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Partial Bell Polynomials and Inverse Relations

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Abstract

Chou, Hsu and Shiue gave some applications of Faà di Bruno's formula for the characterization of inverse relations. In this paper, we use partial Bell polynomials and binomial-type sequence of polynomials to develop complementary inverse relations.

1 Introduction

Recall that the (exponential) partial Bell polynomials $B_{n,k}$ are defined by their generating function

$$\sum_{n=k}^{\infty} B_{n,k}(x_1, x_2, \cdots)\frac{t^n}{n!} = \frac{1}{k!}\Big(\sum_{m=1}^{\infty} x_m \frac{t^m}{m!}\Big)^k \tag{1}$$

and are given explicitly by the formula

$$B_{n,k}(x_1, x_2, \ldots) = \sum_{\pi(n,k)} \frac{n!}{k_1!k_2!\cdots}\Big(\frac{x_1}{1!}\Big)^{k_1}\Big(\frac{x_2}{2!}\Big)^{k_2}\cdots,$$

where $\pi(n, k)$ is the set of all nonnegative integers $(k_1, k_2, \ldots)$ such that

$$k_1 + k_2 + k_3 + \cdots = k \quad \text{and} \quad k_1 + 2k_2 + 3k_3 + \cdots = n,$$

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Comtet [3] has studied the partial and complete Bell polynomials and has given their basic properties. Riordan [6] has shown the applications of the Bell polynomials in combinatorial analysis and Roman [7] in umbral calculus. Chou, Hsu and Shiue [2] have used these polynomials to characterize some inverse relations. They have proved that, for any function F having power formal series with compositional inverse $F^{\langle -1\rangle}$, the following inverse relations hold

$$y_n = \sum_{j=1}^{n} D_{x=a}^{j} F(x) B_{n,j}(x_1, x_2, \ldots),$$

$$x_n = \sum_{j=1}^{n} D_{x=f(a)}^{j} F^{\langle -1\rangle}(x) B_{n,j}(y_1, y_2, \ldots).$$

In this paper, we link their results to those of Mihoubi [5, 6, 7] on partial Bell polynomials and binomial-type sequence of polynomials.

2 Bell polynomials and inverse relations

Using the compositional inverse function with binomial-type sequence of polynomials, we determine some inverse relations and the connections with the partial Bell polynomials.

Theorem 1. *Let $\{f_n(x)\}$ be a binomial-type sequence of polynomials with exponential generating function $(f(t))^x$. Then the compositional inverse function of*

$$h(t) = t(f(t))^x = \sum_{n\geq 1} n f_{n-1}(x) \frac{t^n}{n!}$$

is given by

$$h^{\langle -1\rangle}(t) = \sum_{n\geq 1} f_{n-1}(-nx) \frac{t^n}{n!}.$$

Proof. To obtain the compositional inverse function of h it suffices to solve the equation $z = tf(z)^{-x}$. The Lagrange inversion formula ensures that the last equation has an unique solution defined around zero by

$$z = h^{\langle -1\rangle}(t) = \sum_{n\geq 1} D_{z=0}^{n-1} (f(z)^{-nx}) \frac{t^n}{n!} = \sum_{n\geq 1} f_{n-1}(-nx) \frac{t^n}{n!}.$$

□

Corollary 2. *Let $\{f_n(x)\}$ be a binomial-type sequence of polynomials and let a be a real number. Then the compositional inverse function of*

$$h(t;a) = \sum_{n\geq 1} \frac{nx}{a(n-1)+x} f_{n-1}(a(n-1)+x) \frac{t^n}{n!}$$

is given by

$$h^{\langle -1\rangle}(t;a) = -\sum_{n\geq 1} \frac{nx}{a(n-1)-nx} f_{n-1}(a(n-1)-nx) \frac{t^n}{n!}.$$

Proof. This result follows by replacing $\{f_n(x)\}$ in Theorem 1, by the binomial-type sequence of polynomials $\{f_n(x;a)\}$, where

$$f_n(x;a) = \frac{x}{an+x} f_n(an+x), \tag{2}$$

see Mihoubi [5, 6, 7]. ☐

Theorem 3. *Let $\{f_n(x)\}$ be a binomial-type sequence of polynomials and a be a real number. Then the following inverse relations hold*

$$\begin{aligned} y_n &= \textstyle\sum_{j=1}^{n} \frac{xj}{a(j-1)+x} f_{j-1}(a(j-1)+x) B_{n,j}(x_1, x_2, \ldots) \\ x_n &= -\textstyle\sum_{j=1}^{n} \frac{xj}{a(j-1)-jx} f_{j-1}(a(j-1)-jx) B_{n,j}(y_1, y_2, \ldots). \end{aligned} \tag{3}$$

