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On the Shifted Product of Binary Recurrences

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Abstract

The present paper studies the diophantine equation $G_nH_n + c = x_{2n}$ and related questions, where the integer binary recurrence sequences $\{G\}$, $\{H\}$ and $\{x\}$ satisfy the same recurrence relation, and c is a given integer. We prove necessary and sufficient conditions for the solubility of $G_nH_n + c = x_{2n}$. Finally, a few relevant examples are provided.

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1 Introduction

Let the binary recurrences $\{a\}_{n=0}^{\infty}$, $\{b\}_{n=0}^{\infty}$, $\{c\}_{n=0}^{\infty}$ and $\{d\}_{n=0}^{\infty}$ satisfy the recurrence relation

$$X_{n+2} = 6X_{n+1} - X_n, \qquad n \in \mathbb{N}, \tag{1}$$

with the initial values

$$a_0 = -1,\ a_1 = 1; \quad b_0 = 1,\ b_1 = 5; \quad c_0 = 0,\ c_1 = 2; \quad d_0 = 3,\ d_1 = 17;$$

respectively (Sloane: $\{c\}$ is A001542, $\{d\}$ is A001541). Here, and in the sequel, $\mathbb{N}$ denotes the set of non-negative integers. By the recurrences above, we can define a fifth sequence $\{x\}_{n=0}^{\infty}$ via

$$x_{2n} = a_n b_n + 1 \quad , \quad x_{2n+1} = c_n d_n + 1, \qquad n \in \mathbb{N}. \tag{2}$$

The sequence $\{x\}$ (the first few terms are: 0, 1, 6, 35, 204, 1189, ..., Sloane: A001109) also satisfies (1), in spite of the unusual method of its composition, which we discovered while studying the terms of $\{x\}$, called balancing numbers. A positive integer $x \geq 2$ is called a *balancing number* for the integer y if

$$1 + \cdots + (x-1) = (x+1) + \cdots + (y-1) \tag{3}$$

holds. This definition was introduced by Finkelstein [3] (he called balancing numbers *numerical centers*) when he solved a puzzle from the book [1]. For other properties of balancing numbers see, for example [2]. Clearly, (3) leads to the Pell equation $(2y-1)^2 - 2(2x)^2 = 1$, and the possible values of x can be described by a suitable recurrence relation of type (1). It is known that all real binary recurrences satisfying (1) form a vector space of dimension 2 over $\mathbb{R}$. Definition (2) (and later (6)) does not fit the vector space structure. The question arises which other integer binary recurrences can possess the interesting property described by (2) or its generalization.

The phenomena appearing in (2) is a specific case of a more general property. Let $t \in \mathbb{N}^+$ and put $\alpha = t + \sqrt{t^2+1}$, $\beta = t - \sqrt{t^2+1}$. Define the sequences $\{T\}_{n=0}^{\infty}$ and $\{U\}_{n=0}^{\infty}$ by $\alpha^n = T_n + U_n\sqrt{t^2+1}$. It is easy to see that the sequences $\{T\}$ and $\{U\}$ both satisfy the recurrence relation $X_n = 2tX_{n-1} + X_{n-2}$ with the initial values $T_0 = 1$, $T_1 = t$ and $U_0 = 0$, $U_1 = 1$, respectively. The corresponding explicit formulae

$$T_n = \frac{\alpha^n + \beta^n}{2} \quad \text{and} \quad U_n = \frac{\alpha^n - \beta^n}{2\sqrt{t^2+1}}$$

show that

$$U_{n \mp (-1)^n} T_{n \pm (-1)^n} \mp t = \frac{1}{2} U_{2n}$$

holds. Subsequently, the choice $a_n = T_{2n-1}$, $b_n = U_{2n+1}$, $c_n = T_{2n+2}$, $d_n = U_{2n}$ implies $a_n b_n + t = U_{4n}/2 \in \mathbb{N}$ and $c_n d_n + t = U_{4n+2}/2 \in \mathbb{N}$. That is, the example given at the start of the introduction is a particular case of this construction with $t = 1$. Observe, that $X_n = (4t^2+2)X_{n-1} - X_{n-2}$ holds for all the sequences $\{a\}$, $\{b\}$, $\{c\}$ and $\{d\}$, and also for $\{x_n\} = \{U_{2n}/2\}$. Thus, we can provide infinitely many, but not all examples similar to (2) (see in Section 4) since the coefficients $A = 4t^2 + 2$ and $B = -1$ do not represent the most general case.

