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Combinatorial Identities Involving the Möbius Function

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Abstract

We derive some identities and inequalities concerning the Möbius function. Our main tool is phi functions for intervals of positive integers and their unions.

1 Introduction

The *Möbius function* μ is an important arithmetic function in number theory and combinatorics that appears in various identities. We mention the following identities which are well known and can be found in books on elementary number theory and arithmetic functions. Let n be a positive integer. Then

$$\sum_{d|n} \mu(d) = \begin{cases} 1, & \text{if } n = 1; \\ 0, & \text{if } n > 1. \end{cases}$$

If $\tau(n)$ is the number of divisors of n, then

$$\sum_{d|n} \mu(d)\tau(n/d) = 1.$$

If $n = \prod_{i=1}^{r} p_i^{k_i}$ is the prime decomposition of n, then

$$\sum_{d|n} \mu(d)\lambda(d) = 2^r,$$

where λ denotes the *Liouville lambda* function defined as follows:
If $m = \prod_{i=1}^{s} p_i^{l_i}$ is the prime decomposition of m, then

$$\lambda(m) = (-1)^{\sum_{i=1}^{s} l_i}.$$

For a survey on combinatorial identities we refer to Hall [6] and Riordan [9] and their references. In this note we shall prove the following two theorems on identities involving the Möbius mu function.

Theorem 1. *Let m and n be positive integers such that $n > 1$. Then we have*

$$(a) \quad \sum_{d|n} \mu(d) 2^{\lfloor \frac{m}{d} \rfloor - \lfloor \frac{m-1}{d} \rfloor} = \sum_{d|n} \mu(d) \left(\left\lfloor \frac{m}{d} \right\rfloor - \left\lfloor \frac{m-1}{d} \right\rfloor \right) = \begin{cases} 0, & \textit{if } \gcd(m,n) > 1; \\ 1, & \textit{if } \gcd(m,n) = 1. \end{cases}$$

$$(b) \quad \sum_{d|n} \mu(d) 2^{\lfloor \frac{m+1}{d} \rfloor - \lfloor \frac{m-1}{d} \rfloor} =$$

$$\begin{cases} 1, & \textit{if } \gcd(m,n) > 1 \textit{ and } \gcd(m+1,n) > 1; \\ 2, & \textit{if } \gcd(m,n) = 1 \textit{ and } \gcd(m+1,n) > 1 \textit{ or } \gcd(m,n) > 1 \textit{ and } \gcd(m+1,n) = 1; \\ 3, & \textit{if } \gcd(m,n) = \gcd(m+1,n) = 1. \end{cases}$$

Theorem 2. *Let m and n be positive integers such that $n > 1$. Then we have:*

$$(a) \quad \sum_{d|n} \mu(d) \left(\left\lfloor \frac{m+1}{d} \right\rfloor - \left\lfloor \frac{m-1}{d} \right\rfloor \right) =$$

$$\begin{cases} 0, & \textit{if } \gcd(m,n) > 1 \textit{ and } \gcd(m+1,n) > 1; \\ 1, & \textit{if } \gcd(m,n) = 1 \textit{ and } \gcd(m+1,n) > 1 \textit{ or } \gcd(m,n) > 1 \textit{ and } \gcd(m+1,n) = 1; \\ 2, & \textit{if } \gcd(m,n) = \gcd(m+1,n) = 1. \end{cases}$$

$$(b) \quad \sum_{d|n} \mu(d) \binom{\lfloor \frac{m+1}{d} \rfloor - \lfloor \frac{m-1}{d} \rfloor + 1}{2} =$$

$$\begin{cases} 1, & \textit{if } \gcd(m,n) > 1 \textit{ and } \gcd(m+1,n) > 1; \\ 2, & \textit{if } \gcd(m,n) = 1 \textit{ and } \gcd(m+1,n) > 1 \textit{ or } \gcd(m,n) > 1 \textit{ and } \gcd(m+1,n) = 1; \\ 3, & \textit{if } \gcd(m,n) = \gcd(m+1,n) = 1. \end{cases}$$

