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Congruence Properties of the Function that Counts Compositions into Powers of 2

Giedrius Alkauskas[1]
Institute of Mathematics, Department of Integrative Biology
Universität für Bodenkultur
Gregor Mendel-Straße 33
A-1180 Wien
Austria

and

Vilnius University, Department of Mathematics and Informatics
Naugarduko 24
LT-03225 Vilnius
Lithuania
giedrius.alkauskas@gmail.com

Abstract

Let $\vartheta(n)$ denote the number of compositions (ordered partitions) of a positive integer n into powers of 2. It appears that the function $\vartheta(n)$ satisfies many congruences modulo 2^N. For example, for every integer a there exists (as k tends to infinity) the limit of $\vartheta(2^k + a)$ in the $2-$adic topology. The parity of $\vartheta(n)$ obeys a simple rule. In this paper we extend this result to higher powers of 2. In particular, we prove that for each positive integer N there exists a finite table which lists all the possible cases of this sequence modulo 2^N. One of our main results claims that $\vartheta(n)$ is divisible by 2^N for almost all n, however large the value of N is.

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1 Introduction

Let $\vartheta(n)$ denote the number of *compositions* of a positive integer n into powers of 2 (compositions are sometimes called *ordered partitions*): this is the amount of finite sequences $\{q_1, q_2, \dots, q_\ell\}$ of non-negative integers such that $n = 2^{q_1} + 2^{q_2} + \cdots + 2^{q_\ell}$. Thus, for example, $3 = 1+1+1 = 2+1 = 1+2$ give all possible compositions, hence $\vartheta(3) = 3$. This sequence appears in Sloane's encyclopedia [8] as A023359. Several properties of this sequence are also listed there. Let us call this function *the binary composition function.* It is easy to see (with the help of calculus of residues) that

$$\vartheta(n) \sim \frac{c}{\rho^{n+1}},$$

where ρ is the unique zero of $f(x) = 1 - \sum_{k=0}^{\infty} x^{2^k}$ in the interval $(0,1)$, and $c = -\frac{1}{f'(\rho)}$. Nevertheless, in this note we are mainly concerned with 2−adic rather than with real asymptotics.

The binary partition function $b(n)$ (which counts the partitions of n into non-negative powers of 2 neglecting the order of the summands) was investigated by many authors, beginning with L. Euler (1750), and in the 20th century A. Tanturi (1918), K. Mahler (1940) (who explored asymptotic behavior). See the sequences A018819 and A000123 in Sloane's Encyclopedia [8]; one can find numerous references there. Congruence properties of $b(n)$ modulo powers of 2 were first observed by R. F. Churchhouse [5] (the main congruence was given without a proof as a conjecture). This conjecture was later proved by H. Gupta [6] and independently by Ø. Rødseth [7]. This result can also be found in Andrews' monograph [3]. The paper by the author [1] gives another proof of this fact along with one possible generalization of this congruence. As an aside [1], for every positive integer s which is not divisible by 8 there exists a finite algorithm to verify the fact that infinitely many terms of the sequence $b(n)$ are divisible by s. Calculations confirmed this for $2 \le s \le 14$, $s \ne 8$ (as was noticed by Churchhouse himself, $b(n)$ is never divisible by 8). Moreover, for every power of 2 there exists a finite table which lists all the possible remainders of $b(n)$ modulo this power. For example, modulo 32 one of the entries is [1]

$$b(4n+2) \equiv 2 + 4w(n) + 8w\big(\lfloor n/2 \rfloor\big) + 16\tau(n) \pmod{32}).$$

Here $\lfloor \star \rfloor$ stands for the "floor" function, $w(n)$ represents the Thue-Morse sequence with initial conditions $w(0) = 0$, $w(1) = 1$, and $\tau(n)$ stands for the Rudin-Shapiro sequence with conditions $\tau(0) = 0$ and $\tau(3) = 1$.

We will now formulate the Rødseth-Gupta theorem.

Theorem. *If n is odd positive integer, then for any integer $s \ge 1$ we have*

$$b(2^{s+2}n) \equiv b(2^s n) \pmod{2^{\mu(s)}}, \quad \textit{were } \mu(s) = \Big\lfloor \frac{3s+4}{2} \Big\rfloor;$$

moreover, this congruence is exact.

On the other hand, the binary composition function has not yet been arithmetically investigated. The only papers (apart from The On-Line Encyclopedia of Integer Sequences) where this sequence appears are the papers by the author [2] and by Chinn and Niederhausen [4]. The authors of the latter are concerned with finding an exact formula for the number of binary compositions of n into exactly $n-k$ parts for small k.