Now consider a slightly different property having a similar flavor. Assume that $A > 0$ and $B \neq 0$ are integers with non-vanishing $D = A^2 + 4B$, further let $\mathcal{B}$ denote the set of all integer binary recurrences $\{X\}_{n=0}^{\infty}$ satisfying the recurrence relation

$$X_{n+2} = AX_{n+1} + BX_n, \qquad n \in \mathbb{N}. \tag{4}$$

If $\{u\}_{n=0}^{\infty}$ and $\{v\}_{n=0}^{\infty}$ are elements of $\mathcal{B}$ with the initial values $u_0 = 0$, $u_1 = 1$ and $v_0 = 2$, $v_1 = A$, respectively, then the equality

$$u_n v_n = u_{2n} \tag{5}$$

holds for any non-negative integer n. Identity (5) is well-known, and all books or papers dealing, for instance, with the Fibonacci sequence mention the formula $F_n L_n = F_{2n}$, where F_n and L_n are the nth term of the Fibonacci (Sloane: A000045) and Lucas sequences (Sloane: A000032), respectively. Note, that the sequences $\{F\}$ and $\{L\}$ both satisfy (4) with $A = B = 1$.

In this paper, we examine how we can construct a binary recurrence $\{x\}$ as a shifted product of two binary recurrences satisfying the same recurrence relation. More precisely, given the integers c and l, we consider the equation

$$G_n H_n + c = x_{2n+l}, \tag{6}$$

where $\{G\}_{n=0}^{\infty}$, $\{H\}_{n=0}^{\infty}$ and $\{x\}_{n=0}^{\infty}$ belong to the class of binary recurrences given by (4). There is no restriction in assuming that $l = 0$, since l causes only a translation in the subscript.

Moreover, suppose that c is a fixed integer, and the binary recursive sequences $\{G\}$ and $\{J\}_{n=0}^{\infty}$ also belong to $\mathcal{B}$. Analogously to (2), we investigate and determine the binary recurrences $\{H\}$ and $\{K\}_{n=0}^{\infty}$ of $\mathcal{B}$, which generate a sequence $\{x\} \in \mathcal{B}$ satisfying

$$x_{2n} = G_n H_n + c \quad , \quad x_{2n+1} = J_n K_n + c, \qquad n \in \mathbb{N}.$$

It is worth noting that if one forgets about the integrality conditions, then obviously c can be considered to be either 0 or 1. Indeed, if $c \neq 0$, then we can replace $\{x_n\}$ by $\{x_n/c\}$, further $\{G_n\}$, $\{H_n\}$, $\{J_n\}$ and $\{K_n\}$ by $\{G_n/\sqrt{c}\}$, $\{H_n/\sqrt{c}\}$, $\{J_n/\sqrt{c}\}$ and $\{K_n/\sqrt{c}\}$, respectively, which results in assuming that $c = 1$.

2 Preliminaries

For any complex numbers α, β, $\ldots$ and for any sequence $\{X\}_{n=0}^{\infty} \in \mathcal{B}$ put $\alpha_X = X_1 - \alpha X_0$, $\beta_X = X_1 - \beta X_0$, etc. Recall, that $A > 0$ and $B \neq 0$ are integers and $D = A^2 + 4B \neq 0$.

Lemma 1. *Assume that $\{G\} \in \mathcal{B}$. Then the zeros $\alpha = (A + \sqrt{D})/2$ and $\beta = (A - \sqrt{D})/2$ of the companion polynomial $p(x) = x^2 - Ax - B$ of $\{G\}$ are distinct and nonzero. Further*

- $\alpha\beta = -B, \quad \alpha+\beta = A, \quad \alpha-\beta = \sqrt{D}$,
- $D > 0$ *implies* $\beta < \alpha$ *and* $1 < \alpha$.

Moreover,

$$G_n = \frac{\beta_G\alpha^n - \alpha_G\beta^n}{\sqrt{D}}. \tag{7}$$

Proof. All formulae and statements of Lemma 1 are known. Nevertheless, the first two conditions are immediate from the determination of α and β, while (7) can be derived from the basic theorem concerning the linear recurrences (see, for instance, [4]). Note that α and β are conjugate zeros of $p(x)$ if they are not integers. $\square$

Remark 2. Later we will use the results of Lemma 1 without any comment.

In the present and next sections we assume that $c \in \mathbb{Z}$, $l = 0$. Further $\{G\} \in \mathcal{B}$, $\{H\} \in \mathcal{B}$ and $\{x\} \in \mathcal{B}$ satisfy (6).

