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# Integral Representations and Binomial Coefficients

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**Abstract**

In this article, we present two extensions of Sofo's theorems on integral representations of ratios of reciprocals of double binomial coefficients. From the two extensions, we get several new relations between integral representations and binomial coefficients.

## 1 Introduction

Recently, Sofo [10] extended the result in relation to the integral representations of ratios of reciprocals of the double binomial coefficients with the help of Beta function in integral form. In [10], Sofo investigated integral representations for

$$\sum_{n=0}^{\infty} f_n(a, b, j, k, t),$$

which is a function of the reciprocal double binomial coefficients and derivatives, and then Sofo reproved many results in [4, 5, 12].

For the completion of this article, we reproduce the Γ-function defined through Euler integral

$$\Gamma(x) = \int_0^\infty u^{x-1} e^{-u} \mathrm{d}u \quad \text{with} \quad \Re(x) > 0,$$

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and Beta function

$$B(s,t)=\int_0^1 z^{s-1}(1-z)^{t-1}dz=\frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}\quad \text{for } \Re(s)>0 \text{ and } \Re(t)>0$$

which is very useful in the work of simplification and representation of binomial sums in closed integral form. The reader may refer to [1, 2, 7, 13]. Throughout this paper, $\mathbb{N}$ and $\mathbb{R}$ denote the natural numbers and the real numbers, respectively.

In fact, Sofo first gave the following theorem in [9] about the relation between the integral representations and double binomial coefficients.

**Theorem 1.** *For $t\in\mathbb{R}$ and $a,\ b,\ n,\ j$ and $k\in\mathbb{N}$ subject to $|t|\le 1$ and $j,\ k>0$, then*

$$\sum_{n=0}^{\infty}\frac{t^n}{\binom{an+j}{j}\binom{bn+k}{k}}=jk\int_0^1\int_0^1\frac{(1-x)^{j-1}(1-y)^{k-1}}{1-tx^ay^b}dxdy$$

$$={}_{j+k+1}F_{j+k}\left[\begin{matrix}1,\ \frac{1}{a},\ \frac{2}{a},\ \cdots,\ \frac{j}{a},\ \frac{1}{b},\ \frac{2}{b},\ \cdots,\ \frac{k}{b}\\ \frac{a+1}{a},\ \frac{a+2}{a},\cdots,\ \frac{a+j}{a},\ \frac{b+1}{b},\ \frac{b+2}{b},\cdots,\ \frac{b+k}{b}\end{matrix}\ \Big|\ t\right].$$

The ${}_{j+k+1}F_{j+k}(t)$ in this theorem is the general hypergeometric series, the readers can refer to Bailey [3] and Slater [11].

In recent work [10], Sofo extended Theorem 1 by the following theorem.

**Theorem 2.** *For $t\in\mathbb{R}$ and $a,\ b,\ c,\ d,\ n,\ j$ and $k\in\mathbb{N}$ subject to $a\ge b,\ c\ge d,\ j,\ k>0$ and*

$$\Big|t\frac{b^b(a-b)^{a-b}d^d(c-d)^{c-d}}{a^ac^c}\Big|\le 1,$$

*then*

$$\begin{aligned}S(a,b,c,d,j,k,t)&=\sum_{n=0}^{\infty}\frac{t^n}{\binom{an+j}{bn}\binom{bn+k}{dn}}\\&=(a-b)(c-d)\int_0^1\int_0^1\frac{(1-x)^{j-1}(1-y)^{k-1}U(1+U)}{(1-U)^3}\,dxdy\\&+\Big[(a-b)k+(c-d)j\Big]\int_0^1\int_0^1\frac{(1-x)^{j-1}(1-y)^{k-1}U}{(1-U)^2}\,dxdy\\&+jk\int_0^1\int_0^1\frac{(1-x)^{j-1}(1-y)^{k-1}}{1-U}\,dxdy,\end{aligned}$$

*where*

$$U:=tx^b(1-x)^{a-b}y^d(1-y)^{c-d}.$$

The purpose of this paper is to present two extensions of Sofo's Theorems 1 and 2, and then get some new results on the relations between integral representations and binomial coefficients.