Proof. For any function F having power formal series with compositional inverse $F^{\langle -1 \rangle}$, Chou, Hsu and Shiue [2, Remark 1] have proved that

$$\begin{aligned} y_n &= \textstyle\sum_{j=1}^{n} D_{x=a}^{j} F(x) B_{n,j}(x_1, x_2, \ldots) \\ x_n &= \textstyle\sum_{j=1}^{n} D_{x=f(a)}^{j} F^{\langle -1 \rangle}(x) B_{n,j}(y_1, y_2, \ldots). \end{aligned}$$

To prove (3), it suffices to take

$$F(t) := \sum_{n=1}^{\infty} \frac{nx}{a(n-1)+x} f_{n-1}(a(n-1)+x) \frac{t^n}{n!}$$

and then use Corollary 2. ☐

Now, let $f_n(x)$ in Theorem 3 be one of the next binomial-type sequence of polynomials

$$\begin{aligned} &f_n(x) = x^n, \\ &f_n(x) = (x)_{(n)} := x(x-1)\cdots(x-n+1),\ n \geq 1, \quad \text{with } (x)_{(o)} = 1, \\ &f_n(x) = (x)^{(n)} := x(x+1)\cdots(x+n-1),\ n \geq 1, \quad \text{with } (x)^{(o)} = 1, \\ &f_n(x) = n! \binom{x}{n}_q := \textstyle\sum_{j=o}^{n} B_{n,j}(\binom{1}{1}_q, \ldots, i!\binom{1}{i}_q, \ldots)(x)_{(j)}, \\ &f_n(x) = B_n(x) := \textstyle\sum_{j=o}^{n} S(n,k) x^k, \end{aligned}$$

where $B_n(.)$, $S(n,k)$ and $\binom{k}{n}_q$ are, respectively, the single variable Bell polynomials, the Stirling numbers of second kind and the coefficients defined by

$$(1 + x + x^2 + \cdots + x^q)^k = \sum_{n \geq 0} \binom{k}{n}_q x^n,$$

see Belbachir, Bouroubi and Khelladi [4]. We deduce the following results:

Corollary 4. *Let a and x be real numbers. Then the following inverse relations hold:*
For $f_n(x)=x^n$, we get

$$\begin{aligned} y_n &= \textstyle\sum_{j=1}^{n} xj(a(j-1)+x)^{j-2}B_{n,j}(x_1,x_2,\ldots),\\ x_n &= -\textstyle\sum_{j=1}^{n} xj(a(j-1)-jx)^{j-2}B_{n,j}(y_1,y_2,\ldots). \end{aligned} \tag{4}$$

For $f_n(x)=(x)_{(n)}$, we get

$$\begin{aligned} y_n &= \textstyle\sum_{j=1}^{n} \frac{xj}{a(j-1)+x}(a(j-1)+x)_{(j-1)}B_{n,j}(x_1,x_2,\ldots),\\ x_n &= -\textstyle\sum_{j=1}^{n} \frac{xj}{a(j-1)-jx}(a(j-1)-jx)_{(j-1)}B_{n,j}(y_1,y_2,\ldots). \end{aligned} \tag{5}$$

For $f_n(x)=(x)^{(n)}$, we get

$$\begin{aligned} y_n &= \textstyle\sum_{j=1}^{n} \frac{xj}{a(j-1)+x}(a(j-1)+x)^{(j-1)}B_{n,j}(x_1,x_2,\ldots),\\ x_n &= -\textstyle\sum_{j=1}^{n} \frac{xj}{a(j-1)-jx}(a(j-1)-jx)^{(j-1)}B_{n,j}(y_1,y_2,\ldots). \end{aligned} \tag{6}$$

For $f_n(x)=n!\binom{x}{n}_q$, we get

$$\begin{aligned} y_n &= \textstyle\sum_{j=1}^{n} \frac{x(j-1)!}{a(j-1)+x}\binom{a(j-1)+x}{j-1}_q B_{n,j}(x_1,x_2,\ldots),\\ x_n &= -\textstyle\sum_{j=1}^{n} \frac{xj!}{a(j-1)-jx}\binom{a(j-1)-jx}{j-1}_q B_{n,j}(y_1,y_2,\ldots). \end{aligned} \tag{7}$$

