}$$

With regard to the $(x, y) \in R_{11}$, we partition A into either 1 or 2 parts and assign one part to both x and y. This then leaves $(\emptyset, \emptyset)$ to be included in R_{11}. There are clearly $1 = S(n, 1)$ ways to accomplish the 1-part enumeration, $C(2, 1) \cdot S(n, 2) = 2 \cdot S(n, 2)$ ways to accomplish the 2-part enumeration, and, of course, 1 way to construct $(\emptyset, \emptyset)$. With $k = 1$ in Eqn. (14), we then see that

$$|R_{11}| = S(n + 1, 2) + 1. \tag{20}$$

$R_{12} = R_7 \cup R_8$	$R_{24} = R_7 \cup R_9 \cup R_{10}$
$R_{13} = R_7 \cup R_9$	$R_{25} = R_7 \cup R_9 \cup R_{11}$
$R_4 = R_7 \cup R_{10}$	$R_3 = R_7 \cup R_{10} \cup R_{11}$
$R_{14} = R_7 \cup R_{11}$	$R_{26} = R_8 \cup R_9 \cup R_{10}$
$R_{15} = R_8 \cup R_9$	$R_{27} = R_8 \cup R_9 \cup R_{11}$
$R_{16} = R_8 \cup R_{10}$	$R_{28} = R_8 \cup R_{10} \cup R_{11}$
$R_{17} = R_8 \cup R_{11}$	$R_{29} = R_9 \cup R_{10} \cup R_{11}$
$R_{18} = R_9 \cup R_{10}$	$R_{30} = R_7 \cup R_8 \cup R_9 \cup R_{10}$
$R_{19} = R_9 \cup R_{11}$	$R_{31} = R_7 \cup R_8 \cup R_9 \cup R_{11}$
$R_{20} = R_{10} \cup R_{11}$	$R_{32} = R_7 \cup R_8 \cup R_{10} \cup R_{11}$
$R_{21} = R_7 \cup R_8 \cup R_9$	$R_{33} = R_7 \cup R_9 \cup R_{10} \cup R_{11}$
$R_{22} = R_7 \cup R_8 \cup R_{10}$	$R_{34} = R_8 \cup R_9 \cup R_{10} \cup R_{11}$
$R_{23} = R_7 \cup R_8 \cup R_{11}$	$R_{35} = R_7 \cup R_8 \cup R_9 \cup R_{10} \cup R_{11}$

Table 5: Definitions of R_3, R_4, and R_{12} - R_{35}

Table 5 lists the definitions of relations R_3 and R_4, as we understand them here, and R_{12} - R_{35}. Table 6 lists the cardinalities of R_3, R_4 and $R_7 - R_{35}$ with references to corresponding integer sequences in Sloane's *Encyclopedia* [5]. Again, as R_7 - R_{11} are defined to be pairwise disjoint, the cardinalities of R_3, R_4, and R_{12} - R_{35} are easily deduced. For example, $|R_3|$ just equals $|R_7| + |R_{10}| + |R_{11}|$.

Before concluding here we note that the cardinalities of R_{30}, R_{26}, R_{35}, and R_{34} may be easily computed using binomial and multichoose coefficients. The *multichoose coefficient* $MC(n, k)$ [1] counts the number of ways to select k elements from an n-element set where any given element may be selected multiple times. For example, it is easily seen that

$$|R_{30}| = C(2^n, 2), \tag{21}$$

as we are effectively selecting 2 distinct elements of the 2^n of $P(A)$ and assigning one to x and the other to y (in some specified manner) for all the $(x, y) \in R_{30}$.

By similar reasoning, we have that

$$|R_{26}| = C(2^n - 1, 2), \tag{22}$$

as $\emptyset$ is excluded from consideration in enumerating the given relation.

It is then also easily seen that

$$|R_{35}| = MC(2^n, 2) \tag{23}$$

and

$$|R_{34}| = MC(2^n - 1, 2) + 1, \tag{24}$$

as $\emptyset$ only occurs in $(\emptyset, \emptyset)$ in R_{34}.