If we consider compositions of a positive integer n with no limitation on the non-negative summands, then the amount of these is equal to 2^{n-1}. On the other hand, compositions with summands coming from a certain set reveal new congruence phenomena. For example, one of our main results is the following surprising fact. Let us denote by $s_2(n)$ the amount of 1's in the binary expansion of n. This is the sequence A000120.

Theorem 1. *Suppose that $n \geq 1$, $N \geq 1$, and $s_2(n+2^{N-1}) \geq 2^N$. Then*

$$\vartheta(n) \equiv 0 \pmod{2^N}.$$

Let us say that a property $\mathcal{A}$ is satisfied for *almost all natural numbers*, if

$$\lim_{M\to\infty} \frac{\#\{n \leq M : n \text{ satisfies property } \mathcal{A}\}}{M} = 1.$$

Corollary 2. *Let $N \in \mathbb{N}$. Then for almost all natural numbers the congruence $\vartheta(n) \equiv 0 \pmod{2^N}$ is satisfied.*

Proof. This is clear: for a fixed $M \in \mathbb{N}$, almost all natural numbers will have more than M 1's in the binary expansion. $\square$

2 Congruence properties

Now we will derive some basic facts about $\vartheta(n)$. Les us make a convention $\vartheta(0) = 1$ and $\vartheta(-n) = 0$ for $n \in \mathbb{N}$. Binary compositions of n can be divided into disjoined subsets, each of which consists of compositions with the first summand equal to 2^k, $1 \leq 2^k \leq n$. Thus, this gives the recurrence relation, which also appears in [2, 4, 8]:

$$\vartheta(n) = \sum_{k\geq 0} \vartheta(n-2^k). \tag{1}$$

Hence, the generating function is give by

$$\sum_{n=0}^{\infty} \vartheta(n)x^n = (1 - \sum_{k=0}^{\infty} x^{2^k})^{-1}.$$

From the recurrence relation we can already determine the parity of $\vartheta(n)$. This is the only property which admits easy proof directly from (1).

Proposition 3. *The number $\vartheta(n)$ is odd if and only if $n = 2^u - 1$, $u \geq 0$.*

Proof. Suppose we have already proved this statement for all positive integers $\leq n-1$. From the recurrence relation and inductive hypothesis we inherit that the parity of $\vartheta(n)$ equals the parity of the amount of odd terms among $\vartheta(n-2^u)$, $u \geq 0$. This term is odd iff (according to the induction hypothesis) $n-2^u = 2^v - 1$; that is, it happens only iff $n+1 = 2^u + 2^v$. Hence, if $s_2(n+1) > 2$, this cannot occur. If $s_2(n+1) = 2$, so $n+1 = 2^u + 2^v$, $u \neq v$, we have exactly two odd summands: $\vartheta(n-2^u)$ and $\vartheta(n-2^v)$, and therefore the sum is even. Finally, we have only one odd summand iff $n = 2^u - 1$, and this summand is $\vartheta(n-2^{u-1}) = \vartheta(2^{u-1}-1)$. We finish by induction. □

The congruence properties of the binary composition function modulo higher powers of 2 were observed by the author [2]. One of these congruences claim that

$$\vartheta(2^k) \equiv 8 \pmod{16} \text{ for } k \geq 3. \tag{2}$$

Unfortunately, despite many efforts to manipulate with (1), this and similar claims were not proved in [2] but rather extrapolated from numerical data. This failure suggests the fact that the recurrence (1) alone is insufficient in proving these congruences. Luckily, one can derive some other recurrence relations, much more convenient and powerful.

Proposition 4. *For $n \geq 1$, we have*

$$\vartheta(2n) \;=\; \vartheta^2(n) + \sum_{\substack{a+b=2n-2^s\\ a,b<n,s\geq 1}} \vartheta(a)\cdot\vartheta(b). \tag{3}$$

In general, for $m, n \geq 1$, the following equality holds

$$\vartheta(m+n) \;=\; \vartheta(m)\cdot\vartheta(n) + \sum_{\substack{a+b=m+n-2^s\\ a<m,b<n,s\geq 1}} \vartheta(a)\cdot\vartheta(b). \tag{4}$$

Thus, in case $m = n$ this gives (3), and in case $m = 1$ this reduces exactly to (1).