For $f_n(x)=B_n(x)$, we get

$$\begin{aligned} y_n &= \textstyle\sum_{j=1}^{n} \frac{xj}{a(j-1)+x}B_{j-1}(a(j-1)+x)B_{n,j}(x_1,x_2,\ldots),\\ x_n &= -\textstyle\sum_{j=1}^{n} \frac{xj}{a(j-1)-jx}B_{j-1}(a(j-1)-jx)B_{n,j}(y_1,y_2,\ldots). \end{aligned} \tag{8}$$

Example 5. For $a=0,\ x=1$ in (5), we obtain

$$\begin{aligned} y_n &= \textstyle\sum_{j=0}^{n}\binom{n}{j}x_jx_{n-j} \quad \text{with} \quad x_0=\frac{1}{2},\\ x_n &= \textstyle\sum_{j=0}^{n-1}(-1)^j\frac{(2j)!}{(j)!}B_{n,j+1}(y_1,y_2,\ldots). \end{aligned} \tag{9}$$

For $x_n=2^{(n-2)/2}$, $x_n=\dfrac{1}{2}$ or $x_n=\frac{1}{2^{n+1}}$ in (9), we get

$$\sum_{j=0}^{n-1}\frac{(2j)!}{(j)!}(-4)^{n-j}S(n,j+1)=(-1)^n2^{n+1}.$$

For $x_1=1,\ x_n=0$, $n\geq 2$, in (9), we get

$$\sum_{j=n-[n/2]}^{n+1}(-1)^j\binom{n+1}{j}\binom{2j}{n}=0,\quad n\geq 0.$$

Take $x = a = x_1 = 1,\ x_2 = 2$ and $x_n = 0$, $n \geq 3$, in (5) and from the identity of Ceralosi [1]

$$\sum_{j=1}^{n} j! B_{n,j}(1!, 2!, 0, 0, \ldots) = n! F_n, \quad n \geq 1,$$

we obtain

$$\sum_{j=1}^{n} (-1)^j j! B_{n,j}(1! F_1, 2! F_2, \ldots) = 0, \quad n \geq 3,$$

where $F_n, n = 0, 1, 2, \cdots$, are the Fibonacci numbers.
Take $x = 1,\ a = x_1 = 0,\ x_2 = 2$ and $x_n = n!$, $n \geq 3$, in (5), from the identity of Ceralosi [1]

$$\sum_{j=1}^{n} j! B_{n,j}(0, 2!, 3!, \ldots) = n! F_{n-2}, \quad n \geq 2,$$

we obtain

$$\sum_{j=1}^{n} (-1)^{j-1} j! B_{n,j}(0, 2! F_0, 3! F_1, \ldots) = n!, \quad n \geq 2.$$

Theorem 6. *Let r, s be nonnegative integers, $rs \neq 0$, and let $\{u_n\}$ be a sequence of real numbers with $u_1 = 1$. Then*

$$\begin{aligned} & y_n = s \textstyle\sum_{j=1}^{n} \frac{j}{U_j} \binom{U_j+j-1}{U_j}^{-1} B_{U_j+j-1, U_j}(1, u_2, u_3, \ldots) B_{n,j}(x_1, x_2, \ldots), \\ & x_1 = y_1 \quad \text{and for } n \geq 2 \text{ we have} \\ & x_n = y_n - s \textstyle\sum_{j=2}^{n} \frac{j}{V_j} \binom{V_j+j-1}{V_j}^{-1} B_{V_j+j-1, V_j}(1, u_2, u_3, \ldots) B_{n,j}(y_1, y_2, \ldots), \end{aligned} \tag{10}$$

where $U_j = (r + 2s)(j - 1) + s$ and $V_j = (r + s)(j - 1) - s$.

