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A Curious Bijection on Natural Numbers

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Abstract

We give a greedy algorithm for describing an enumeration of the set of all natural numbers such that the sum of the first n terms of the sequence is divisible by n for each natural number n. We show that this leads to a bijection f of the set of all natural numbers onto itself that has some nice properties. We also show that the average function of the first n terms of the sequence satisfies a functional equation which completely describes all the properties of the function f. In particular, f turns out to be an *involution* on the set of all natural numbers.

1 Introduction

The following problem was posed by A. Shapovalov [1]:

> *Does there exist a sequence of positive integers containing each positive integer exactly once such that the sum of the first k terms is divisible by k for each $k = 1, 2, 3, \ldots$?*

The published solution, though ingenious, is not intuitive. Shapovalov inductively defines a sequence $\langle a_n \rangle$, which appears as a sequence A019444 in Sloane's Encyclopedia of Integer Sequences [3], as follows. Put $a_1 = 1$ and having defined $a_1, a_2, \ldots, a_k$, first compute $n_k = (a_1 + a_2 + \cdots + a_k)/k$. Now define

$$a_{k+1} = \begin{cases} n_k, & \text{if } n_k \text{ is not already in the set } \{a_1, a_2, \ldots, a_k\}; \\ n_k + k + 1, & \text{if } n_k \text{ is in the set } \{a_1, a_2, \ldots, a_k\}. \end{cases}$$

It is not clear, a priori, that n_k is an integer – which is necessary to conclude that k divides $a_1 + a_2 + \cdots + a_k$. The proposer first proves this by induction and then proceeds to prove that such a sequence indeed has the required property.

Equivalently one has to find a bijection f on $\mathbb{N}$, the set of all natural numbers, such that n divides $f(1) + f(2) + \cdots + f(n)$ for each n. A natural approach is the so called the *greedy algorithm*. One can start with $f(1) = 1$ and having defined $f(j)$ for $1 \leq j \leq n$, the choice for $f(n+1)$ is the least positive integer l, not in the set $\{f(1), f(2), \ldots, f(n)\}$, such that $(n+1)$ divides $f(1) + f(2) + \cdots + f(n) + l$. Since $f(1) = 1$, the value of $f(2)$ cannot be equal to 1 although $f(1) + 1$ is divisible by 2. The natural choice is $f(2) = 3$, for then 3 is not yet assumed by the function f and $1 + 3 = 4$ is divisible by 2. Since $f(1) + f(2) + 2 = 6$ is divisible by 3 and 2 is not in the set $\{f(1), f(2)\}$, the algorithm proposes $f(3) = 2$. Now, although $f(1) + f(2) + f(3) + 2 = 8$ is divisible by 4, it is not possible to choose $f(4) = 2$ since 2 is already in the set $\{f(1), f(2), f(3)\}$. However if we add 4 more, then $8 + 4 = 12$ is also divisible by 4 and $2 + 4 = 6$ is not in the set $\{f(1), f(2), f(3)\}$. Thus it is natural to choose $f(4) = 6$. A similar reasoning shows that $f(5)$ may be chosen 8 and $f(6)$ may be chosen 4. This process can be continued to compute $f(n)$ for at least small values of n. The following table gives an idea how the *greedy algorithm* works and the nature of $f(n)$ for small n. The table also includes the *average function* $h(n) = \dfrac{f(1) + f(2) + \cdots + f(n)}{n}$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$f(n)$	1	3	2	6	8	4	11	5	14	16	7	19	21	9	24	10	27	29	12
$h(n)$	1	2	2	3	4	4	5	5	6	7	7	8	9	9	10	10	11	12	12

There are some interesting properties of the functions $f(n)$ and $h(n)$ as evident from the table, some are simple and some are more deep. In fact we have the following results.

Lemma 1. *Let f and h be functions defined on $\mathbb{N}$ as above. Then*

(**I**) *h is a nondecreasing function on $\mathbb{N}$ and $h(n) \leq n$;*

(**II**) *$h(n+1) = h(n)$ or $h(n) + 1$, for all $n \in \mathbb{N}$;*

(**III**) *$h(n+1) = h(n) \iff f(n+1) = h(n)$, for all $n \in \mathbb{N}$;*

(**IV**) *$h(n+1) = h(n) + 1 \iff f(n+1) = h(n) + n + 1$, for all $n \in \mathbb{N}$.*

Much deeper properties of the functions f and h are given by Theorem 1.

Theorem 1. *The functions f and h also satisfy, for all $n \in \mathbb{N}$,*

(**V**) *$h\big(h(n)\big) + h\big(n+1\big) = n + 2$;*

(**VI**) *$f\big(f(n)\big) = n$;*

(**VII**) *$h\big(h(n) + n\big) = n + 1$.*

A close look at this way of defining f and Shapovalov's sequence $\langle a_n \rangle$ shows that $f(n) = a_n$. Indeed a_{k+1} is defined using the previous average n_k: $a_{k+1} = n_k$ if n_k is not already a_j for some $j \le k$; and $a_{k+1} = n_k + k + 1$ if n_k is used up to define some a_j. This is precisely forced in greedy algorithm, but one need to prove it. The table also exhibits some nice properties of h and f. The most intriguing are (**V**), (**VI**) and (**VII**). In particular the property (**VI**) shows that f is indeed a bijection. We prove all these in the sequel.

The involutive property of f is mentioned in [3]. Apart from this, no other property is mentioned in the literature to the best knowledge of author.