Proof. As a matter of fact, this identity is valid for any function which counts compositions of n into positive integers $1 = a_1 < a_2 < a_3 < \cdots$, only 2^s must be replaced with a_s in the formula. To prove the identity, consider any composition of $m+n$:

$$m + n = 2^{q_1} + 2^{q_2} + \cdots + 2^{q_\ell}.$$

Let s be the largest non-negative integer such that $\sum_{i=1}^{s} 2^{q_i} \leq m$ (if $s = 0$, the empty sum is 0 by convention). The number of compositions of $m+n$ where $\sum_{i=1}^{s} 2^{q_i} = m$ for some s is obviously equal to $\vartheta(m)\cdot\vartheta(n)$. If $a = \sum_{i=1}^{s} 2^{q_i} < m$, then $b = \sum_{i=s+2}^{\ell} 2^{q_i} < n$. Thus, $m + n - a - b = 2^{q_{s+1}}$. Thus, if $a < m$ and $b < n$ are fixed, the amount of such compositions is equal to $\vartheta(a)\cdot\vartheta(b)$. This proves the formula (4). □

Now we are able to derive the following

Proposition 5. *The sequence* $\vartheta(n)$ *can be completely described modulo* 4.
(i) For $n \geq 3$*, let* $\tau_3(n)$ *denote the number of solutions of* $n+1 = 2^s + 2^v + 2^u$ *with* $s \geq v > u \geq 0$*, let* $\tau_2(n)$ *denote the number of solutions of* $n+1 = 2^s + 2^u$*, and let* $\tau_1(n)$ *denote the number of solutions of* $n+1 = 2^s$*. Then*

$$\vartheta(2n) \equiv 2\tau_3(n) + \tau_2(n) + \tau_1(n) \pmod 4.$$

(ii) In a similar way, $\vartheta(2^k - 1) \equiv 3 \pmod 4$ *for* $k \geq 2$*, and* $\vartheta(2^k + 2^l - 1) \equiv 2 \pmod 4$ *for* $k > l \geq 1$*. In all other cases,* $\vartheta(2n-1) \equiv 0 \pmod 4$ *(the first occurrence is* $n = 7$*).*

Proof. *(i)* Note that (3) can be rewritten as

$$\vartheta(2n) \;=\; \vartheta^2(n) + 2\sum_{\substack{a+b=2n-2^s\\ 0\leq a<b<n,s\geq 1}} \vartheta(a)\cdot\vartheta(b) + \sum_{s\geq 1}\vartheta^2(n-2^{s-1}). \tag{5}$$

Consider this equality modulo 4. Obviously, $\vartheta^2(n) \equiv \tau_1(n) \pmod 4$, since this is implied by Proposition 3. In the second sum, only the terms with $a = 2^u - 1$ and $b = 2^v - 1$, $u, v \geq 0$, do contribute to the final result. In this case $2n = 2^s + 2^u - 1 + 2^v - 1$. Thus, suppose n is of this form. Since $a < b$, we have $u < v$, $u \geq 1$, and also, since $2^v - 1 < n$, it is easy to see that $s \geq v$. The number of solutions is thus $\tau_2(n)$. Finally, the second sum contributes exactly $\tau_2(n)$.

(ii) Equally, (4) for $m = n-1$ reads as

$$\begin{aligned}
\vartheta(2n-1) &= \vartheta(n-1)\cdot\vartheta(n) + \sum_{\substack{a+b=2n-1-2^s\\ a<n-1,b<n,s\geq 1}} \vartheta(a)\cdot\vartheta(b)\\
&= \vartheta(n-1)\cdot\vartheta(n) + \sum_{s\geq 1}\vartheta(n-2^s)\cdot\vartheta(n-1) + 2\sum_{\substack{a+b=2n-1-2^s\\ a<b<n-1,s\geq 1}} \vartheta(a)\cdot\vartheta(b)\\
&= 2\vartheta(n-1)\cdot\vartheta(n) - \vartheta^2(n-1) + 2\sum_{\substack{a+b=2n-1-2^s\\ a<b<n-1,s\geq 1}} \vartheta(a)\cdot\vartheta(b).
\end{aligned} \tag{6}$$

Here we used (1). In the sum, we have zero contribution to $\vartheta(2n-1)$ modulo 4 unless $a = 2^u - 1$, $b = 2^v - 1$, and $2^u - 1 + 2^v - 1 = 2n - 1 - 2^s$. Thus, $u = 0$ and $n = 2^{s-1} + 2^{v-1}$. Since $b = 2^v - 1 < n - 1$, this implies $s > v$. Thus, there exists at most one such solution, and this happens exactly when $s_2(n) = 2$. If $n = 2^k$, $k \geq 1$, we get that $\vartheta(2^{k+1} - 1) \equiv -\vartheta^2(2^k - 1) \equiv 34)$. ∎

The following table summarizes the results of Proposition 5.