Our proofs are combinatorial with phi functions for integer subsets as a main tool. For the sake of completeness we include the following result which is a natural extension of El Bachraoui [3, Theorem 2 (a)] on Möbius inversion for arithmetical functions in several variables. For simplicity we let

$$(\overline{m}_a, \overline{n}_b) = (m_1, m_2, \ldots, m_a, n_1, n_2, \ldots, n_b)$$

and

$$\left(\frac{\overline{m}_a}{d}, \left[\frac{\overline{n}_b}{d} \right] \right) = \left(\frac{m_1}{d}, \frac{m_2}{d}, \ldots, \frac{m_a}{d}, \left[\frac{n_1}{d} \right], \left[\frac{n_2}{d} \right], \ldots, \left[\frac{n_b}{d} \right] \right).$$

Theorem 3. *If F and G are arithmetical of $a+b$ variables, then*

$$G(\overline{m}_a, \overline{n}_b) = \sum_{d \mid \gcd(m_1, m_2, \ldots, m_a)} F\left(\frac{\overline{m}_a}{d}, \left[\frac{\overline{n}_b}{d}\right]\right)$$

if and only if

$$F(\overline{m}_a, \overline{n}_b) = \sum_{d \mid \gcd(m_1, m_2, \ldots, m_a)} \mu(d) G\left(\frac{\overline{m}_a}{d}, \left[\frac{\overline{n}_b}{d}\right]\right).$$

2 Phi functions

Throughout let k, l, m, l_1, l_2, m_1, m_2 and n be positive integers such that $l \leq m$ and $l_1 \leq m_1 \leq l_2 \leq m_2$, let $[l, m] = \{l, l+1, \ldots, m\}$, and let A be a nonempty finite set of positive integers. The set A is called *relatively prime to* n if $\gcd(A \cup \{n\}) = \gcd(A, n) = 1$.

Definition 4. Let

$$\Phi(A, n) = \#\{X \subseteq A : \ X \neq \emptyset \text{ and } \gcd(X, n) = 1\}$$

and

$$\Phi_k(A, n) = \#\{X \subseteq A : \ \#X = k \text{ and } \gcd(X, n) = 1\}.$$

Nathanson, among other things, introduced $\Phi(n)$ and $\Phi_k(n)$ (in our terminology $\Phi([1, n], n)$ and $\Phi_k([1, n], n)$ respectively) along with their formulas in Nathanson [7]. Formulas for $\Phi([m, n], n)$ and $\Phi_k([m, n], n)$ can be found in El Bachraoui [3] and Nathanson and Orosz [8] and formulas for $\Phi([1, m], n)$ and $\Phi_k([1, m], n)$ are obtained in El Bachraoui [4]. Ayad and Kihel in [1, 2] considered extensions to sets in arithmetic progression and obtained formulas for $\Phi([l, m], n)$ and $\Phi_k([l, m], n)$ as consequences. Recently the following formulas for $\Phi([1, m_1] \cup [l_2, m_2])$ and $\Phi_k([1, m_1] \cup [l_2, m_2])$ have been found in El Bachraoui [5].

Theorem 5. *We have*

$$(a) \quad \Phi([1, m_1] \cup [l_2, m_2], n) = \sum_{d \mid n} \mu(d) 2^{\lfloor \frac{m_1}{d} \rfloor + \lfloor \frac{m_2}{d} \rfloor - \lfloor \frac{l_2 - 1}{d} \rfloor},$$

$$(b) \quad \Phi_k([1, m_1] \cup [l_2, m_2], n) = \sum_{d \mid n} \mu(d) \binom{\lfloor \frac{m_1}{d} \rfloor + \lfloor \frac{m_2}{d} \rfloor - \lfloor \frac{l_2 - 1}{d} \rfloor}{k}.$$

3 Phi functions for $[l_1, m_1] \cup [l_2, m_2]$

We need two lemmas.