Among several properties of the functions h and f as stated in Theorem 1, is it possible to single out a particular property which tells about the remaining ones? It turns out that (**V**) is indeed one such property. In fact it completely describes all the rest. It is not surprising in view of the fact that the functional equation

$$h\big(h(n)\big) + h(n+1) = n+2,$$

for all $n \in \mathbb{N}$, uniquely determines the function h. The surprising fact is that this function is related to the *Golden Ratio* α, given by $\alpha = (1+\sqrt{5})/2$. In fact, one can show that $h(n) = \lfloor n\alpha \rfloor - n + 1$, where $\lfloor x \rfloor$ denotes the greatest integer not exceeding x; [2]. However, without actually computing $h(n)$ explicitly, it is possible to show that the above relation directly leads to a bijection of the desired type. The following theorem elucidates these facts.

Theorem 2. *Suppose $h : \mathbb{N} \to \mathbb{N}$ is a function such that*

$$h\big(h(n)\big) + h(n+1) = n+2, \tag{1}$$

holds for all $n \in \mathbb{N}$. Define $f : \mathbb{N} \to \mathbb{N}$ by $f(1) = 1$ and

$$f(n+1) = \begin{cases} h(n), & \text{if } h(n+1) = h(n); \\ h(n) + n + 1, & \text{if } h(n+1) = h(n) + 1. \end{cases}$$

Then f is a bijection on $\mathbb{N}$ such that for each $n \in \mathbb{N}$, the sum $f(1) + f(2) + \cdots + f(n)$ is divisible by n.

Thus the functional equation (1) on $\mathbb{N}$ uniquely determines a function h on $\mathbb{N}$ and this leads to a bijection f on $\mathbb{N}$ such that for every natural number n the sum $f(1) + f(2) + \cdots + f(n)$ is divisible by n, which was sought by Shapovalov.

2 Proof of Lemma 1

We prove some simple properties of f and h in this section, as stated in Lemma 1.

Proof. We use induction on n. These are easy to verify for $n = 1, 2$ using the table of values of f and h given earlier. Suppose these hold for all j, where $1 \le j \le n$. Thus we have $h(j) \le j$, for $1 \le j \le n$; $h(j+1) = h(j)$ or $h(j) + 1$, for $1 \le j \le n-1$. This in particular implies that $h(1) \le h(2) \le \cdots \le h(n)$. Moreover $f(j+1) = h(j) \iff h(j+1) = h(j)$, for

$1 \le j \le n-1$ and $f(j+1) = h(j)+j+1 \Longleftrightarrow h(j+1) = h(j)+1$, for $1 \le j \le n-1$. Let l be the least positive integer such that $(n+1)$ divides $f(1)+f(2)+\cdots+f(n)+l$. Thus

$$(n+1)k = f(1)+f(2)+\cdots+f(n)+l = nh(n)+l,$$

for some k. Since $nh(n)+h(n)$ is divisible by $(n+1)$, the definition of l implies that $l \le h(n)$. Suppose, if possible, $l < h(n)$. Then $nh(n)+l$ and $nh(n)+h(n)$ are both divisible by $(n+1)$, and

$$nh(n) < nh(n)+l < nh(n)+h(n).$$

Thus we see that $h(n)-l$ is divisible by $n+1$. This forces $h(n)-l \ge n+1$ contradicting the induction hypothesis $h(n) \le n$. We conclude that $l = h(n)$. In other words, the least positive integer l such that $(n+1)$ divides $f(1)+f(2)+\cdots+f(n)+l$ is equal to $h(n)$, the average of $f(1)+f(2)+\cdots+f(n)$.

If $h(n)$ does not belong to the set $\{f(1), f(2), \ldots, f(n)\}$, then the definition of $f(n+1)$, via the greedy algorithm, shows that $f(n+1) = h(n)$. If $h(n)$ is already in the set $\{f(1), f(2), \ldots, f(n)\}$, then we consider $h(n)+n+1$. Obviously $f(1)+f(2)+\cdots+f(n)+h(n)+n+1$ is a multiple of $(n+1)$. It is our intention to prove that $h(n)+n+1$ is not in the set $\{f(1), f(2), \ldots, f(n)\}$. Suppose to the contrary that $h(n)+n+1$ lies in the set $\{f(1), f(2), \ldots, f(n)\}$. Then $h(n)+n+1 = f(j)$ for some $j \le n$. Induction hypothesis shows that $f(j) = h(j-1)$ or $h(j-1)+j$. If the first alternative holds, then

$$h(n)+n+1 = f(j) = h(j-1) \le j-1 < n,$$

a clear contradiction. If on the other hand $f(j) = h(j-1)+j$, then

$$h(n)+n+1 = h(j-1)+j.$$

This forces $h(n)-h(j-1) = j-n-1 < 0$, which is impossible since $h(j-1) \le h(n)$ by induction hypothesis. Thus $h(n)+n+1$ is not an element of $\{f(1), f(2), \ldots, f(n)\}$. It follows that $h(n)+n+1$ is the least positive integer l not in the set $\{f(1), f(2), \ldots, f(n)\}$ such that $(n+1)$ divides $f(1)+f(2)+\cdots+f(n)+l$. By definition $f(n+1) = h(n)+n+1$.

Thus $f(n+1) = h(n)$ or $h(n)+n+1$. In the first case

$$h(n+1) = \frac{f(1)+f(2)+\cdots+f(n)+h(n)}{n+1} = \frac{nh(n)+h(n)}{n+1} = h(n).$$

Similarly $h(n+1) = h(n)+1$ in the second case. On the other hand, $h(n+1) = h(n)$ forces $f(n+1) = h(n)$ and $h(n+1) = h(n)+1$ implies that $f(n+1) = h(n)+n+1$. It may also be observed that $h(n+1) = h(n)$ or $h(n)+1$ gives $h(n) \le h(n+1)$ and $h(n+1) \le n+1$. This proves inductive step and the properties (**I**)-(**IV**) are true for all values of n. $\square$

3 Some more consequences

Here are some more consequences of the definition and the properties proved in Lemma 1. These ar useful in completing the proof of Theorem 1.