Similar results may be deduced for the remaining relations. However, most are not as pretty as these, so we pass them over here.

Relation	Cardinality	Sequence
R_7	$S(n+1,2)$	A000225(n)
R_8	$S(n+1,3)$	A000392$(n+1)$
R_9	$3 \cdot S(n+1,4)$	A032263$(n+1)$
R_{10}	$2 \cdot S(n+1,3)$	A028243$(n+1)$
R_{11}	$S(n+1,2)+1$	A000079(n)
R_{12}	$S(n+1,3)+S(n+1,2)$	A003462(n)
R_{13}	$3 \cdot S(n+1,4)+S(n+1,2)$	A134018(n)
R_4	$2 \cdot S(n+1,3)+S(n+1,2)$	A001047(n)
R_{14}	$S(n+2,2)$	A000225$(n+1)$
R_{15}	$3 \cdot S(n+1,4)+S(n+1,3)$	A016269$(n-2)$
R_{16}	$3 \cdot S(n+1,3)$	A094033$(n+1)$
R_{17}	$S(n+1,3)+S(n+1,2)+1$	A007051(n)
R_{18}	$3 \cdot S(n+1,4)+2 \cdot S(n+1,3)$	A036239(n)
R_{19}	$3 \cdot S(n+1,4)+S(n+1,2)+1$	A134019(n)
R_{20}	$2 \cdot S(n+1,3)+S(n+1,2)+1$	A083323(n)
R_{21}	$3 \cdot S(n+1,4)+S(n+1,3)+S(n+1,2)$	A133789(n)
R_{22}	$S(n+2,3)$	A000392$(n+2)$
R_{23}	$S(n+1,3)+S(n+2,2)$	A094374(n)
R_{24}	$3 \cdot S(n+1,4)+2 \cdot S(n+1,3)+S(n+1,2)$	A053154(n)
R_{25}	$3 \cdot S(n+1,4)+S(n+2,2)$	A134045(n)
R_3	$2 \cdot S(n+1,3)+S(n+2,2)$	A000244(n)
R_{26}	$3 \cdot (S(n+1,4)+S(n+1,3))$	A134057(n)
R_{27}	$3 \cdot S(n+1,4)+S(n+1,3)+S(n+1,2)+1$	A084869(n)
R_{28}	$S(n+2,3)+1$	A134063(n)
R_{29}	$3 \cdot S(n+1,4)+2 \cdot S(n+1,3)+S(n+1,2)+1$	A134064(n)
R_{30}	$3 \cdot S(n+1,4)+S(n+2,3)$	A006516(n)
R_{31}	$3 \cdot S(n+1,4)+S(n+1,3)+S(n+2,2)$	A134165(n)
R_{32}	$S(n+2,3)+S(n+1,2)+1$	A053156$(n+1)$
R_{33}	$3 \cdot S(n+1,4)+2 \cdot S(n+1,3)+S(n+2,2)$	A134168(n)
R_{34}	$3 \cdot S(n+1,4)+S(n+2,3)+1$	A134169(n)
R_{35}	$3 \cdot S(n+1,4)+S(n+2,3)+S(n+1,2)+1$	A007582(n)

Table 6: The cardinalities of R_3, R_4, and R_7 – R_{35}

3.3 Seven sums associated with R_{35}

Recall that B is just $P(A) \times P(A)$. As $|P(A)| = 2^n$ it is then obvious that $|B| = 4^n$. Another way to see this is that in enumerating the $(x, y) \in B$ there are 4 possible outcomes for each of the elements of A:

1. include in both x and y,
2. exclude from both x and y,
3. exclude from x and include in y, or
4. include in x and exclude from y.