1. Sequence $\vartheta(n)$ modulo 4, $n \geq 2$		
n	$\vartheta(n) \pmod 4$	Condition
$2^k + 2^l + 2^m - 2$	2	$k > l > m \geq 1$
$2^k - 2$	2	$k \geq 2$
$3 \cdot 2^k - 2$	2	$k \geq 1$
Other even numbers	0	
$2^k - 1$	3	$k \geq 2$
$2^k + 2^l - 1$	2	$k > l \geq 1$
Other odd numbers	0	

This table should list all even numbers n such that $s_2(n+2) \leq 3$. However, two types of these numbers, namely $\{2^k, k \geq 3\}$ and $\{2^k+2^l-2, k > l+1 \geq 3\}$, fall under the qualification "other even numbers", while the type $\{n = 2^k + 2^l, k > l \geq 2\}$ is a special case of the first type listed in the table.

Let us now inspect the recurrences (5) and (6) more carefully. We will use the following well-known implication which, as a matter of fact, makes the investigations of quadratic forms over 2−adic number field rather exceptional in p−adic analysis. Let $U, V \in \mathbb{Z}$, $N \geq 1$. Then

$$\text{if } U \equiv V \pmod{2^N} \Rightarrow U^2 \equiv V^2 \pmod{2^{N+1}}.$$

Suppose we know the sequence $\vartheta(n)$ modulo 2^N. In this case the recurrences (5), (6) and the above fact show us that the sequence $\vartheta(n)$ is completely describable modulo 2^{N+1} as well. Further, note that Table 1 lists only those even and odd numbers n such that $s_2(n+2) \leq 3$. The recurrence (5) shows then that the corresponding table for $\vartheta(n)$ modulo 8 will list only those even numbers n such that $s_2(n+4) \leq 7$. Exactly the same conclusion follows for odd n. Here is one tricky point. In fact, consider odd number $2n-1$ and the multiplier $\vartheta(n)$ of the term $2\vartheta(n-1)\vartheta(n)$ in (6). This multiplier matters if $s_2(n+2) \leq 3$. This shows that odd numbers $2n-1$ such that $s_2((2n-1)+5) \leq 3$ should also be considered as candidates to be listed in the table for $\vartheta(n)$ modulo 8. Indeed, it can happen that $s_2((2n-1)+5) \leq 3$ and $s_2((2n-1)+4) \geq 8$ are simultaneously satisfied. But then elementary considerations show that $s_2(n+1) \geq 7$. Thus, $\vartheta(n-1) \equiv 0 \pmod 4$ and, due to this multiplier, the term $2\vartheta(n-1)\vartheta(n)$ does not contribute to $\vartheta(2n-1) \pmod 8$. We can proceed by induction on N. Therefore, a careful analysis of (5) and (6) implies the following

Theorem 6. *For each positive integer N there exists a finite table (analogous to the Table 1) which lists a finite number of possibilities for $\vartheta(n)$ modulo 2^N. The table encompasses only finite number of classes of those positive integers n such that*

$$s_2(n + 2^{N-1}) < 2^N. \tag{7}$$

If the entry $n = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_\ell} - 2^{N-1}$ is in this table, the corresponding residue depends solely on ℓ and the exact shape of the collection of inequalities or equalities (the amount of these collections is also finite) satisfied by $k_1, k_2, \ldots, k_\ell$. These inequalities or equalities are of the form $k_i = k_{i+1} + d_i$, or $k_i \geq k_{i+1} + d_i$, for a fixed collection of $d_i \in \mathbb{N}$. For those positive integers n which are not in this table, $\vartheta(n) \equiv 0 \pmod{2^N}$.

This result has numerous corollaries. One immediate corollary is Theorem 1. Also, Theorem 6 says that to prove the congruence (2) or even to improve it to

$$\vartheta(2^k) \equiv 8 \pmod{32} \text{ for } k \geq 8$$

(which does hold), one needs to perform only a finite number of calculations: all what is demanded is to check that this congruence holds for k up to a given bound, to be rest assured that this holds throughout. Indeed, according to the Theorem 6, two numbers 2^{k+1} and 2^k will eventually qualify for the same entry in the table for k large enough (or be both left out of the table). For the very same reason, this allows to make the following claim. Let $a \in \mathbb{Z}$. Then there exists

$$\lim_{k\to\infty} \vartheta(2^k + a) = \Theta(a) \in \mathbb{Z}_2;$$

here $\mathbb{Z}_2$ stands for the ring of 2−adic integers, and the limit is taken in 2−adic topology. A generalization of this is the following

