Journal of Integer Sequences, Vol. 12 (2009),
Article 09.4.6

A Note on the Lonely Runner Conjecture

Ram Krishna Pandey
Département de Mathématiques
Université Jean Monnet
23, rue Dr. Paul Michelon
42023 Saint-Etienne
France
ram.krishna.pandey@univ-st-etienne.fr

Abstract

Suppose n runners having nonzero distinct constant speeds run laps on a unit-length circular track. The *Lonely Runner Conjecture* states that there is a time at which all the n runners are simultaneously at least $1/(n+1)$ units from their common starting point. The conjecture has been already settled up to six ($n \leq 6$) runners and it is open for seven or more runners. In this paper the conjecture has been proved for two or more runners provided the speed of the $(i+1)$th runner is more than double the speed of the ith runner for each i, arranged in an increasing order.

1 Introduction and Summary

The conjecture in its original form stated by Wills [10] and also independently by Cusick [6] is as follows:

For any n positive integers $w_1, w_2, \ldots, w_n$, there is a real number x such that

$$\|w_i x\| \geq \frac{1}{n+1},$$

for each $i = 1, 2, \ldots, n$, where for a real number x, $\|x\|$ is the distance of x from the nearest integer.

Due to the interpretation by Goddyn [4], the conjecture is now known as the "Lonely Runner Conjecture".

> Suppose n runners having nonzero distinct constant speeds run laps on a unit-length circular track. Then there is a time at which all the n runners are simultaneously at least $1/(n+1)$ units from their common starting point.

The term "lonely runner" reflects an equivalent formulation in which there are $n+1$ runners with distinct speeds.

> Suppose $n+1$ runners having nonzero distinct constant speeds run laps on a unit-length circular track. A runner is called lonely if the distance (on the circular track) between him (or her) and every other runner is at least $1/(n+1)$. The conjecture is equivalent to asserting that for each runner there is a time when he (or she) is lonely.

The case $n=2$ is very simple. For $n=3$, Betke and Wills [3] settled the conjecture while Wills was dealing with some Diophantine approximation problem and also independently by Cusick [6] while Cusick was considering n-dimensional "view-obstruction" problem. The case $n=4$ was first proved by Cusick and Pomerence [7] with a proof that requires a work of electronic case checking. Later, Bienia et al. [4] gave a simpler proof for $n=4$. The case $n=5$ was proved by Bohman, Holzman and Kleitman [5]. A simpler proof for the case $n=5$ was given by Renault [9]. Recently, Barajas and Serra ([1], [2]) proved the conjecture for $n=6$. Goddyn and Wong [8] gave some tight instances of the lonely runner. For $n \geq 7$ the conjecture is still open. We prove the conjecture for two or more runners provided the speed of the $(i+1)$th runner is more than double the speed of the ith runner for each i, with the speeds arranged in an increasing order.

2 Main Result

Theorem 1. *Let $M = \{m_1, m_2, \ldots, m_n\}$ where $n \geq 2$, and $(\frac{m_{j+1}}{m_j})(\frac{n-1}{n+1}) \geq 2$ for each $j = 1, 2, \ldots, n-1$. Then there exists a real number x such that*

$$\|m_j x\| \geq \frac{1}{n+1},$$

for each $j = 1, 2, \ldots, n$.

Proof. Consider an interval $I = [u, v] = [\frac{1}{m_1(n+1)}, \frac{n}{m_1(n+1)}]$. Clearly, for $x \in I$, we have $\|m_1 x\| \geq \frac{1}{n+1}$, and $v - u = \frac{1}{m_1}(\frac{n-1}{n+1})$. Let us denote the interval I by I_1. We now construct the intervals $I_2, I_3, \ldots I_n$ satisfying the following properties:

(a) $I_1 \supset I_2 \supset I_3 \supset \ldots \supset I_n$

(b) For $I_j = [u_j, v_j]$, $v_j - u_j = \frac{1}{m_j}(\frac{n-1}{n+1})$

(c) For each $x \in I_j$ we have $\|m_j x\| \geq \frac{1}{n+1}$

Clearly, I_1 satisfies (b) and (c). Inductively, we now define the jth interval $I_j = [u_j, v_j]$. We have

$$m_j v_{j-1} - m_j u_{j-1} = \frac{m_j}{m_{j-1}} \left(\frac{n-1}{n+1} \right) \geq 2.$$

Therefore, there exists an integer $\ell(j)$ such that

$$m_j u_{j-1} \leq \ell(j) < \ell(j) + 1 \leq m_j v_{j-1} \Rightarrow u_{j-1} \leq \frac{\ell(j)}{m_j} < \frac{\ell(j)+1}{m_j} \leq v_{j-1}.$$

Define,

$$I_j = [u_j, v_j] = \left[\frac{\ell(j) + \frac{1}{n+1}}{m_j}, \frac{\ell(j) + \frac{n}{n+1}}{m_j} \right].$$

It can be seen easily that the interval I_j satisfies all (a), (b) and (c). Since the intersection of the intervals $I_1, I_2, \ldots I_n$ is nonempty therefore, we have the theorem. $\square$

In the theorem we have seen that the n runners having their speeds $r_1, r_2, \ldots, r_n$ with $(\frac{r_{j+1}}{r_j})(\frac{n-1}{n+1}) \geq 2$ satisfy the Lonely Runner Conjecture.

3 Acknowledgements

I am very much thankful to the referee for his/her useful suggestions to present the paper in a better form.

References

[1] J. Barajas and O. Serra, Regular chromatic number and the lonely runner problem, *Electron. Notes Discrete Math.*, **29** (2007), 479–483.

[2] J. Barajas and O. Serra, The lonely runner with seven runners, *Electron. J. Combin.*, **15** (2008), Paper # R48.

[3] U. Betke and J. M. Wills, Untere Schranken für zwei diophantische Approximations-Funktionen, *Monatsh. Math.*, **76** (1972), 214–217.

[4] W. Bienia, L. Goddyn, P. Gvozdjak, A. Sebő and M. Tarsi, Flows, view obstructions and the lonely runner, *J. Combin. Theory Ser. B*, **72** (1998), 1–9.

[5] T. Bohman, R. Holzman and D. Kleitman, Six lonely runners, *Electron. J. Combin.*, **8** (2001), Paper # R3.

[6] T. W. Cusick, View-obstruction problems in n-dimensional geometry, *J. Combin. Theory Ser. A*, **16** (1974), 1–11.

[7] T. W. Cusick and C. Pomerance, View-obstruction problems III, *J. Number Theory* , **19** (1984), 131–139.

[8] L. Goddyn and E. B. Wong, Tight instances of the lonely runner, *Integers*, **6** (2006), Paper A38.

[9] J. Renault, View-obstruction: a shorter proof for six lonely runners, *Discrete Math.*, **287** (2004), 93–101.

[10] J. M. Wills, Zwei Sätze uber inhomogene diophantische Approximation von Irrationalzahlen, *Monatsh. Math.*, **71** (1967), 263–269.

2000 *Mathematics Subject Classification*: Primary 11B50; Secondary 11B75, 11A99.
Keywords: integers, distance from the nearest integer.

Received April 16 2009; revised version received June 4 2009. Published in *Journal of Integer Sequences*, June 5 2009.

Return to Journal of Integer Sequences home page.