Lemma 3.

$$x_1 = \frac{G_1H_1 - BG_0H_0 + c(1-B)}{A}.$$

Proof. Combining $x_0 = G_0H_0 + c$, $x_2 = G_1H_1 + c$ and $x_2 = Ax_1 + Bx_0$, we immediately get the desired statement. $\square$

Now we define some important constants which will be useful in studying our problems. Put

$$E_\alpha = \frac{\beta_G\beta_H}{D} - \frac{\beta_x}{\sqrt{D}}, \qquad E_\beta = \frac{\alpha_G\alpha_H}{D} + \frac{\alpha_x}{\sqrt{D}}, \qquad E_{\alpha\beta} = \frac{\beta_G\alpha_H + \alpha_G\beta_H}{D}. \tag{8}$$

Further, let

$$\Delta = (2G_1 - AG_0)H_1 - (2BG_0 + AG_1)H_0 \in \mathbb{Z}. \tag{9}$$

Remark 4. We call $\{\widetilde{G}\}_{n=0}^{\infty}$ the companion sequence of $\{G\} \in \mathcal{B}$ if $\{\widetilde{G}\}$ is also in $\mathcal{B}$ and $\widetilde{G}_0 = 2G_1 - AG_0$, $\widetilde{G}_1 = 2BG_0 + AG_1$. Thus, Δ can be simplified as $\widetilde{G}_0H_1 - \widetilde{G}_1H_0$.

Sometimes it is more convenient to write (9) in the form $\Delta = 2G_1H_1 - 2BG_0H_0 - A(G_1H_0 + G_0H_1)$.

Lemma 5. *The quantities from (8) can also be given in the forms*

$$E_\alpha = \frac{\beta}{AD}\left(\Delta - \frac{1-\beta^2}{\beta}\sqrt{D}c\right), \qquad E_\beta = \frac{\alpha}{AD}\left(\Delta + \frac{1-\alpha^2}{\alpha}\sqrt{D}c\right), \qquad E_{\alpha\beta} = \frac{\Delta}{D}. \tag{10}$$

Proof. Recall the definition of E_α and $\beta_G = G_1 - \beta G_0$, $\beta_H = H_1 - \beta H_0$, $\beta_x = x_1 - \beta x_0$, $x_0 = G_0H_0 + c$. These formulae, together with Lemma 3, yield

$$E_\alpha = \frac{\beta}{AD}\left(2G_1H_1 - 2BG_0H_0 - A(G_1H_0 + G_0H_1) - \frac{1-\beta^2}{\beta}\sqrt{D}c\right).$$

In order to complete the proof on E_α, observe that the first three terms in the paranthesis above make up Δ (see Remark 4).

If $\alpha \notin \mathbb{Z}$, then E_β is a conjugate of E_α; hence, the second part of the lemma is obvious. Otherwise, we can use the same procedure we applied to E_α.

Finally, one can easily get $E_{\alpha\beta} = \Delta/D$ from the parts of definition (8) concerned with $E_{\alpha\beta}$. $\square$

3 The equation $G_nH_n + c = x_{2n}$

Now we proceed to the solution of the first problem. The following result provides necessary conditions for the solubility of equation (6).

Theorem 6. *Let $c \in \mathbb{Z}$. If there exist sequences $\{G\} \in \mathcal{B}$, $\{H\} \in \mathcal{B}$ and $\{x\} \in \mathcal{B}$ such that $G_nH_n + c = x_{2n}$ then one of the following cases holds.*

- *If $c = 0$ then $\Delta = 0$;*
- *if $c \neq 0$ then either $B = -1$, $\Delta = Dc$; or $\beta = \pm 1$, $\Delta = 0$.*

Remark 7. The assertion $B = -1$, together with $\beta \in \mathbb{R}$ implies $0 < \beta < 1$. Then the two instances in the second part of Theorem 6 are not-overlapping.

Proof. $G_nH_n + c = x_{2n}$ is equivalent to

$$E_\alpha \alpha^{2n} + E_\beta \beta^{2n} - E_{\alpha\beta}(\alpha\beta)^n + c = 0, \qquad n \in \mathbb{N}. \tag{11}$$

Since (11) is true for all n, therefore it is true for $n = 0,\ 1,\ 2,\ 3$. Consequently, (11) at $n = 0,\ 1,\ 2,\ 3$ is a homogeneous linear system of four equations in the unknowns $E_\alpha,\ E_\beta,\ E_{\alpha\beta}$ and c. The determinant

$$\mathcal{D} = \begin{vmatrix} 1 & 1 & -1 & 1 \\ \alpha^2 & \beta^2 & -\alpha\beta & 1 \\ \alpha^4 & \beta^4 & -(\alpha\beta)^2 & 1 \\ \alpha^6 & \beta^6 & -(\alpha\beta)^3 & 1 \end{vmatrix}$$

of the coefficient matrix is the negative of the Vandermonde of $\alpha^2,\ \beta^2,\ \alpha\beta$ and 1. Hence,

$$\mathcal{D} = -V(\alpha^2, \beta^2, \alpha\beta, 1) = \alpha\beta(\alpha - \beta)^3(\alpha + \beta)(1 - \alpha^2)(1 - \beta^2)(1 - \alpha\beta). \tag{12}$$