Proof. Let n, r, s be nonnegative integers, $nr(nr + s) \geq 1$, $z_n(r) := \frac{B_{(r+1)n, nr}(1, u_2, u_3, \ldots)}{nr \binom{(r+1)n}{nr}}$, and consider the binomial-type sequence of polynomials $\{f_n(x)\}$ defined by

$$f_n(x) := \sum_{j=1}^{n} B_{n,j}(z_1(r), z_2(r), \ldots) x^j \ \text{ with } f_o(x) = 1,$$

see Roman [8]. Then from the identity

$$\textstyle\sum_{j=1}^{n} B_{n,j}(z_1(r), z_2(r), \ldots) s^j = \dfrac{s}{nr + s} \dbinom{(r+1)n + s}{nr + s}^{-1} B_{(r+1)n+s, nr+s}(1, u_2, u_3, \ldots),$$

see Mihoubi [5, 6, 7], we get

$$f_n(s) = \frac{s}{nr + s} \binom{(r+1)n + s}{nr + s}^{-1} B_{(r+1)n+s, nr+s}(1, u_2, u_3, \ldots). \tag{11}$$

To obtain (10), we set $a = 0,\ x = s$ in (3) and use the expression of $f_n(s)$ given by (11), with $r + 2s$ instead of r. $\square$

Example 7. From the well-known identity $B_{n,k}(1!, 2!, \ldots, i!, \ldots) = \binom{n-1}{k-1}\frac{n!}{k!}$, we get

$$y_n = s\sum_{j=1}^{n} j!\frac{((r+2s+1)(j-1)+s-1)!}{((r+2s)(j-1)+s)!}B_{n,j}(x_1, x_2, \ldots),$$

$x_1 = y_1$ and for $n \geq 2$ we have

$$x_n = y_n - s\sum_{j=2}^{n} j!\frac{((r+s+1)(j-1)+s-1)!}{((r+s)(j-1)+s)!}B_{n,j}(y_1, y_2, \ldots).$$

Similar relations can be obtained for the Stirling numbers of the first kind, the unsigned Stirling numbers of the first kind and the Stirling numbers of the second kind by setting $u_n = (-1)^{n-1}(n-1)!$, $u_n = (n-1)!$ and $u_n = 1$ for all $n \geq 1$, respectively.

Corollary 8. *Let u, r, s be nonnegative integers, a, α be real numbers, $\alpha urs \neq 0$, and $\{f_n(x)\}$ be a binomial-type sequence of polynomials. Then*

$$y_n = s\sum_{j=1}^{n} \frac{j}{\alpha^{T_j-u(j-1)}T_j}D_{z=0}^{T_j}(e^{\alpha z}f_{j-1}(T_jx+z;a))B_{n,j}(x_1, x_2, \ldots),$$

$x_1 = y_1$ *and for $n \geq 2$ we have*

$$x_n = y_n - s\sum_{j=2}^{n} \frac{j}{\alpha^{R_j-u(j-1)}R_j}D_{z=0}^{R_j}(e^{\alpha z}f_{j-1}(R_jx+z;a))B_{n,j}(y_1, y_2, \ldots),$$

where $T_j = (u+r+2s)(j-1)+s$ and $R_j = (u+r+s)(j-1)-s$.

Proof. Set in Theorem 6

$$u_n = \frac{n}{(u(n-1)+1)\alpha}D_{z=0}^{u(n-1)+1}(e^{\alpha z}f_{n-1}((u(n-1)+1)x+z;a))$$

and use the first identity of Mihoubi [6, Theorem 2]. □

Corollary 9. *Let u, r, s be nonnegative integers, $urs \neq 0$, a be real number and let $\{f_n(x)\}$ be a binomial-type sequence of polynomials. Then*

$$y_n = s\sum_{j=1}^{n} \frac{j!}{\alpha^{T_j-u(j-1)}(T_j+j-1)!T_j}D_{z=0}^{T_j}f_{T_j+j-1}(T_jx+z;a)B_{n,j}(x_1, x_2, \ldots),$$

$x_1 = y_1$ *and for $n \geq 2$ we have*

$$x_n = y_n - s\sum_{j=2}^{n} \frac{j!}{\alpha^{R_j-u(j-1)}(R_j+j-1)!R_j}D_{z=0}^{R_j}f_{R_j+j-1}(R_jx+z;a)B_{n,j}(y_1, y_2, \ldots),$$

where $T_j = (u+r+2s)(j-1)+s$ and $R_j = (u+r+s)(j-1)-s$.