## 2 The first extension of Sofo's theorems

In this section, we prove an extension of Sofo's theorems, which will lead to several new relations between double integral representations and binomial coefficients.

**Theorem 3** (The first extension). *For $t \in \mathbb{R}$ and $a,\ b,\ c,\ d,\ n,\ i,\ j,\ k$ and $l \in \mathbb{N}$ subject to $a \geq b,\ c \geq d,\ j,\ k > 0,\ i,\ l \geq 0$ and*

$$\left|t\frac{b^b(a-b)^{a-b}d^d(c-d)^{c-d}}{a^a c^c}\right| \leq 1,$$

*then*

$$\begin{aligned} &Q(a,b,c,d,i,j,k,l,t) = \sum_{n=0}^{\infty} \frac{t^n}{\binom{an+j}{bn+i}\binom{cn+k}{dn+l}} \\ &= (a-b)(c-d)\int_0^1\int_0^1 \frac{x^i(1-x)^{j-i-1}y^l(1-y)^{k-l-1}U(1+U)}{(1-U)^3}\,dxdy \\ &+\big[(a-b)(k-l)+(c-d)(j-i)\big]\int_0^1\int_0^1 \frac{x^i(1-x)^{j-i-1}y^l(1-y)^{k-l-1}U}{(1-U)^2}\,dxdy \\ &+(j-i)(k-l)\int_0^1\int_0^1 \frac{x^i(1-x)^{j-i-1}y^l(1-y)^{k-l-1}}{1-U}\,dxdy, \end{aligned}$$

*where*

$$U := tx^b(1-x)^{a-b}y^d(1-y)^{c-d}.$$

Obviously, when $a = b$, $c = d$ and $i = l = 0$, Theorem 3 reduces to Theorem 1 which is due to Sofo [9]. Letting $i = l = 0$, $a \neq b$ and $c \neq d$ in Theorem 3, we obtain Theorem 2 which is given by Sofo [10].

*Proof.* The summation of double binomial coefficients in this theorem can be expressed as follows:

$$\begin{aligned} Q(a,b,c,d,i,j,k,l,t) &= \sum_{n=0}^{\infty} \frac{\Gamma(bn+i+1)\big[(a-b)n+j-i\big]\Gamma\big((a-b)n+j-i\big)}{\Gamma(an+j+1)} \\ &\times \frac{\Gamma(dn+l+1)\big[(c-d)n+k-l\big]\Gamma\big((c-d)n+k-l\big)}{\Gamma(cn+k+1)}t^n \\ &= \sum_{n=0}^{\infty} \Big[(a-b)n+j-i\Big]\Big[(c-d)n+k-l\Big]t^n \\ &\times B\big(bn+i+1,(a-b)n+j-i\big)B\big(dn+l+1,(c-d)n+k-l\big). \end{aligned}$$

Expanding the beta functions by the integral function, we express the $Q(a, b, c, d, i, j, k, l, t)$ as follows.

$$
\begin{aligned}
Q(a,b,c,d,i,j,k,l,t) &= \sum_{n=0}^{\infty} \Big[(a-b)n + j - i\Big]\Big[(c-d)n + k - l\Big] t^n \\
&\times \int_0^1 x^{bn+i}(1-x)^{(a-b)n+j-i-1} dx \int_0^1 y^{dn+l}(1-y)^{(c-d)n+k-l-1} dy \\
&= \sum_{n=0}^{\infty} \Big\{(a-b)(c-d)n^2 + \big[(a-b)(k-l) + (c-d)(j-i)\big]n + (j-i)(k-l)\Big\} \\
&\times \int_0^1 \int_0^1 x^i(1-x)^{j-i-1} y^l (1-y)^{k-l-1} \Big[t x^b (1-x)^{a-b} y^d (1-y)^{c-d}\Big]^n dxdy
\end{aligned}
$$