If the homogeneous system has only the trivial solution $c = 0$ and $E_\alpha = E_\beta = E_{\alpha\beta} = 0$, then, by Lemma 5, we obtain

$$E_\alpha = \frac{\beta}{AD}\Delta = 0, \qquad E_\beta = \frac{\alpha}{AD}\Delta = 0, \qquad E_{\alpha\beta} = \frac{1}{D}\Delta = 0.$$

The coefficient of Δ is nonzero in any of the three equalities. Subsequently, $\Delta = 0$ follows.

If the system has infinitely many solutions, then $\mathcal{D} = 0$. Recalling (12) and the conditions $\alpha\beta \neq 0$, $\alpha \neq \beta$ (see Lemma 1), we distinguish four cases.

1. $\alpha + \beta = 0$**.** We deduce a degenerate recursion characterized by $\alpha/\beta = -1$. Thus $A = 0$, and we have arrived at a contradiction.

2. $1 - \alpha^2 = 0$**.** Now $\alpha = \pm 1 \in \mathbb{R}$. This is impossible since $\alpha > 1$ holds.

3. $1 - \beta^2 = 0$**.** Both cases $\beta = \pm 1$ lead to the solutions $E_\alpha = E_{\alpha\beta} = 0$ and $E_\beta = -c$, where c is a free variable. Clearly, by (10), $E_{\alpha\beta}$ implies $\Delta = 0$.

4. $1-\alpha\beta=0$**.** Obviously, $B=-1$. Moreover $D=A^2-4\neq 0$ yields $A\neq 2$. The infinitely many solutions of the system can be described by $E_\alpha=E_\beta=0$ and

$$c=-\frac{\alpha\beta(\alpha-\beta)^2}{(\alpha^2-1)(\beta^2-1)}E_{\alpha\beta}. \tag{13}$$

Since $\alpha\beta=B$ and $(\alpha-\beta)^2=A^2-4$, relation (13) together with $(\alpha^2-1)(\beta^2-1)=(A\alpha-2)(A\beta-2)=-A^2+4$ gives $E_{\alpha\beta}=c$, where c is a free variable. Then, again by (10), $\Delta=Dc$.

□

Theorem 6 provides necessary conditions for the sequences $\{G\}$, $\{H\}$ and $\{x\}$ satisfying $G_nH_n+c=x_{2n}$. Now we show that the conditions are, essentially, sufficient as well. More precisely, we prove the following theorem.

Theorem 8. *Suppose that the integers c, G_0, G_1, H_0 and H_1 are fixed. Put $x_0=G_0H_0+c$. If*

$$\begin{aligned}
&c=0,\ \Delta=0;\ \textit{or}\\
&c\neq 0,\ B=-1,\ \Delta=Dc;\ \textit{or}\\
&c\neq 0,\ \beta=\pm 1,\ \Delta=0,
\end{aligned}$$

hold, as well as

$$\begin{aligned}
x_1 &= \frac{G_1H_1-BG_0H_0}{A}\in\mathbb{Z}, \quad \textit{or}\\
x_1 &= \frac{G_1H_1+G_0H_0+2c}{A}\in\mathbb{Z}, \quad \textit{or}\\
x_1 &= \frac{G_1H_1-BG_0H_0+(1-B)c}{A}\in\mathbb{Z},
\end{aligned}$$

respectively, then the terms of the sequences $\{G\}\in\mathcal{B}$, $\{H\}\in\mathcal{B}$ and $\{x\}\in\mathcal{B}$ satisfy

$$G_nH_n+c=x_{2n}, \qquad n\in\mathbb{N}.$$

Remark 9. Theorem 8 asserts necessarily that x_1 is an integer, otherwise $\{x\}$ would not be an integer sequence although it satisfies $G_nH_n+c=x_{2n}$. For example, let $c=0$, $A=14$, $B=-5$, $G_0=2$, $G_1=3$, $H_0=-1$, $H_1=1$. Then $\Delta=0$, $x_1=-1/2$, and

$$\begin{aligned}
\{G\}&:\quad 2,\ 3,\ 32,\ 433,\ \ldots\ ;\\
\{H\}&:\quad -1,\ 1,\ 19,\ 261,\ \ldots\ ;\\
\{x\}&:\quad -2=G_0H_0,\ -\frac{1}{2},\ 3=G_1H_1,\ \frac{89}{2},\ 608=G_2H_2,\ \frac{16579}{2},\ 113013=G_3H_3,\ \ldots\ .
\end{aligned}$$