Lemma 6. *Let*

$$\Psi(l_1, m_1, l_2, m_2, n) = \#\{X \subseteq [l_1, m_1] \cup [l_2, m_2] : \ l_1, l_2 \in X \text{ and } \gcd(X, n) = 1\},$$

$$\Psi_k(l_1, m_1, l_2, m_2, n) = \#\{X \subseteq [l_1, m_1] \cup [l_2, m_2] : \ l_1, l_2 \in X, \ |X| = k, \text{ and } \gcd(X, n) = 1\}.$$

Then

$$(a)\quad \Psi(l_1,m_1,l_2,m_2,n) = \sum_{d|\gcd(l_1,l_2,n)} \mu(d) 2^{\lfloor \frac{m_1}{d}\rfloor + \lfloor \frac{m_2}{d}\rfloor - \frac{l_1+l_2}{d}},$$

$$(b)\quad \Psi_k(m_1,l_2,m_2,n) = \sum_{d|\gcd(l_1,l_2,n)} \mu(d) \binom{\lfloor \frac{m_1}{d}\rfloor + \lfloor \frac{m_2}{d}\rfloor - \frac{l_1+l_2}{d}}{k-2}.$$

Proof. (a) Assume first that $m_2 \leq n$. Let $\mathcal{P}(l_1,m_1,l_2,m_2)$ denote the set of subsets of $[l_1,m_1]\cup[l_2,m_2]$ containing l_1 and l_2 and let $\mathcal{P}(l_1,m_1,l_2,m_2,d)$ be the set of subsets X of $[l_1,m_1]\cup[l_2,m_2]$ such that $l_1,l_2 \in X$ and $\gcd(X,n) = d$. It is clear that the set $\mathcal{P}(l_1,m_1,l_2,m_2)$ of cardinality $2^{m_1+m_2-l_1-l_2}$ can be partitioned using the equivalence relation of having the same gcd (dividing l_1, l_2 and n). Moreover, the mapping $A \mapsto \frac{1}{d}X$ is a one-to-one correspondence between $\mathcal{P}(l_1,m_1,l_2,m_2,d)$ and the set of subsets Y of $[l_1/d, \lfloor m_1/d\rfloor]\cup[l_2/d,\lfloor m_2/d\rfloor]$ such that $l_1/d, l_2/d \in Y$ and $\gcd(Y,n/d) = 1$. Then

$$\#\mathcal{P}(l_1,m_1,l_2,m_2,d) = \Psi(l_1/d,\lfloor m_1/d\rfloor, l_2/d, \lfloor m_2/d\rfloor, n/d).$$

Thus

$$2^{m_1+m_2-l_1-l_2} = \sum_{d|(l_1,l_2,n)} \#\mathcal{P}(l_1,m_1,l_2,m_2,d) = \sum_{d|(l_1,l_2,n)} \Psi(l_1/d,\lfloor m_1/d\rfloor, l_2/d, \lfloor m_2/d\rfloor, n/d),$$

which by Theorem 3 is equivalent to

$$\Psi(l_1,m_1,l_2,m_2,n) = \sum_{d|(l_1,l_2,n)} \mu(d) 2^{\lfloor m_1/d\rfloor + \lfloor m_2/d\rfloor - (l_1+l_2)/d}.$$

Assume now that $m_2 > n$ and let a be a positive integer such that $m_2 \leq n^a$. As $\gcd(X,n) = 1$ if and only if $\gcd(X,n^a) = 1$ and $\mu(d) = 0$ whenever d has a nontrivial square factor, we have

$$\begin{aligned}\Psi(l_1,m_1,l_2,m_2,n) &= \Psi(l_1,m_1,l_2,m_2,n^a)\\ &= \sum_{d|(l_1,l_2,n^a)} \mu(d) 2^{\lfloor m_1/d\rfloor + \lfloor m_2/d\rfloor - (l_1+l_2)/d}\\ &= \sum_{d|(l_1,l_2,n)} \mu(d) 2^{\lfloor m_1/d\rfloor + \lfloor m_2/d\rfloor - (l_1+l_2)/d}.\end{aligned}$$