Let $x \triangle y$ denote the *symmetric difference* of sets x and y. It is not hard to see, then, that the sums

$$\begin{aligned} S(n) &= \sum_{(x,y)\in B} |x \triangle y|, \\ T(n) &= \sum_{(x,y)\in B} |x \cap y|, \\ U(n) &= \sum_{(x,y)\in B} |x \cup y| \end{aligned}$$

may be computed by counting relevant possible outcomes for each of the n decisions made in the given enumeration. By correcting these sums for double-counting, the values of the sums

$$\begin{aligned} S_Q(n) &= \sum_{(x,y)\in R_{35}} |x \triangle y|, \\ T_Q(n) &= \sum_{(x,y)\in R_{35}} |x \cap y|, \\ U_Q(n) &= \sum_{(x,y)\in R_{35}} |x \cup y| \end{aligned}$$

may then be easily deduced. This correction is necessary because R_{35} effectively results from removing every $(y, x) \in B$ when $(x, y) \in B$ and $x \neq y$. In other words, let

$$R_{36} = \{(x, y) : x, y \in P(A), x \neq y\}$$

and recall that $R_{11} = \{(x, y) : x, y \in P(A), x = y\}$. This just means that

$$R_{36} = B \setminus R_{11}.$$

As it is easily deduced that

$$R_{30} = \{(x, y) \in R_{35}, x \neq y\} = R_{35} \setminus R_{11},$$

we then have that

$$\frac{1}{2}|R_{36}| = |R_{30}|. \tag{25}$$

Let m be the number of relevant possible outcomes for each decision made in the enumeration of B. It is not hard to see that $m = 2$ for $S(n)$ and $S_Q(n)$, as only outcomes 3 and 4 can make a difference with regard to $|x \triangle y|$ for each of the $(x, y) \in B$. By similar reasoning, we have $m = 1$ for $T(n)$ and $T_Q(n)$ and $m = 3$ for $U(n)$ and $U_Q(n)$. It is not hard to see from this, then, that

$$S(n) = n \cdot \frac{2}{4} \cdot 4^n = 2 \cdot n \cdot 4^{n-1}, \tag{26}$$

$$T(n) = n \cdot \frac{1}{4} \cdot 4^n = n \cdot 4^{n-1}, \tag{27}$$

$$U(n) = n \cdot \frac{3}{4} \cdot 4^n = 3 \cdot n \cdot 4^{n-1}. \tag{28}$$

Now with

$$P(n) = \sum_{x \in P(A)} |x| \tag{29}$$

we may deduce that

$$P(n) = n \cdot \frac{1}{2} \cdot 2^n = n \cdot 2^{n-1} \tag{30}$$

by applying the same reasoning used to establish Eqns. (26) – (28). Furthermore, it is obvious that $|x \triangle x| = 0$ and $|x \cap x| = |x \cup x| = |x|$ for any given finite set x. It is then easily seen that

$$\sum_{(x,y) \in R_{11}} |x \triangle y| = 0, \tag{31}$$

and that with

$$X(n) = \sum_{(x,y) \in R_{11}} |x \cap y| \tag{32}$$

$$= \sum_{(x,y) \in R_{11}} |x \cup y| \tag{33}$$

we have

$$X(n) = P(n) = n \cdot 2^{n-1}, \tag{34}$$

as, again, $x = y$ for all $(x, y) \in R_{11}$. A little thought then shows that

$$S_Q(n) = \frac{1}{2} \cdot (S(n) - 0) + 0, \tag{35}$$

$$T_Q(n) = \frac{1}{2} \cdot (T(n) - X(n)) + X(n), \tag{36}$$

$$U_Q(n) = \frac{1}{2} \cdot (U(n) - X(n)) + X(n), \tag{37}$$

which upon substitution and simplification yields

$$S_Q(n) = n \cdot 4^{n-1}, \tag{38}$$

$$T_Q(n) = \frac{1}{2} \cdot n \cdot (4^{n-1} + 2^{n-1}), \tag{39}$$

$$U_Q(n) = \frac{1}{2} \cdot n \cdot (3 \cdot 4^{n-1} + 2^{n-1}). \tag{40}$$

Table 7 lists the values of the sums computed here with references to corresponding sequences in Sloane's *Encyclopedia* [5].