Proof. We will use the notation of the previous sections. By (9), it is easy to verify that $G_1H_0+G_0H_1=(2G_1H_1-2BG_0H_0-\Delta)/A$. Combining it with

$$\beta_G\beta_H=G_1H_1+\beta^2G_0H_0-\beta(G_1H_0+G_0H_1) \quad\text{and}\quad \alpha_G\alpha_H=G_1H_1+\alpha^2G_0H_0-\alpha(G_1H_0+G_0H_1),$$

respectively, we obtain

$$\beta_G\beta_H = \frac{\sqrt{D}(G_1H_1 - \beta^2 G_0H_0) + \beta\Delta}{A} \quad \text{and} \quad \alpha_G\alpha_H = \frac{\sqrt{D}(-G_1H_1 + \alpha^2 G_0H_0) + \alpha\Delta}{A}. \tag{14}$$

By $x_0 = G_0H_0 + c$ and Lemma 3, we also have

$$\beta_x = x_1 - \beta x_0 = \frac{G_1H_1 - \beta^2 G_0H_0 + (1-\beta^2)c}{A} \quad \text{and} \quad \alpha_x = x_1 - \alpha x_0 = \frac{G_1H_1 - \alpha^2 G_0H_0 + (1-\alpha^2)c}{A}. \tag{15}$$

Thus, by (14) and (15),

$$G_nH_n + c = \frac{\beta_G\alpha^n - \alpha_G\beta^n}{\sqrt{D}}\,\frac{\beta_H\alpha^n - \alpha_H\beta^n}{\sqrt{D}} + c = \frac{\beta_G\beta_H\alpha^{2n} + \alpha_G\alpha_H\beta^{2n} - \Delta(\alpha\beta)^n + cD}{D},$$

which is equivalent to

$$\frac{\beta_x\alpha^{2n} - \alpha_x\beta^{2n}}{\sqrt{D}} + \underbrace{\frac{1}{A}\left(\frac{\beta\Delta}{D} - \frac{1-\beta^2}{\sqrt{D}}c\right)\alpha^{2n} + \frac{1}{A}\left(\frac{\alpha\Delta}{D} + \frac{1-\alpha^2}{\sqrt{D}}c\right)\beta^{2n} - \left(\frac{\Delta}{D}(\alpha\beta)^n - c\right)}_{W}. \tag{16}$$

Here, the first summand is just x_{2n}. Now we show that the remaining part of (16), denoted by W, vanishes under certain conditions. First, suppose that $c = 0$ and $\Delta = 0$. Then, obviously $W = 0$.

Assume now $c \neq 0$, and consider the case $B = -1$ (i.e., $\alpha\beta = 1$) with $\Delta = cD$. It follows that

$$W = \frac{c}{A}\left(\beta - \frac{1-\beta^2}{\sqrt{D}}\right)\alpha^{2n} + \frac{c}{A}\left(\alpha + \frac{1-\alpha^2}{\sqrt{D}}\right)\beta^{2n} = \frac{c}{A\sqrt{D}}\left(\alpha^{2n} - \beta^{2n}\right)(\alpha\beta - 1).$$

But the last factor vanishes since $\alpha\beta = 1$. Consequently, W is zero again.

In the third case, we have $\beta = \pm 1$ and $\Delta = 0$. Thus,

$$W = \frac{c}{A}\left(-\frac{1-\beta^2}{\sqrt{D}}\right)\alpha^{2n} + \frac{c}{A}\left(\frac{1-\alpha^2}{\sqrt{D}}\right)\beta^{2n} + c = \frac{c}{A}\left(\frac{1-\alpha^2}{\sqrt{D}}\right) + c = \frac{c}{A\sqrt{D}}(1-\beta^2) = 0.$$

The proof is therefore complete. □

4 The system $x_{2n} = G_nH_n + c$, $x_{2n+1} = J_nK_n + c$

The investigation here is based on the results of the previous sections. We have to handle only two more problems, namely the question of the translation by 1 in the subscript, and the question of putting two subsequences of $\{x\}$ together.

According to the definition of Δ, we introduce the notation $\Delta_2 = (2J_1 - AJ_0)K_1 - (2BJ_0 + AJ_1)K_0 \in \mathbb{Z}$. The next statement is an immediate consequence of Theorem 6.

Theorem 10. *Suppose that $c \in \mathbb{Z}$ and the sequences $\{G\}$, $\{H\}$, $\{J\}$, $\{K\}$ and $\{x\}$, all are in $\mathcal{B}$, satisfy $x_{2n} = G_nH_n + c$ and $x_{2n+1} = J_nK_n + c$. Suppose furthermore that the following two conditions*

- *If $c = 0$ then $\Delta = \Delta_2 = 0$,*
- *if $c \neq 0$ then either $B = -1$, $\Delta = \Delta_2 = Dc$ or $\beta = \pm 1$, $\Delta = \Delta_2 = 0$*

hold.