- *The function h does not assume the same value at three consecutive integers and hence it does not assme the same value at three distinct integers. Consequently $h(n+2) > h(n)$ for all $n \geq 1$.*

  Since h is a nondecreasing function of n, we have $h(n) \leq h(n+1) \leq h(n+2)$. If $h(n+2) = h(n+1)$, then $f(n+2) = h(n+1)$ and hence $h(n+1)$ is not in the set $\{f(1), f(2), \ldots, f(n), f(n+1)\}$, by definition. But $h(n+1) = h(n)$ implies that $f(n+1) = h(n) = h(n+1)$ showing that $h(n+1)$ is an element of the set $\{f(1), f(2), \ldots, f(n), f(n+1)\}$. (In fact equal to $f(n+1)$.) This contradiction proves the statement.

- *The function h is surjective.*

  This follows from the fact that h is nondecreasing, it increases in steps of 0 or 1 and $h(n+2) > h(n)$ for all $n \geq 1$.

- *The function f is surjective.*

  Take any $m \in \mathbb{N}$, and let $m = h(k)$. Such a k exists because h is surjective. Then either $h(k)$ belongs to the set $\{f(1), f(2) \ldots, f(k)\}$ or $f(k+1) = h(k)$. Thus m is in the range of f.

- *The function f is one-one.*

  Suppose $f(m) = f(k)$ for some $m < k$. The property (**IV**) of f shows that $f(k) = h(k-1)$ or $h(k-1) + k$. If $f(k) = h(k-1)$, then $f(m) = h(k-1)$ belongs to the set $\{f(1), f(2), \ldots, f(m)\}$ which itself is a subset of $\{f(1), f(2), \ldots, f(k-1)\}$, as $m \leq k-1$. By definition of f, it follows $f(k) = h(k-1) + k$ contradicting $f(k) = h(k-1)$.

  Suppose on the other hand $f(k) = h(k-1)+k$. Now $f(m) = h(m-1)$ or $h(m-1)+m$. Since $m-1 < k-1$, we have $h(m-1) \leq h(k-1)$ and hence

  $$h(m-1) = f(m) = f(k) = h(k-1) + k$$

  is impossible. If $h(k-1) + k = h(m-1) + m$, we have

  $$h(k-1) - h(m-1) = m - k < 0,$$

  which again is impossible since $h(m-1) \leq h(k-1)$. Thus $f(m) \neq f(k)$ if $m \neq k$.

- *The inequality $h(n) \leq n-2$ holds for all $n \geq 6$.*

  This follows from the observation $h(6) = 4$ and and by an easy induction using the fact that h increases in steps of 0 or 1.

- *If $f(n+1) > h(n)$, then $f(j) > h(n)$ for all $j \geq n+1$.*

  In fact

  $$\begin{aligned} f(n+2) \geq h(n+1) &= \frac{f(1) + f(2) + \cdots + f(n+1)}{n+1} \\ &= \frac{nh(n) + f(n+1)}{n+1} \\ &> h(n). \end{aligned}$$

An easy induction proves that $f(j) > h(n)$ for all $j \geq n+1$.

- *There are no integers $k \geq 2$ and l such that $f(k-1) = l$ and $f(k) = l+1$. In other words, f does not assume consecutive values at consecutive integers.*

  Suppose such a pair $k \geq 2$ and l exist. We may assume $k > 2$, for this result is immediate for $k = 2$. Thus $f(k) = f(k-1) + 1$ and hence

$$\begin{aligned} kh(k) &= f(1) + f(2) + \cdots + f(k) \\ &= f(1) + f(2) + \cdots + f(k-2) + 2f(k-1) + 1 \\ &= (k-2)h(k-2) + 2f(k-1) + 1. \end{aligned}$$

  This implies that

$$k\big(h(k) - h(k-2)\big) = -2h(k-2) + 2f(k-1) + 1.$$

  Now $h(k)-h(k-2) = 1$ or 2, since h does no assume the same value at three consecutive integers. It cannot be equal to 2, for then left side is even and right side is odd. Thus $h(k) = h(k-2) + 1$ and hence

$$\begin{aligned} k &= -2h(k-2) + 2f(k-1) + 1 \\ &= 2\big(f(k-1) - h(k-2)\big) + 1. \end{aligned}$$

  But $f(k-1) = h(k-2)$ or $h(k-2) + k - 1$. In both the cases $k = 1$ contradicting $k > 2$.

4 Proof of Theorem 1

We prove the deeper properties of h and f as stated in Theorem 1. The following properties hold for all values of $n \in \mathbb{N}$:

(V) $h\big(h(n)\big) + h\big(n+1\big) = n+2$;

(VI) $f\big(f(n)\big) = n$;

(VII) $h\big(h(n)+n\big) = n+1$.