Sum	Value	Sequence
$S(n)$	$2 \cdot n \cdot 4^{n-1}$	A002699(n)
$T(n)$	$n \cdot 4^{n-1}$	A002697(n)
$U(n)$	$3 \cdot n \cdot 4^{n-1}$	none
$P(n)$	$n \cdot 2^{n-1}$	A001787(n)
$S_Q(n)$	$n \cdot 4^{n-1}$	A002697(n)
$T_Q(n)$	$\frac{1}{2} \cdot n \cdot (4^{n-1} + 2^{n-1})$	A082134(n)
$U_Q(n)$	$\frac{1}{2} \cdot n \cdot (3 \cdot 4^{n-1} + 2^{n-1})$	A133224(n)

Table 7: Seven sums associated with R_{35}

3.4 Wieder connections

As before, let $P(A)$ be the power set of an n-element set A. In Wieder's analysis [6] of coalition structures, four basic sets of k-subsets of $P(A)$ are defined:

conjoint usual k-combinations,

conjoint strict k-combinations,

disjoint usual k-combinations, and

disjoint strict k-combinations.

Let $C_{u,k}$, $C_{s,k}$, $D_{u,k}$, and $D_{s,k}$ respectively denote these sets and fix $k = 2$. $C_{u,2}$ is just the set of all 2-subsets $\{x, y\}$ of $P(A)$; $C_{s,2}$ is the subset of $C_{u,2}$ for which neither x nor y may be equal to $\emptyset$ or A. Similarly, $D_{u,2}$ is the set of all 2-subsets $\{x, y\}$ of $P(A)$ where $x \cap y = \emptyset$; $D_{s,2}$ is the subset of $D_{u,2}$ for which neither x nor y may be equal to $\emptyset$. So we have that

$$C_{s,2} \subset C_{u,2}; \qquad D_{s,2} \subset D_{u,2} \subset C_{u,2};$$
$$D_{s,2} \subset C_{s,2}; \qquad D_{u,2} \not\subseteq C_{s,2}, D_{u,2} \cap C_{s,2} \neq \emptyset.$$

Now recall that $R_{30} = R_{35} \setminus R_{11}$ is an asymmetric binary relation on $P(A)$. This just means that either (x, y) or (y, x) is an element of R_{30} such that $x \neq y$. So we may map $\{x, y\} = \{y, x\}$ in $C_{u,2}$ to whichever of either (x, y) or (y, x) occurs in this relation. Moreover, it is clear that this mapping is invertible. In other words, there is a one-to-one correspondence between $C_{u,2}$ and R_{30}. Let

$$R_{37} = (P(A) \times A) \setminus \{(\emptyset, A), (A, A)\}$$

so that

$$R_{38} = R_{26} \setminus R_{37}.$$

Then we have that

$$\begin{aligned} R_{38} &\subset R_{30}; & R_8 &\subset R_{12} \subset R_{30}; \\ R_8 &\subset R_{38}; & R_{12} &\not\subseteq R_{38}, R_{12} \cap R_{38} \neq \emptyset. \end{aligned}$$

It is then easily shown that the following one-to-one correspondences also obtain:

$$\begin{aligned} C_{s,2} &\leftrightarrow R_{38}, \\ D_{u,2} &\leftrightarrow R_{12}, \\ D_{s,2} &\leftrightarrow R_8. \end{aligned}$$

It is clear that we could extend this analysis to any remaining sets of 2-subsets of $P(A)$ and corresponding asymmetric binary relations on the same. More generally, we might consider sets of k-subsets (or even k-multisubsets) on any given well-defined finite set and corresponding asymmetric k-ary relations. Whatever the merits might be of such investigations, we leave as an open question here.

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