Recall, that $A > 0,\ B \neq 0$ with $D \neq 0$. Now we are going to formulate the reciprocal of Theorem 10.

Theorem 11. *Let $c \in \mathbb{Z}$ be given. Suppose that $\{G_n\}$, $\{H_n\}$, $\{J_n\}$, $\{K_n\}$ all are elements of the set $\mathcal{B}$, and the integers $G_i,\ H_i,\ J_i,\ K_i$ $(i = 0,\ 1)$ fulfill one of the following conditions*

- *$\Delta = \Delta_2 = 0$ if $c = 0$;*
- *$B = -1$ and $\Delta = \Delta_2 = Dc$ if $c \neq 0$;*
- *$\beta = \pm 1$ and $\Delta = \Delta_2 = 0$ if $c \neq 0$.*

If $x_0 = G_0H_0 + c$, $x_1 = J_0K_0 + c$, $x_2 = G_1H_1 + c$ and $x_3 = J_1K_1 + c$ satisfy $x_2 = Ax_1 + Bx_0$ and $x_3 = Ax_2 + Bx_1$, then the sequence $\{x_n\}$ given by

$$x_{2n} = G_nH_n + c \quad , \quad x_{2n+1} = J_nK_n + c, \qquad n \in \mathbb{N}$$

belongs to $\mathcal{B}$.

Proof. First we prove that, under the given conditions, the relation $x_{2n+2} = Ax_{2n+1} + Bx_{2n}$ holds, i.e.,

$$G_{n+1}H_{n+1} + c = A(J_nK_n + c) + B(G_nH_n + c).$$

By (7), it is equivalent to

$$\begin{aligned}&\frac{\beta_G\alpha^{n+1} - \alpha_G\beta^{n+1}}{\sqrt{D}}\frac{\beta_H\alpha^{n+1} - \alpha_H\beta^{n+1}}{\sqrt{D}} + c = \\ &= A\left(\frac{\beta_J\alpha^n - \alpha_J\beta^n}{\sqrt{D}}\frac{\beta_K\alpha^n - \alpha_K\beta^n}{\sqrt{D}} + c\right) + B\left(\frac{\beta_G\alpha^n - \alpha_G\beta^n}{\sqrt{D}}\frac{\beta_H\alpha^n - \alpha_H\beta^n}{\sqrt{D}} + c\right),\end{aligned}$$

which in turn is equivalent to

$$\alpha^{2n}\underbrace{\left((\alpha^2 - B)\beta_G\beta_H - A\beta_J\beta_K\right)}_{W_1} + \beta^{2n}\underbrace{\left((\beta^2 - B)\alpha_G\alpha_H - A\alpha_J\alpha_K\right)}_{W_2} + \underbrace{(-B)\left(2B\Delta + A\Delta_2\right) + (1 - A - B)Dc}_{W_3} = 0. \quad (17)$$

The value of W_3 seems to be the simplest to determine. Obviously, $\Delta = \Delta_2 = 0$, and $c = 0$ provide immediately $W_3 = 0$. If $c \neq 0$ and $\Delta = \Delta_2 = Dc$, $B = -1$, then again $W_3 = 0$. Otherwise, when $c \neq 0$ and $\Delta = \Delta_2 = 0$, $\beta = \pm 1$, we must distinguish two cases. When $\beta = 1$, we obtain $W_3 = 0$, while $\beta = -1$ we get $W_3 = 2(1 - \alpha)Dc$.

We next compute W_1. Here condition $\alpha^2 - B = A\alpha$, together with the definition of β_G, β_H, β_J and β_K, provides

$$W_1 = A\left(\alpha G_1H_1 - \beta BG_0H_0 + B(G_1H_0 + G_0H_1)\right) - \frac{A}{\alpha}\left(\alpha J_1K_1 - \beta BJ_0K_0 + B(J_1K_0 + J_0K_1)\right). \tag{18}$$