Exchanging the double integral function and summation, we get

$$
\begin{aligned}
Q(a,b,c,d,i,j,k,l,t) &= \int_0^1 \int_0^1 x^i(1-x)^{j-i-1} y^l (1-y)^{k-l-1} \Big\{ \sum_{n=0}^{\infty} (a-b)(c-d)n^2 U^n \\
&+ \big[(a-b)(k-l) + (c-d)(j-i)\big] n U^n + (j-i)(k-l) U^n \Big\} dxdy \\
&= (a-b)(c-d) \int_0^1 \int_0^1 \frac{x^i(1-x)^{j-i-1} y^l (1-y)^{k-l-1} U(1+U)}{(1-U)^3} dxdy \\
&+ \big[(a-b)(k-l) + (c-d)(j-i)\big] \int_0^1 \int_0^1 \frac{x^i(1-x)^{j-i-1} y^l (1-y)^{k-l-1} U}{(1-U)^2} dxdy \\
&+ (j-i)(k-l) \int_0^1 \int_0^1 \frac{x^i(1-x)^{j-i-1} y^l (1-y)^{k-l-1}}{1-U} dxdy,
\end{aligned}
$$

where we have applied the derivation operator in the last equality to evaluate the summations. Here the requirement $|tb^b(a-b)^{a-b}d^d(c-d)^{c-d}/a^a c^c| \le 1$ is for convergence. $\square$

## 2.1 Example

Letting $a = c = j = 2,\ b = d = i = k = t = 1$ and $l = 0$ in Theorem 3, we have the following result:

$$
\begin{aligned}
&Q\left(2,1,2,1,1,2,1,0,1\right) = \sum_{n=0}^{\infty} \frac{1}{\binom{2n+2}{n+1}\binom{2n+1}{n}} \\
&= \int_0^1 \int_0^1 \frac{x^2y(1-x)(1-y)\left[1+xy(1-x)(1-y)\right]}{\left[1-xy(1-x)(1-y)\right]^3}\, dxdy \\
&+ 2\int_0^1 \int_0^1 \frac{x^2y(1-x)(1-y)}{\left[1-xy(1-x)(1-y)\right]^2}\, dxdy \\
&+ \int_0^1 \int_0^1 \frac{x}{1-xy(1-x)(1-y)}\, dxdy \\
&= \int_0^1 \int_0^1 \frac{x(1+xy-x^2y-xy^2+x^2y^2)}{\left[1-xy(1-x)(1-y)\right]^3}\, dxdy.
\end{aligned}
$$

## 2.2 Example

Letting $a = 3,\ c = k = 2$ and $b = d = i = j = l = t = 1$ in Theorem 3, we get

$$
\begin{aligned}
&Q\left(3,1,2,1,1,1,2,1,1\right) = \sum_{n=0}^{\infty} \frac{1}{\binom{3n+1}{n+1}\binom{2n+2}{n+1}} \\
&= 2\int_0^1 \int_0^1 \frac{x^2y^2(1-x)(1-y)\left[1+xy(1-x)^2(1-y)\right]}{\left[1-xy(1-x)^2(1-y)\right]^3}\, dxdy \\
&+ 2\int_0^1 \int_0^1 \frac{x^2y^2(1-x)(1-y)}{\left[1-xy(1-x)^2(1-y)\right]^2}\, dxdy \\
&= 4\int_0^1 \int_0^1 \frac{x^2y^2(1-x)(1-y)}{\left[1-xy(1-x)(1-y)^2\right]^3}\, dxdy.
\end{aligned}
$$