We use induction to prove these statements. For $n = 1$, it is a routine verification:

$$f(1) = 1, h(1) = 1, h(2) = 2, h(1) + 1 = 2;$$

hence

$$h\big(h(n)\big) + h\big(n+1\big) = h\big(h(1)\big) + h\big(2\big) = h(1) + h(2) = 3 = n+2;$$
$$f\big(f(n)\big) = f\big(f(1)\big) = f\big(1\big) = 1 = n;$$
$$h\big(h(n)+n\big) = h\big(h(1)+1\big) = h\big(2\big) = 2 = n+1.$$

These may also be verified for $n = 2, 3, 4, 5, 6$. So we assume that $m > 6$ and (**V**)-(**VII**) are true for all $n \leq m$. We prove them for $n = m+1$. We have to prove three statements:

(**a**) $h\big(h(m+1)\big) + h\big(m+2\big) = m+3;$

(**b**) $f\big(f(m+1)\big) = m+1;$

(**c**) $h\big(h(m+1)+m+1\big) = m+2.$

Proof of (a): We have either $h(m+1) = h(m)$ or $h(m+1) = h(m)+1$. If $h(m+1) = h(m)+1$, we have two possibilities: $h(m+2) = h(m+1)$ or $h(m+2) = h(m+1)+1$. We consider these three cases separately.

Case (i). Suppose $h(m+1) = h(m)$. Since h does not take the same value at three consecutive integers, it follows that $h(m+2)$ cannot be equal to $h(m+1)$. Hence $h(m+2) = h(m+1)+1$. Induction hypothesis gives

$$\begin{aligned} h\big(h(m+1)\big) + h\big(m+2\big) &= h\big(h(m)\big) + h\big(m+1\big) + 1 \\ &= (m+2)+1 = m+3. \end{aligned}$$

Case (ii). Suppose on the other hand $h(m+1) = h(m)+1$, but $h(m+2) = h(m+1)$. Here we have either $h\big(h(m)+1\big) = h\big(h(m)\big)$ or $h\big(h(m)+1\big) = h\big(h(m)\big)+1$. If $h\big(h(m)+1\big) = h\big(h(m)\big)$, then it must be the case that $h\big(h(m)+2\big) = h\big(h(m)+1\big)+1$; otherwise h assumes the same value at three consecutive integers $h(m)$, $h(m)+1$ and $h(m)+2$. Thus using the property (**IV**), we obtain

$$\begin{aligned} f\big(h(m)+2\big) &= h\big(h(m)+1\big) + h(m) + 2 \\ &= h\big(h(m)+1\big) + h(m+1) + 1 \\ &= h\big(h(m)\big) + h(m+1) + 1 \\ &= (m+2)+1 \\ &= m+3, \end{aligned}$$

where induction hypothesis is invoked. But $h(m)+2 \le m$ for $m \ge 6$. Again by induction hypothesis, we get

$$f\Big(f\big(h(m)+2\big)\Big) = h(m)+2.$$

It follows that

$$\begin{aligned} f(m+3) &= h(m)+2 \\ &= h(m+1)+1 = h(m+2)+1. \end{aligned}$$

But this is impossible, since either $f(m+3) = h(m+2)$ or $h(m+2)+m+3$. We conclude that $h\big(h(m)+1\big) = h\big(h(m)\big)$ is not possible.

Thus we must have $h\big(h(m)+1\big) = h\big(h(m)\big)+1$. In this case

$$\begin{aligned} h\big(h(m+1)\big) + h\big(m+2\big) &= h\big(h(m)+1\big) + h(m+1) \\ &= h\big(h(m)\big) + 1 + h(m+1) \\ &= (m+2)+1 = m+3, \end{aligned}$$

where again induction hypothesis that $h\big(h(m)\big) + h(m+1) = m+2$ is used.

Case (iii). Suppose $h(m+1) = h(m)+1$ and $h(m+2) = h(m+1)+1$. Here again we have two possibilities: $h\big(h(m)+1\big) = h\big(h(m)\big)$ or $h\big(h(m)+1\big) = h\big(h(m)\big)+1$.

If $h\big(h(m)+1\big) = h\big(h(m)\big)$ holds, we have

$$\begin{aligned} h\big(h(m+1)\big) + h\big(m+2\big) &= h\big(h(m)+1\big) + h(m+1) + 1 \\ &= h\big(h(m)\big) + h(m+1) + 1 \\ &= (m+2) + 1 = m+3, \end{aligned}$$

using $h\big(h(m)\big) + h(m+1) = m+2$.

If on the other hand $h\big(h(m)+1\big) = h\big(h(m)\big)+1$, then the property (**IV**) shows that

$$\begin{aligned} f\big(h(m)+1\big) &= h\big(h(m)\big) + h(m) + 1 \\ &= h\big(h(m)\big) + h(m+1) \\ &= m+2. \end{aligned}$$

Since $h(m+1) \le m$, induction hypothesis gives

$$h(m+1) = f\Big(f\big(h(m+1)\big)\Big) = f(m+2).$$

But then (**III**) implies that $h(m+2) = h(m+1)$ contradicting $h(m+2) = h(m+1)+1$. Thus the case $h\big(h(m)+1\big) = h\big(h(m)\big)+1$ cannot occur when $h(m+1) = h(m)+1$ and $h(m+2) = h(m+1)+1$.

This completes the proof of (**V**) for $n = m+1$.

Proof of (b): Here again there are two cases: $f(m+1) = h(m)$ or $f(m+1) = h(m)+m+1$.

Case (i). Suppose $f(m+1) = h(m)$. By the property (**III**), we have $h(m+1) = h(m)$. Put $j = h(m)-1$. Since $h(m+1) = h(m)$ and h does not assume the same value at three consecutive integers, we must have $h(m) = h(m-1)+1$. Thus $j+1 = h(m) = f(m+1)$ and

$$\begin{aligned} h(j+1) + j &= h\big(h(m)\big) + h(m) - 1 \\ &= h\big(h(m)\big) + h(m+1) - 1 \\ &= m+1; \end{aligned}$$

we have used induction hypothesis that $h\big(h(m)\big) + h(m+1) = m+2$.