We can replace the terms $G_1H_0 + G_0H_1$ and $J_1K_0 + J_0K_1$ in (18) by

$$\frac{2G_1H_1 - 2BG_0H_0 - \Delta}{A} \quad \text{and} \quad \frac{2J_1K_1 - 2BJ_0K_0 - \Delta_2}{A},$$

respectively. Thus,

$$\begin{aligned} W_1 &= (A\alpha + 2B)G_1H_1 - B(\beta A + 2B)G_0H_0 - B\Delta - \\ &\quad \frac{1}{\alpha}\left((A\alpha + 2B)J_1K_1 - B(\beta A + 2B)J_0K_0 - B\Delta_2\right) \\ &= \left(\alpha\sqrt{D}G_1H_1 + \beta B\sqrt{D}G_0H_0 - B\Delta\right) - \\ &\quad \frac{1}{\alpha}\left(\alpha\sqrt{D}J_1K_1 + \beta B\sqrt{D}J_0K_0 - B\Delta_2\right), \end{aligned}$$

since $\alpha A + 2B = \alpha\sqrt{D}$ and $B(\beta A + 2B) = -\beta B\sqrt{D}$. Using $G_0H_0 = x_0 - c$, etc., it follows that

$$W_1 = \sqrt{D}\left((\alpha x_2 - x_3) + \frac{\beta B}{\alpha}(\alpha x_0 - x_1) + \left(1 - \alpha - \beta B + \frac{\beta B}{\alpha}\right)c\right) + \left(\frac{\Delta_2}{\alpha} - \Delta\right)B.$$

We use $x_3 = Ax_2 + Bx_1$ and $x_2 = Ax_1 + Bx_0$ to get $\alpha x_2 - x_3 = \beta^2(\alpha x_0 - x_1)$. Consequently, we get $(\alpha x_2 - x_3) + \frac{\beta B}{\alpha}(\alpha x_0 - x_1) = 0$. Since $1 - \alpha - \beta B + \beta B/\alpha = (\alpha - 1)(\beta^2 - 1)$, we conclude that

$$W_1 = \sqrt{D}(\alpha - 1)(\beta^2 - 1)c + B\left(\frac{\Delta_2}{\alpha} - \Delta\right). \tag{19}$$

Similar arguments applied to W_2 give

$$W_2 = -\sqrt{D}(\beta - 1)(\alpha^2 - 1)c + B\left(\frac{\Delta_2}{\beta} - \Delta\right). \tag{20}$$

Note that if $\alpha \notin \mathbb{Z}$ then W_1 and W_2 are conjugates, therefore (20) comes directly from (19).

Clearly, $W_1 = W_2 = 0$ if $c = 0$ and $\Delta = \Delta_2 = 0$ hold. Now consider the second possibility, when $c \neq 0$, $\Delta = \Delta_2 = Dc$ and $B = -1$. The last condition shows that β is the reciprocal of α. Thus, the coefficients $(\alpha - 1)(\beta^2 - 1)$ and $(\beta - 1)(\alpha^2 - 1)$ in (19) and (20), respectively, coincide $(\beta-1)\sqrt{D}$ and $-(\alpha-1)\sqrt{D}$, respectively. Hence, $W_1 = W_2 = 0$ holds again. Finally, $c \neq 0$ and $\Delta = \Delta_2 = 0$ yield $W_1 = \sqrt{D}c(1-\alpha)(1-\beta^2)$ and $W_2 = -\sqrt{D}c(1-\beta)(1-\alpha^2)$. This leads to $\beta = \pm 1$. Subsequently, $W_1 = 0$, while $W_2 = 0$, or $W_2 = -2\sqrt{D}(1 - \alpha^2)$ depending on the sign of β.

Consider now W_1, W_2 and W_3 again. Ignoring the trivial cases, it is sufficient to look at only the case $\beta = -1$. Now $W_1 = 0$, $W_2 = -2\sqrt{D}c(1-\alpha^2)$ and $W_3 = 2(1-\alpha)Dc$. Therefore,

$$x_{2n+2} - Ax_{2n+1} - Bx_{2n} = 0\cdot\alpha^{2n} - 2\sqrt{D}c(1-\alpha^2)(-1)^{2n} + 2(1-\alpha)Dc = -2\sqrt{D}c(1-\alpha)(1+\beta) = 0.$$

So, we have showed that $x_{2n+2} = Ax_{2n+1} + Bx_{2n}$. The argument for $x_{2n+3} = Ax_{2n+2} + Bx_{2n}$ is entirely similar to the procedure we have just applied. $\square$

5 Examples

Example 12. This example indicates some difficulties arising if $c \in \mathbb{Z}$, the sequence $\{G\} \in \mathcal{B}$ is fixed, and one intends to determine a sequence $\{H\} \in \mathcal{B}$ (and $\{x\} \in \mathcal{B}$) such that $G_n H_n + c = x_{2n}$.