Corollary 7. *Let $P(x) = \sum_{i=0}^{d} a_i x^i$ be a polynomial with non-negative integral coefficients. Then for every integer $N \in \mathbb{N}$ there exists $k_0 \in \mathbb{N}$ such that*

$$\vartheta\Big(P(2^{k+1})\Big) \equiv \vartheta\Big(P(2^k)\Big) \pmod{2^N} \textit{ for } k \geq k_0.$$

According to Theorem 1, if a polynomial $P(x) = \sum_{i=0}^{d} a_i x^i$, $a_d \geq 1$, has at least one negative coefficient $a_i < 0$ with $i \geq 1$, then $\vartheta(P(2^k)) \to 0$ in 2−adic topology.

We finish with providing the table for $\vartheta(n)$ modulo 8.

2. Sequence $\vartheta(n)$ modulo 8, $n \geq 7$

n	$\vartheta(n) \pmod 8$	Condition
$2^k + 2^l + 2^m - 2$	6	$k > l > m \geq 1$
$2^k - 2$	2	$k \geq 2$
$3 \cdot 2^k - 2$	6	$k \geq 1$
Other even numbers	0	
$2^k - 1$	7	$k \geq 3$
$2^k + 1$	6	$k \geq 4$
$2^k + 2^l - 1$	2	$k > l \geq 2$
$2^k - 3$	4	$k \geq 4$
$3 \cdot 2^k - 3$	4	$k \geq 3$
$2^k + 2^l + 2^m - 3$	4	$k > l > m \geq 2$
Other odd numbers	0	

Thus, for example, $4|\vartheta(2n) \Rightarrow 8|\vartheta(2n)$. The complete table for $2^N = 16$ has the following entries, which are not members of larger classes:

$$\vartheta(7 \cdot 2^k - 2) \equiv 14 \pmod{16} \text{ for } k \geq 1, \quad \vartheta(5 \cdot 2^k - 2) \equiv 8 \pmod{16} \text{ for } k \geq 3.$$

One could wonder, for example, whether these have combinatoric proofs.

Concerning the condition (7), it is certainly not sharp for $N > 2$, as the above table for $2^N = 8$ suggests. Employing this table, we can show that in fact 2^N on the right hand side can be replaced by $2^{N-1} + 2^{N-3} - 1$ for $N \geq 4$. This might be also far from the optimal for larger N.

3. Sequence $\vartheta(n)$

n	$\vartheta(n)$	mod 64	n	$\vartheta(n)$	mod 64	n	$\vartheta(n)$	mod 64
1	1	000001	25	882468	100100	49	751322695068	011100
2	2	000010	26	1558798	001110	50	1327134992166	100110
3	3	000011	27	2753448	101000	51	2344248747712	000000
4	6	000110	28	4863696	010000	52	4140876568224	100000
5	10	001010	29	8591212	101100	53	7314436562436	000100
6	18	010010	30	15175514	011010	54	12920206953182	011110
7	31	011111	31	26805983	011111	55	22822229201360	010000
8	56	111000	32	47350056	101000	56	40313142631496	001000
9	98	100010	33	83639030	110110	57	71209059135432	001000
10	174	101110	34	147739848	001000	58	125783547796216	111000
11	306	110010	35	260967362	000010	59	222183821668104	001000
12	542	011110	36	460972286	111110	60	392465083678728	001000
13	956	111100	37	814260544	000000	61	693249583836156	111100
14	1690	011010	38	1438308328	101000	62	1224554757801706	101010
15	2983	100111	39	2540625074	110010	63	2163051215343439	001111
16	5272	011000	40	4487755390	111110	64	3820809588459176	101000
17	9310	011110	41	7927162604	101100	65	6749070853108302	001110
18	16448	000000	42	14002525142	010110	66	11921546029897416	001000
19	29050	111010	43	24734033936	010000	67	21058196429732338	110010
20	51318	110110	44	43690150992	010000	68	37197158469308174	001110
21	90644	010100	45	77174200244	110100	69	65704990586807960	011000
22	160118	110110	46	136320361910	110110	70	116061171489076784	110000
23	282826	001010	47	240796030130	110010	71	205010234490786986	101010
24	499590	000110	48	425341653750	110110	72	362129691668018062	001110

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(Concerned with sequences A000120, A000123, A018819, and A023359.)

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