Suppose, if possible, $h(j+1) = h(j)$. Then $h(j)+j = h(j+1)+j = m+1$ and hence

$$h\big(h(m)-1\big) + h(m) - 1 = m+1.$$

Now $h(m)-1 = h(m-1)$ and hence

$$h\big(h(m)-1\big) + h(m) - 1 = h\big(h(m-1)\big) + h(m) - 1 = (m+1) - 1 = m,$$

by induction hypothesis. Thus we obtain an absurd conclusion that $m+1 = m$. It follows that $h(j+1) = h(j)+1$ and hence one obtains from (**IV**)

$$f(j+1) = h(j) + j + 1 = h(j+1) + j = m+1.$$

It follows that $f\big(f(m+1)\big) = m+1$.

Case (ii). Suppose $f(m+1) = h(m) + m + 1$, so that $h(m+1) = h(m) + 1$. Here put $j = h(m) + m$. Thus $j + 1 = h(m) + m + 1 = f(m+1)$, and using induction hypothesis one obtains

$$h(j) = h\big(h(m) + m\big) = m + 1.$$

We show that $h(j+1) = h(j)$ in this case. Suppose the contrary that $h(j+1) = h(j) + 1$. Then (**IV**) shows that

$$f(j+1) = h(j) + j + 1 > h(j).$$

The property of f shows that (section **3**), $f(l) > h(j)$ for all $l \geq j+1$. Since f is surjective, it must be the case that

$$h(j) \in \Big\{f(1), f(2), \ldots, f(j)\Big\}.$$

Thus $h(j) = f(r)$ for some $r \leq j$. If $m+1 < r < j$, then $f(r) = h(j) = m+1 < r$. However $f(r) = h(r-1)$ or $h(r-1) + r$. The bound $f(r) < r$ implies that $f(r) = h(r-1)$ which corresponds to $h(r) = h(r-1)$. Thus it follows $h(r-1) = h(r) = f(r) = h(j)$, where $r - 1 < r < j$. However this contradicts the property of h that it does not assume the same value at three integers. If $r = m+1$, then again $h(j) = f(r) = f(m+1) = j+1$ contradicting $h(j) \leq j$. If $r < m+1$, then $f(r) = h(j) = m+1$ and hence $f(m+1) = f\big(f(r)\big) = r$ by induction hypothesis. This implies that $f(m+1) = r \leq m$. However $f(m+1) = h(m)$ or $h(m) + m + 1$. Using $f(m+1) \leq m$, it may be concluded that $f(m+1) = h(m)$. But then $h(m+1) = h(m)$, contradicting $h(m+1) = h(m) + 1$. The only choice left is $r = j$. Thus $h(j) = f(r) = f(j)$ and this gives

$$f(j) = h(j) = h\big(h(m) + m\big) = m + 1 \leq j.$$

Using $f(j) = h(j-1)$ or $h(j-1)+j$, it may be concluded that $f(j) = h(j-1)$. Using (**III**), $h(j) = h(j-1) = f(j)$ and

$$f(j) = h(j-1) = h(j) = h\big(h(m) + m\big) = m + 1.$$

Using induction hypothesis,

$$h\big(h(m-1) + m - 1\big) = m.$$

Comparing this with $h\big(h(m) + m - 1\big) = h(j-1) = h(j) = m+1$, it follows that $h(m) = h(m-1) + 1$. Thus, we obtain

$$f(m) = h(m-1) + m = h(m) - 1 + m = j - 1.$$

Using induction hypothesis, we also have $f\big(f(m)\big) = m$. Thus two relations $f(j-1) = m$ and $f(j) = m+1$ are obtained. But this is impossible since f does not assume consecutive values at consecutive integers. We conclude that $h(j+1) = h(j)$. In this case

$$f(j+1) = h(j) = m + 1,$$

and hence $f\big(f(m+1)\big) = m+1$.

Proof of (c): Here again there are two possibilities: $h(m+1) = h(m)$ or $h(m)+1$.

Case (i). Suppose $h(m+1) = h(m)+1$. Then

$$\begin{aligned} h\big(h(m+1)+m+1\big) &= h\big(h(m)+m+2\big) \\ &= h\big(h(m)+m\big)+1 \ \ or \ \ h\big(h(m)+m\big)+2. \end{aligned}$$

Suppose $h\big(h(m)+m+2\big) = h\big(h(m)+m\big)+2$ holds. Then induction hypothesis gives $h\big(h(m)+m\big) = m+1$ and

$$h\big(h(m+1)+m+1\big) = h\big(h(m)+m\big)+2 = m+1+2 = m+3.$$

We observe that

$$h(m)+m \le h(m)+m+1 \le h(m+1)+m+1.$$

Since h is surjective, we conclude $h\big(h(m)+m+1\big) = m+2$.

If $h\big(h(m)+m\big) = h\big(h(m)+m-1\big)+1$, then $h\big(h(m)+m-1\big) = m$. Moreover the condition $h\big(h(m)+m+1\big) = m+2 = h\big(h(m)+m\big)+1$ implies that $h\big(h(m)+m\big)$ lies in the set $\big\{f(1), f(2), \ldots, f\big(h(m)+m\big)\big\}$. Thus $h\big(h(m)+m\big) = f(r)$ for some $r \le h(m)+m$. Therefore

$$f(r) = h\big(h(m)+m\big) = m+1 = f\big(f(m+1)\big),$$

from (**b**). Since f is one-one, it follows that $r = f(m+1)$. Thus the bound $f(m+1) = r \le h(m)+m$ is obtained. However $f(m+1) = h(m)$ or $h(m)+m+1$. The bound $f(m+1) \le h(m)+m$ shows that $f(m+1) = h(m)$. But then $h(m+1) = h(m)$ contradicting $h(m+1) = h(m)+1$. Thus the relation $h\big(h(m)+m\big) = h\big(h(m)+m-1\big)$ must be true.