(b) For the same reason as before, we may assume that $m_2 \leq n$. Noting that the correspondence $X \mapsto \frac{1}{d}X$ defined above preserves the cardinality and using an argument similar to the one in part (a), we obtain the following identity

$$\binom{m_1+m_2-l_1-l_2}{k-2} = \sum_{d|(l_1,l_2,n)} \Psi_k(l_1/d,\lfloor m_1/d\rfloor, l_2/d, \lfloor m_2/d\rfloor, n/d)$$

which by Theorem 3 is equivalent to

$$\Psi_k(l_1,m_1,l_2,m_2,n) = \sum_{d|(l_1,l_2,n)} \mu(d) \binom{\lfloor m_1/d\rfloor + \lfloor m_2/d\rfloor - (l_1+l_2)/d}{k-2}.$$

$\square$

By arguments similar to the ones in the proof of Lemma 6 we have:

Lemma 7. *Let*

$$\psi(l,m,n) = \#\{X \subseteq [l,m] :\ l \in X \ \textit{and}\ \gcd(X,n)=1\},$$
$$\psi_k(l,m,n) = \#\{X \subseteq [l,m] :\ l \in X,\ \#X = k,\ \textit{and}\ \gcd(X,n)=1\}.$$

Then

$$\psi(l,m,n) = \sum_{d \mid \gcd(l,n)} \mu(d) 2^{\lfloor m/d \rfloor - l/d},$$
$$\psi_k(l,m,n) = \sum_{d \mid \gcd(l,n)} \mu(d) \binom{\lfloor m/d \rfloor - l/d}{k}.$$

We are now ready to prove the main theorem of this section.

Theorem 8. *We have*

$$(a) \quad \Phi([l_1,m_1] \cup [l_2,m_2], n) = \sum_{d \mid n} \mu(d) 2^{\lfloor \frac{m_1}{d} \rfloor + \lfloor \frac{m_2}{d} \rfloor - \lfloor \frac{l_1-1}{d} \rfloor - \lfloor \frac{l_2-1}{d} \rfloor},$$

$$(b) \quad \Phi_k([l_1,m_1] \cup [l_2,m_2], n) = \sum_{d \mid n} \mu(d) \binom{\lfloor \frac{m_1}{d} \rfloor + \lfloor \frac{m_2}{d} \rfloor - \lfloor \frac{l_1-1}{d} \rfloor - \lfloor \frac{l_2-1}{d} \rfloor}{k}.$$

Proof. (a) Clearly

$$\Phi([l_1,m_1] \cup [l_2,m_2], n) =$$

$$\Phi([1,m_1] \cup [l_2,m_2], n) - \sum_{i=1}^{l_1-1} \sum_{j=l_2}^{m_2} \Psi(i,m_1,j,m_2,n) - \sum_{i=1}^{l_1-1} \psi(i,m_1,n) = \tag{1}$$

$$\sum_{d \mid n} \mu(d) 2^{\lfloor \frac{m_1}{d} \rfloor + \lfloor \frac{m_2}{d} \rfloor - \lfloor \frac{l_2-1}{d} \rfloor} - \sum_{i=1}^{l_1-1} \sum_{j=l_2}^{m_2} \sum_{d \mid (i,j,n)} \mu(d) 2^{\lfloor \frac{m_1}{d} \rfloor + \lfloor \frac{m_2}{d} \rfloor - \frac{i+j}{d}} - \sum_{i=1}^{l_1-1} \sum_{d \mid (i,n)} \mu(d) 2^{\lfloor \frac{m_1}{d} \rfloor - \frac{i}{d}},$$

where the second identity follows by Theorem 5, Lemma 6, and Lemma 7. Rearranging the triple summation in identity (1), we get