Let $A = 3$, $B = -1$ and $c = 7$, $G_0 = 1$, $G_1 = 2$ (Sloane: A001519). Thus, $D = 5$, $\Delta = Dc = 35$. To find $\{H\}$, it is necessary to solve the linear diophantine equation

$$H_1 - 4H_0 = 35$$

(see Remark 4). Clearly, it is solvable. For example, put $H_0 = t \in \mathbb{Z}$ and $H_1 = 4t + 35$. Now, by Theorem 8, we must verify that x_1 is an integer. Here, $x_1 = 3t + 28$. The sequences are given by

$$\begin{array}{rcl} \{G\} & : & 1,\ 2,\ 5,\ 13,\ \ldots\ ; \\ \{H\} & : & t,\ 4t+35,\ 11t+105,\ 29t+280,\ \ldots\ ; \\ \{x\} & : & t+7,\ \ 3t+28,\ \ 8t+77,\ \ 21t+203,\ \ 55t+532,\ \ 144t+1393,\ \ 377t+3467,\ \ \ldots\ . \end{array}$$

It is easy to see that for any $G_1 \in \mathbb{Z}$ there exist suitable sequences $\{H\}$. Indeed, for an arbitrary G_1 we obtain the diophantine equation

$$(2G_1 - 3)H_1 + (2 - 3G_1)H_0 = 35. \tag{21}$$

But (21) is solvable in H_1 and H_0 since $\gcd(2G_1 - 3, 2 - 3G_1)$ divides 5.

Note, that there are fewer problems in the other two cases of Theorem 8, since, instead of Dc, the right hand side of the corresponding linear diophantine equations is zero. Therefore, we only have to guarantee $x_1 \in \mathbb{Z}$.

Example 13. We give a class of recurrences $\{G\}$ having no pair $\{H\}$ with the desired property. Suppose that $B = -1$, $c = 1$, and to facilitate the calculations, let $G_1 = G_0$.

Thus $\gcd(2G_1 - AG_0, -2G_0 + AG_1) = G_0 \gcd(2 - A, -2 + A) = G_0(A - 2)$. But $D = A^2 - 4 \neq 0$ excludes $A = 2$, therefore

$$G_0(A-2) \nmid (A^2 - 4) \quad \Longleftrightarrow \quad G_0 \nmid (A+2).$$

Hence, if we choose an integer G_0 which does not divide $A + 2$, then the linear diophantine equation

$$G_0(2 - A)H_1 + G_0(A - 2)H_0 = A^2 - 4$$

is not solvable in integers H_1 and H_0.

Example 14. Consider now the first example in the Introduction, where the composition of four sequences was defined. Let $A = 6$, $B = -1$ and $c = 1$. Let also $G_0 = -1$, $G_1 = 1$, $J_0 = 0$, and $J_1 = 2$. We have $D = 32$, $\Delta = 32$. Both of the diophantine equations

$$\begin{aligned} 8H_1 - 8H_0 &= 32 \\ 4K_1 - 12K_0 &= 32 \end{aligned}$$

are solvable. Take the parametrizations $H_0 = t \in \mathbb{Z}$, $H_1 = t+4$ and $K_0 = s \in \mathbb{Z}$, $K_1 = 3s+8$. To join the subsequences with even and odd subscripts, we must check the recurrence relation $x_{n+2} = 6x_{n+1} - x_n$. Since the four sequences

$$\begin{aligned}
\{G\} &: \quad -1,\ 1,\ 7,\ 41,\ \ldots\ ; \\
\{H\} &: \quad t,\ t+4,\ 5t+24,\ 29t+140,\ \ldots\ ; \\
\{J\} &: \quad 0,\ 2,\ 12,\ 70,\ \ldots\ ; \\
\{K\} &: \quad s,\ 3s+8,\ 17s+48,\ 99s+280,\ \ldots
\end{aligned}$$

generate

$$\{x\} \ : \quad -t+1,\ \ 1,\ \ t+5,\ \ 6s+17,\ \ 35t+169,\ \ 204s+577,\ \ \ldots\ ,$$

therefore, by the given recurrence rule, $6(t+5)-1 = 6s+17$, or equivalently $s = t+2$ must hold. So we obtain

$$\{x\} \ : \quad -t+1,\ \ 1,\ \ t+5,\ \ 6t+29,\ \ 35t+169,\ \ 204t+985,\ \ \ldots$$

where the particular case $t = 1$ gives the sequence

$$\{x\} \ : \quad 0,\ \ 1,\ \ 6,\ \ 35,\ \ 204,\ \ 1189,\ \ \ldots \qquad .$$

of balancing numbers.

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References

[1] J. P. Adams, *Puzzles for Everybody*, Avon Publications, 1955.

[2] A. Behera and G. Panda, On the square roots of triangular numbers, *Fibonacci Quart.*, **37** (1999), 98–105.

[3] R. P. Finkelstein, The house problem, *American Math. Monthly*, **72** (1965), 1082–1088.

[4] T. N. Shorey and R. Tijdeman, *Exponential Diophantine Equations*, Cambridge University Press, 1986.

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