We have therefore $h\big(h(m)+m-1\big) = h\big(h(m)+m\big) = m+1$ and $h\big(h(m-1)+m-1\big) = m$. Comparing these two, it may be concluded that $h(m) = h(m-1)+1$. However this corresponds to $f(m) = h(m-1)+m$ and hence by induction hypothesis

$$m = f\big(f(m)\big) = f\big(h(m-1)+m\big).$$

Now we have

$$h\big(h(m-1)+m\big) = h\big(h(m)+m-1\big) = m+1.$$

The properties (**III**) and (**IV**) show that

$$\begin{aligned} f\big(h(m-1)+m\big) &= h\big(h(m-1)+m-1\big) \\ &\quad\ or \ \ h\big(h(m-1)+m-1\big)+h(m-1)+m. \end{aligned}$$

If the first alternative holds, then $h\big(h(m-1)+m\big) = h\big(h(m-1)+m-1\big)$ and hence

$$m+1 = h\big(h(m-1)+m\big) = h\big(h(m-1)+m-1\big) = m,$$

where we have used induction hypothesis in the last equality. This absurdity implies that the first alternative cannot be true.

If the second alternative holds, then again

$$m = f\big(h(m-1)+m\big) = h\big(h(m-1)+m-1\big)+h(m-1)+m > m,$$

which is impossible.

We may thus conclude that $h\big(h(m)+m+2\big) = h\big(h(m)+m\big)+2$ is not valid. But then

$$h\big(h(m)+m+2\big) = h\big(h(m)+m\big)+1 = (m+1)+1 = m+2.$$

Case (ii). Suppose $h(m+1) = h(m)$. Then

$$h\big(h(m+1)+m+1\big) = h\big(h(m)+m+1\big) = h\big(h(m)+m\big) \text{ or } h\big(h(m)+m\big)+1.$$

In the first case

$$f\big(h(m)+m+1\big) = h\big(h(m)+m\big) = m+1 = f\big(f(m+1)\big),$$

by (**b**). Since f is one-one, it follows that $f(m+1) = h(m)+m+1$. But then $h(m+1) = h(m)+1$ contradicting $h(m+1) = h(m)$. Thus the second alternative holds and we obtain

$$h\big(h(m+1)+m+1\big) = h\big(h(m)+m+1\big) = h\big(h(m)+m\big)+1 = m+2.$$

This completes the proofs of (**a**), (**b**) and (**c**). We conclude that (**V**), (**VI**) and (**VII**) hold for all values of n, thus completing the proof of Theorem 1.

5 Proof of Theorem 2

We prove several properties of the function h satisfying the equations (1), which are useful in the proof of Theorem 1. First of all, it is necessary to check that the definition of f makes sense. In other words, it is part of the result that $h(n+1) = h(n)$ or $h(n)+1$ and these are the only possibilities for the growth of h. This is proved as a consequence of the relation (1).

Lemma 2. *Suppose $h : \mathbb{N} \to \mathbb{N}$ satisfies the functional equation (1):*

$$h\big(h(n)\big) + h(n+1) = n+2.$$

Then h is a nondecreasing function on $\mathbb{N}$ and $h(n+1) = h(n)$ or $h(n)+1$ for all $n \in \mathbb{N}$. Moreover $h(n) \leq n$ for all $n \in \mathbb{N}$ and $h(n) \leq n-2$ for all $n \geq 6$.

Proof. We first show that $h(1) = 1$, $h(2) = 2$, $h(3) = 2$, $h(4) = 3$, $h(5) = 4$ and $h(6) = 4$. Putting $n = 1$ in (1), we obtain

$$h\big(h(1)\big) + h(2) = 3.$$

Since all the numbers involved are natural numbers, $h(2) \leq 2$. Thus $h(2) = 1$ or 2. Suppose $h(2) = 1$. Taking $h(1) = k$, the above relation gives $h(k) = 2$. Taking $n = 2$ in (1), we also get

$$h\big(h(2)\big) + h(3) = 4.$$

Thus $h(3) = 4 - h(1) = 4 - k$. Since $h(3) \geq 1$, it follows that $k \leq 3$. Thus $k = 1$, 2 or 3. If $k = 1$, then

$$2 = h\big(h(1)\big) = h(k) = h(1) = k = 1,$$

a contradiction. If $k=2$, then again

$$2 = h\big(h(1)\big) = h(k) = h(2) = 1,$$

giving a contradiction. Finally $k=3$ gives

$$2 = h\big(h(1)\big) = h(k) = h(3) = 4-k = 4-3 = 1,$$

which again is absurd. Thus $h(2)=1$ is not feasible. It follows $h(2)=2$ and $h(k)=1$. Now taking $n=2$ in (1), we obtain

$$h\big(h(2)\big) + h(3) = 4.$$

This leads to

$$h(3) = 4 - h\big(h(2)\big) = 4 - h(2) = 4-2 = 2.$$

Recursively, we get

$$\begin{aligned}
h(4) &= 5 - h\big(h(3)\big) = 5 - h(2) = 5-2 = 3,\\
h(5) &= 6 - h\big(h(4)\big) = 6 - h(3) = 6-2 = 4,\\
h(6) &= 7 - h\big(h(5)\big) = 7 - h(4) = 7-3 = 4.
\end{aligned}$$

Now an easy induction shows that $h(n) \ge 2$ for all $n \ge 2$. Suppose $h(1) = k \ge 2$. Then

$$3 = h\big(h(1)\big) + h(2) = h(k) + h(2) \ge 2+2 = 4,$$

which is impossible. Thus $h(1)=1$ and the the first few values of h are obtained.