### 2.3 Example

Letting $a=j=3,\ b=c=i=k=2$ and $d=l=t=1$ in Theorem 3, we have

$$Q\left(3,2,2,1,2,3,2,1,1\right)=\sum_{n=0}^{\infty}\frac{1}{\binom{3n+3}{2n+2}\binom{2n+2}{n+1}}$$

$$=\int_0^1\int_0^1\frac{x^4y^2(1-x)(1-y)\big[1+x^2y(1-x)(1-y)\big]}{\big[1-x^2y(1-x)(1-y)\big]^3}\,dxdy$$

$$+\,2\int_0^1\int_0^1\frac{x^4y^2(1-x)(1-y)}{\big[1-x^2y(1-x)(1-y)\big]^2}\,dxdy$$

$$+\int_0^1\int_0^1\frac{x^2y}{1-x^2y(1-x)(1-y)}\,dxdy$$

$$=\int_0^1\int_0^1\frac{x^2y\big(1+x^2y-x^3y-x^2y^2+x^3y^2\big)}{\big[1-x^2y(1-x)(1-y)\big]^3}\,dxdy.$$

In fact, $Q(3,2,2,1,2,3,2,1,1)$ can be expressed as the Hakmem series [6] as follows:

$$Q(3,2,2,1,2,3,2,1,1)=\sum_{n=0}^{\infty}\frac{1}{\binom{3n+3}{2n+2}\binom{2n+2}{n+1}}$$

$$=\sum_{n=0}^{\infty}\frac{(n+1)!(n+1)!(n+1)!}{(3n+3)!}=\sum_{n=0}^{\infty}\frac{n!n!n!}{(3n)!}-1$$

$$=-1+\int_0^1\Big\{\frac{2(8+7t^2-7t^3)}{(4-t^2+t^3)^2}$$

$$+\,\frac{4t(1-t)(5+t^2-t^3)}{(4-t^2+t^3)^2\sqrt{(1-t)(4-t^2+t^3)}}\arccos\Big(\frac{2-t^2+t^3}{2}\Big)\Big\}dt$$

## 2.4 Example

Letting $a = 4$, $b = c = j = k = 2$ and $d = i = l = t = 1$ in Theorem 3, we derive the following relation.

$$Q\left(4,2,2,1,1,2,2,1,1\right) = \sum_{n=0}^{\infty} \frac{1}{\binom{4n+2}{2n+1}\binom{2n+2}{n+1}}$$

$$= 2\int_0^1\int_0^1 \frac{x^3y^2(1-x)^2(1-y)\big[1+x^2y(1-x)^2(1-y)\big]}{\big[1-x^2y(1-x)^2(1-y)\big]^3}\,dxdy$$

$$+ 3\int_0^1\int_0^1 \frac{x^3y^2(1-x)^2(1-y)}{\big[1-x^2y(1-x)^2(1-y)\big]^2}\,dxdy$$

$$+ \int_0^1\int_0^1 \frac{xy}{1-x^2y(1-x)^2(1-y)}\,dxdy$$

$$= \int_0^1\int_0^1 \frac{xy\big(1+3x^2y-6x^3y+3x^4y-3x^2y^2+6x^3y^2-3x^4y^2\big)}{\big[1-xy^2(1-x)(1-y)^2\big]^3}\,dxdy.$$

# 3 The second extension of Sofo's theorems

In this section, we give another extension of Sofo's Theorems 1 and 2. The following theorem is about the relation between three binomial coefficients and triple integral representations.

**Theorem 4** (The second extension)**.** *For $t \in \mathbb{R}$ and $a$, $b$, $c$, $d$, $e$, $n$, $j$, $k$ and $m \in \mathbb{N}$ subject to $a \geq b$, $c \geq d$, $j$, $k$, $m > 0$ and*

$$\Big|t\frac{b^b(a-b)^{a-b}d^d(c-d)^{c-d}}{a^ac^c}\Big| \leq 1,$$

*then*

$$T(a,b,c,d,e,j,k,m,t) = \sum_{n=0}^{\infty} \frac{t^n}{\binom{an+j}{bn}\binom{cn+k}{dn}\binom{en+m}{en}}$$

$$= m(a-b)(c-d)\int_0^1\int_0^1\int_0^1 \frac{(1-x)^{j-1}(1-y)^{k-1}(1-z)^{m-1}W(1+W)}{(1-W)^3}\,dxdydz$$

$$+m\big[k(a-b)+j(c-d)\big]\int_0^1\int_0^1\int_0^1 \frac{(1-x)^{j-1}(1-y)^{k-1}(1-z)^{m-1}W}{(1-W)^2}\,dxdydz$$

$$+mkj\int_0^1\int_0^1\int_0^1 \frac{(1-x)^{j-1}(1-y)^{k-1}(1-z)^{m-1}}{1-W}\,dxdydz,$$