Proof. Set in Theorem 6

$$u_n = \frac{n!D_{z=0}^{u(n-1)+1}f_{((u+1)(n-1)+1)}((u(n-1)+1)x+z;a)}{((u+1)(n-1)+1)!(u(n-1)+1)\alpha}$$

and use the second identity of Mihoubi [6, Theorem 2]. □

Theorem 10. *Let d be an integer ≥ 1. The inverse relations hold*

$$\begin{aligned} y_n &= \sum_{j=1}^{n} (-1)^j (dn+j)_{(j-1)} B_{n,j}(x_1, x_2, \ldots), \\ x_n &= \sum_{j=1}^{n} (-1)^j (dn+j)_{(j-1)} B_{n,j}(y_1, y_2, \ldots). \end{aligned} \tag{12}$$

Proof. Let

$$f(t) = t\big(1 + \sum_{n\geq 1} x_n \frac{t^{dn}}{n!}\big) \text{ and } f^{\langle -1\rangle}(t) = t\big(1 + \sum_{n\geq 1} y_n \frac{t^{dn}}{n!}\big).$$

The proof now follows from Comtet [3, Theorem F, p. 151]. □

Example 11. Take $d = 1$ and $x_n = n!$, $n \geq 1$, in Theorem 10, we get

$$f(t) = \frac{t}{1-t} \text{ and } f^{\langle -1\rangle}(t) = \frac{t}{1+t},$$

i.e. $y_n = (-1)^n n!$, and the relations (12) give

$$\sum_{j=1}^{n} (-1)^{n-j} \binom{n}{j} \binom{n+j}{n+1} = n.$$

Take $d = 2$ and $x_n = n!$, $n \geq 1$, we get

$$f(t) = \frac{t}{1-t^2} \text{ and } f^{\langle -1\rangle}(t) = \frac{1}{2t}\big(-1 + \sqrt{1+4t^2}\big),$$

i.e. $y_n = (-1)^n \dfrac{(2n)!}{(n+1)!}$, $n \geq 1$, and the relations (12) give

$$\sum_{j=1}^{n} (-1)^{n-j} \binom{n}{j} \binom{2n+j}{2n+1} = \frac{n}{n+1}\binom{2n}{n} = nC_n,$$

where $C_n, n = 0, 1, 2, \cdots$, are the Catalan numbers.

Theorem 12. *The following inverse relations hold*

$$y_n = \tfrac{1}{nr}\tbinom{(r+1)n}{nr}^{-1} B_{(r+1)n,nr}(1, x_1, x_2, \ldots),\ r \geq 1,$$

$$x_n = (n+1)\textstyle\sum_{j=1}^{n} B_{n,\ j}(y_1, y_2, \ldots)(-1)^{j-1}(nr-1)^{j-1}.$$

Proof. From Mihoubi [7, Theorem 1] we have

$$x_1^k \sum_{j=1}^{n} B_{n,j}(y_1, y_2, \ldots)(k-nr)^{j-1} = \frac{x_1^{nr}}{k}\binom{n+k}{k}^{-1} B_{n+k,k}(x_1, x_2, x_3, \ldots),$$

with $y_n = \frac{1}{nr}\binom{(r+1)n}{nr}^{-1} B_{(r+1)n,nr}(1, x_1, x_2, \ldots)$, $nrk \geq 1$.
It just suffices to set $k = 1$, $x_1 = 1$, and replace x_n by x_{n-1}. □

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References

[1] M. Cerasoli, Two identities between Bell polynomials and Fibonacci numbers. *Boll. Un. Mat. Ital. A*, (5) **18** (1981), 387–394.

[2] W. S. Chou, L. C. Hsu, P. J. S. Shiue, Application of Faà di Bruno's formula in characterization of inverse relations. *J. Comput. Appl. Math* **190** (2006), 151–169.

[3] L. Comtet, *Advanced Combinatorics*. D. Reidel Publishing Company, 1974.

[4] H. Belbachir, S. Bouroubi, A. Khelladi, Connection between ordinary multinomials, generalized Fibonacci numbers, partial Bell partition polynomials and convolution powers of discrete uniform distribution. *Ann. Math. Inform.* **35** (2008), 21–30.

[5] M. Mihoubi, Bell polynomials and binomial type sequences. *Discrete Math.* **308** (2008), 2450–2459.

[6] M. Mihoubi, The role of binomial type sequences in determination identities for Bell polynomials. To appear, *Ars Combin.* Preprint available at `http://arxiv.org/abs/0806.3468v1`.

[7] M. Mihoubi, Some congruences for the partial Bell polynomials. *J. Integer Seq.* **12** (2009), Article 09.4.1.

[8] S. Roman, *The Umbral Calculus*, 1984.

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