The values of $h(n)$ for $n = 1,2,3$ show that

$$\begin{aligned}
h(2) &= 2 = h(1)+1,\\
h(3) &= 2 = h(2).
\end{aligned}$$

Suppose $h(j) = h(j-1)$ or $h(j-1)+1$ for all $j \le n$. Now (1) gives

$$\begin{aligned}
h\big(h(n)\big) + h(n+1) &= n+2,\\
h\big(h(n-1)\big) + h(n) &= n+1.
\end{aligned}$$

Subtraction gives

$$h\big(h(n)\big) - h\big(h(n-1)\big) + h(n+1) - h(n) = 1.$$

If $h(n) = h(n-1)$, then the above relation gives $h(n+1) - h(n) = 1$. If $h(n) = h(n-1)+1$, then

$$h\big(h(n)\big) = h\big(h(n-1)+1\big) = h\big(h(n-1)\big) \quad or \quad h\big(h(n-1)\big)+1,$$

since $h(n-1) \le n-1$ and the induction hypothesis is applicable. In the first case, $h(n+1) = h(n)+1$; in the second case, $h(n+1) = h(n)$. Thus it follows that $h(n+1) = h(n)$ or $h(n)+1$. Hence h increases in steps of 0 or 1.

It also follows that $h(j) \le h(k)$ for $j \le k$. Moreover the proof also reveals that whenever $h(n) = h(n-1)+1$ and $h\big(h(n-1)+1\big) = h\big(h(n-1)\big)+1$, then $h(n+1) = h(n)$.

Equation (1) shows that $h(n+1) \le n+1$. Since $h(1)=1$, it follows that $h(n) \le n$ for all n. If $n \ge 6$, then $h(n) \ge h(6) = 4$. This implies that $h\big(h(n)\big) \ge h(4) = 3$. Thus

$$h(n+1) = n+2 - h\big(h(n)\big) \le n+2-3 = n-1,$$

for all $n \ge 6$. Since $h(6)=4$, it follows that $h(n) \le n-2$ for all $n \ge 6$. $\square$

Lemma 3. *Let h be a function satisfying the equation (1). Then h cannot take the same value at three consecutive integers. Moreover, h cannot take four distinct values at four consecutive integers.*

Proof. Suppose, if possible, $h(n) = h(n+1) = h(n+2)$, for some n. Now (1) gives

$$\begin{aligned} h\big(h(n)\big) + h(n+1) &= n+2, \\ h\big(h(n+1)\big) + h(n+2) &= n+3. \end{aligned}$$

Using $h(n+2) = h(n+1)$, the above relations give $h\big(h(n+1)\big) = h\big(h(n)\big) + 1$. However $h(n+1) = h(n)$ forces $h\big(h(n+1)\big) = h\big(h(n)\big)$. These two are incompatible and hence $h(n) = h(n+1) = h(n+2)$ cannot happen.

Suppose h assumes four distinct values at four consecutive integers. Since h increases in steps of 0 or 1, we may assume say $h(n+1) = h(n)+1$, $h(n+2) = h(n+1)+1 = h(n)+2$, and $h(n+3) = h(n+2)+1 = h(n)+3$. Using (1),

$$\begin{aligned} h\big(h(n)\big) + h(n+1) &= n+2, \\ h\big(h(n+1)\big) + h(n+2) &= n+3, \\ h\big(h(n+2)\big) + h(n+3) &= n+4. \end{aligned}$$

It follows that

$$h\big(h(n)\big) = h\big(h(n+1)\big) = h\big(h(n+2)\big),$$

which reduces to

$$h\big(h(n)\big) = h\big(h(n)+1\big) = h\big(h(n)+2\big).$$

But this contradicts the property of h that it cannot assume the same value at three consecutive integers. Thus it follows that h cannot take four distinct values at four consecutive integers. $\square$

Lemma 4. *For each $n \in \mathbb{N}$,*

$$h(n) = \frac{\big(f(1) + f(2) + \cdots + f(n)\big)}{n}.$$

Thus the sum $f(1) + f(2) + \cdots + f(n)$ is divisible by n, for each $n \in \mathbb{N}$.

Proof. We use induction on n. For $n = 1, 2$, this may be verified using $h(1) = 1$, $h(2) = 2$, $f(1) = 1$ and $f(2) = 3$. Suppose the relation holds for all $j \leq n$. If $h(n+1) = h(n)$, then $f(n+1) = h(n)$ and hence

$$\frac{f(1) + f(2) + \cdots + f(n) + f(n+1)}{n+1} = \frac{nh(n) + h(n)}{n+1} = h(n) = h(n+1).$$

If, on the other hand, $h(n+1) = h(n)+1$, then $f(n+1) = h(n) + n + 1$ and hence

$$\frac{f(1) + f(2) + \cdots + f(n) + f(n+1)}{n+1} = \frac{nh(n) + h(n) + n + 1}{n+1} = h(n) + 1 = h(n+1).$$

This proves inductive step and hence proves the assertion. $\square$

Lemma 5. *The function h, satisfying the equation* (1) *further satisfies the relation*

$$h\big(h(n)+n\big) = n+1,$$

for all $n \in \mathbb{N}$.