*where*

$$W := tx^b(1-x)^{a-b}y^d(1-y)^{c-d}z^e.$$

Clearly, when $a = b$, $c = d$ and $e = 0$, Theorem 4 reduces to Sofo's [9] Theorem 1. Letting $e = 0$, $a \neq b$ and $c \neq d$ in Theorem 4, we obtain Theorem 2 which is presented by Sofo [10]. The proof of this theorem is as same as we have given for Theorem 3. Now we present some examples of this theorem.

## 3.1 Example

Letting $a = b$, $c = d$ and $t = \pm 1$ in Theorem 4, we obtain the following results:

$$T\,(a, a, c, c, e, j, k, m, \pm 1) = \sum_{n=0}^{\infty} \frac{(\pm 1)^n}{\binom{an+j}{an}\binom{cn+k}{cn}\binom{en+m}{en}}$$

$$= jkm \int_0^1 \int_0^1 \int_0^1 \frac{(1-x)^{j-1}(1-y)^{k-1}(1-z)^{m-1}}{1 \mp x^a y^c z^e}\, dxdydz,$$

which is due to Sofo [8].

## 3.2 Example

Letting $a = c = 2$, $b = d = e = j = k = m = t = 1$ in Theorem 4, we have the following result:

$$T\,(2, 1, 2, 1, 1, 1, 1, 1, 1) = \sum_{n=0}^{\infty} \frac{1}{\binom{2n+1}{n}\binom{2n+1}{n}\binom{n+1}{n}}$$

$$= \int_0^1 \int_0^1 \int_0^1 \frac{xyz(1-x)(1-y)\left[1 + xyz(1-x)(1-y)\right]}{\left[1 - xyz(1-x)(1-y)\right]^3}\, dxdydz$$

$$+ 2 \int_0^1 \int_0^1 \int_0^1 \frac{xyz(1-x)(1-y)}{\left[1 - xyz(1-x)(1-y)\right]^2}\, dxdydz$$

$$+ \int_0^1 \int_0^1 \int_0^1 \frac{1}{1 - xyz(1-x)(1-y)}\, dxdydz,$$

$$= \int_0^1 \int_0^1 \int_0^1 \frac{1 + xyz - x^2yz - xy^2z + x^2y^2z}{\left[1 - xyz(1-x)(1-y)\right]^3}\, dxdydz.$$

### 3.3 Example

Letting $a = 3$, $b = c = 2$ and $d = e = k = j = m = t = 1$ in Theorem 4, we establish the following result:

$$T\left(3,2,2,1,1,1,1,1,1\right) = \sum_{n=0}^{\infty} \frac{1}{\binom{3n+1}{2n}\binom{2n+1}{n}\binom{n+1}{n}}$$

$$= \int_0^1 \int_0^1 \int_0^1 \frac{x^2yz(1-x)(1-y)\left[1+x^2yz(1-x)(1-y)\right]}{[1-x^2yz(1-x)(1-y)]^3} \, dxdydz$$

$$+ 2\int_0^1 \int_0^1 \int_0^1 \frac{x^2yz(1-x)(1-y)}{\left[1-x^2yz(1-x)(1-y)\right]^2} \, dxdydz$$

$$+ \int_0^1 \int_0^1 \int_0^1 \frac{1}{1-x^2yz(1-x)(1-y)} \, dxdydz,$$

$$= \int_0^1 \int_0^1 \int_0^1 \frac{1+x^2yz-x^3yz-x^2y^2z+x^3y^2z}{\left[1-x^2yz(1-x)(1-y)\right]^3} \, dxdydz.$$

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