$$\begin{aligned}
\sum_{i=1}^{l_1-1} \sum_{j=l_2}^{m_2} \sum_{d \mid (i,j,n)} \mu(d) 2^{\lfloor \frac{m_1}{d} \rfloor + \lfloor \frac{m_2}{d} \rfloor - \frac{i+j}{d}} &= \sum_{d \mid n} \sum_{\substack{i=1 \\ d \mid i}}^{l_1-1} \sum_{\substack{j=l_2 \\ d \mid j}}^{m_2} \mu(d) 2^{\lfloor \frac{m_1}{d} \rfloor + \lfloor \frac{m_2}{d} \rfloor - \frac{i+j}{d}} \\
&= \sum_{d \mid n} \mu(d) 2^{\lfloor \frac{m_1}{d} \rfloor + \lfloor \frac{m_2}{d} \rfloor} \sum_{i=1}^{\lfloor \frac{l_1-1}{d} \rfloor} 2^{-i} \sum_{j=\lfloor \frac{l_2-1}{d} \rfloor + 1}^{\lfloor \frac{m_2}{d} \rfloor} 2^{-j} \\
&= \sum_{d \mid n} \mu(d) 2^{\lfloor \frac{m_1}{d} \rfloor + \lfloor \frac{m_2}{d} \rfloor - \lfloor \frac{l_2-1}{d} \rfloor} (1 - 2^{-\lfloor \frac{m_2}{d} \rfloor + \lfloor \frac{l_2-1}{d} \rfloor}) \sum_{i=1}^{\lfloor \frac{l_1-1}{d} \rfloor} 2^{-i} \qquad (2) \\
&= \sum_{d \mid n} \mu(d) 2^{\lfloor \frac{m_1}{d} \rfloor + \lfloor \frac{m_2}{d} \rfloor - \lfloor \frac{l_2-1}{d} \rfloor} (1 - 2^{-\lfloor \frac{m_2}{d} \rfloor + \lfloor \frac{l_2-1}{d} \rfloor})(1 - 2^{-\lfloor \frac{l_1-1}{d} \rfloor}).
\end{aligned}$$

Similarly the double summation in identity (1) gives

$$\sum_{i=1}^{l_1-1} \sum_{d|(i,n)} \mu(d) 2^{\lfloor \frac{m_1}{d} \rfloor - \frac{i}{d}} = \sum_{d|n} \mu(d) 2^{\lfloor \frac{m_1}{d} \rfloor} (1 - 2^{-\lfloor \frac{l_1-1}{d} \rfloor}). \tag{3}$$

Combining identities (1), (2), and (3) we find

$$\Phi([l_1, m_1] \cup [l_2, m_2], n) = \sum_{d|n} \mu(d) 2^{\lfloor \frac{m_1}{d} \rfloor + \lfloor \frac{m_2}{d} \rfloor - \lfloor \frac{l_1-1}{d} \rfloor - \lfloor \frac{l_2-1}{d} \rfloor}.$$

This completes the proof of part (a). Part (b) follows by similar ideas. □

Corollary 9. *(Ayad and Kihel [2]) We have*

$$\Phi([l, m], n) = \sum_{d|n} 2^{\lfloor \frac{m}{d} \rfloor - \lfloor \frac{l-1}{d} \rfloor},$$

$$\Phi_k([l, m], n) = \sum_{d|n} \binom{\lfloor \frac{m}{d} \rfloor - \lfloor \frac{l-1}{d} \rfloor}{k}.$$

Proof. Use Theorem 8 with $l_1 = l$, $m_1 = m - 1$, and $l_2 = m_2 = m$. □

4 Proofs of the main results

Proof of Theorem 1. (a) Apply Corollary 9 to $l = m$ and $k = 1$.
(b) Apply Corollary 9 to the interval $[m, m + 1]$.

Proof of Theorem 2. (a) Apply Theorem 8(b) to $l_1 = m$, $m_1 = m + 1$, $l_2 = m_2 = n$, and $k = 1$ and use the fact that $\sum_{d|n} \mu(d) = 0$ whenever $n > 1$.
(b) Apply Theorem 8(b) to $l_1 = m$, $m_1 = m + 1$, $l_2 = m_2 = n$, and $k = 2$.

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