Proof. For $n = 1, 2$ this may be verified using the values $h(1) = 1$ and $h(2) = 2$. Suppose it holds for all $j \le n$; i.e., $h\big(h(j)+j\big) = j+1$, for all $j \le n$. Consider $h\big(h(n+1)+n+1\big)$. If $h(n+1) = h(n)$, then

$$h\big(h(n+1)+n+1\big) = h\big(h(n)+n+1\big) = h\big(h(n)+n\big) \quad or \quad h\big(h(n)+n\big)+1.$$

Suppose $h\big(h(n)+n+1\big) = h\big(h(n)+n\big)$. Then replacing n in (1) by $h(n)+n$, it takes the form

$$h\Big(h\big(h(n)+n\big)\Big) + h\Big(h(n)+n+1\Big) = h(n)+n+2.$$

If we use induction hypothesis, it reduces to

$$h\big(n+1\big) + h\big(h(n)+n\big) = h(n)+n+2,$$

or to $h(n+1) = h(n)+1$. This contradicts $h(n+1) = h(n)$. We conclude that $h\big(h(n)+n+1\big) = h\big(h(n)+n\big)+1$. Thus

$$h\big(h(n+1)+n+1\big) = h\big(h(n)+n\big)+1 = (n+1)+1 = n+2.$$

Suppose on the other hand $h(n+1) = h(n)+1$. In this case

$$h\big(h(n+1)+n+1\big) = h\big(h(n)+n+2\big) = h\big(h(n)+n\big)+1 \quad or \quad h\big(h(n)+n\big)+2.$$

If $h\big(h(n)+n+2\big) = h\big(h(n)+n\big)+2$, then this implies that $h\big(h(n)+n+2\big) = h\big(h(n)+n+1\big)+1$ and $h\big(h(n)+n+1\big) = h\big(h(n)+n\big)+1$, because h increases in steps of 0 or 1. Using (1),

$$h\Big(h\big(h(n)+n+1\big)\Big) + h\Big(h(n)+n+2\Big) = h(n)+n+3.$$

This may be written in the form

$$h\Big(h\big(h(n)+n\big)+1\Big) + h\Big(h(n)+n\Big)+2 = h(n)+n+3.$$

The induction hypothesis reduces it to

$$h(n+2)+n+3 = h(n)+n+3.$$

Thus $h(n+2) = h(n)$. This forces $h(n+2) = h(n+1) = h(n)$, contradicting Lemma 3. We conclude that $h\big(h(n)+n+2\big) = h\big(h(n)+n\big)+1$. But then

$$h\big(h(n+1)+n+1\big) = h\big(h(n)+n+2\big) = h\big(h(n)+n\big)+1 = (n+1)+1 = n+2.$$

This proves the result for $j = n+1$ and completes the induction. □

We now complete the proof of Theorem 2. We show that f is, in fact, an involution on $\mathbb{N}$, i.e., $f\big(f(n)\big) = n$, for each n.

Proof of Theorem 2. Since $f(1) = 1$, $f(2) = 3$ and $f(3) = 2$, the result is true for $n = 1, 2, 3$. Suppose $n > 3$ and $f\big(f(j)\big) = j$, for all $j \leq n$. We show that $f\big(f(n+1)\big) = n+1$. We consider two cases: $h(n + 1) = h(n)$ and $h(n + 1) = h(n) + 1$.

Case 1. Suppose $h(n + 1) = h(n)$. Then $f(n + 1) = h(n)$. Put $j = h(n) - 1$, and observe that $j > 0$ since $n > 3$. Since $h(n + 1) = h(n)$, Lemma 3 shows that $h(n) = h(n - 1) + 1$. Hence $j = h(n) - 1 = h(n - 1)$, and $j + 1 = h(n) = f(n + 1)$. Observe that

$$\begin{aligned} h(j + 1) + j &= h\big(h(n)\big) + h(n) - 1 \\ &= h\big(h(n)\big) + h(n + 1) - 1 \\ &= (n + 2) - 1 = n + 1. \end{aligned}$$

We show that $h(j + 1) = h(j) + 1$. Suppose the contrary that $h(j + 1) = h(j)$. Then $h(j) + j = h(j + 1) + j = n + 1$. Thus

$$n + 1 = h(j) + j = h\big(h(n - 1)\big) + h(n) - 1 = (n + 1) - 1 = n,$$

which is absurd. It follows that $h(j + 1) = h(j) + 1$ and hence

$$f(j + 1) = h(j) + j + 1 = h(j + 1) + j = n + 1.$$

Thus we obtain $f\big(f(n + 1)\big) = n + 1$.

Case 2. Suppose on the other hand $h(n+1) = h(n)+1$ and hence $f(n+1) = h(n)+n+1$. Put $j = h(n)+n$. Then $h(j) = h\big(h(n)+n\big) = n+1$, by Lemma 5 and $j+1 = h(n)+n+1 = f(n+1)$. We show that $h(j+1) = h(j)$ in this case. If not, we must have $h(j+1) = h(j)+1$ and hence

$$h\big(h(n) + n + 1\big) = h\big(h(n) + n\big) + 1 = n + 2.$$

Now (1) gives

$$h\Big(h\big(h(n) + n\big)\Big) + h\Big(h(n) + n + 1\Big) = h(n) + n + 2.$$

This reduces to

$$h(n + 1) + n + 2 = h(n) + n + 2.$$

Thus $h(n + 1) = h(n)$, contradicting $h(n + 1) = h(n) + 1$. It follows that $h(j + 1) = h(j)$ and using the definition of f, we obtain

$$f(j + 1) = h(j) = n + 1.$$

We obtain $f\big(f(n + 1)\big) = n + 1$.

Thus we see that $f\big(f(n)\big) = n$ for all $n \in \mathbb{N}$. This implies that f is a bijection on $\mathbb{N}$. Lemma 4 shows that n divides $f(1) + f(2) + \cdots + f(n)$ for all $n \in \mathbb{N}$.

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