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On the Summation of Certain Iterated Series

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Abstract

The paper gives a unified treatment of the summation of certain iterated series of the form $\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{n+m}$, where $(a_n)_{n\in\mathbb{N}}$ is a sequence of real numbers. We prove that, under certain conditions, the double iterated series equals the difference of two single series.

1 Introduction

The goal of this paper is to provide a unified treatment of the summation of a special class of double series that have appeared recently. Below, we collect some problems that led to the motivation of this article.

Problem 1 ([6]) Find the sum

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{(-1)^{n+m}}{\left(\lfloor\sqrt{n+m}\rfloor\right)^3},$$

where $\lfloor a\rfloor$ denotes the greatest integer less than or equal to a.

Problem 2 ([5]) Find

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^{n+m}}{n+m}. \tag{1}$$

Problem 3 ([4]) Compute the sum

$$\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}(-1)^{n+m}\frac{\ln(n+m)}{n+m}. \tag{2}$$

Problem 4 ([7]) Find

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)^{n+m}\frac{H_{n+m}}{n+m}, \tag{3}$$

where H_n denotes the nth harmonic number.

Although the problems can be solved by various techniques, in this paper we give a general method for summing the iterated series

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}a_{n+m}, \tag{4}$$

where $(a_n)_{n\in\mathbb{N}}$ is a sequence of real numbers, and we show that, under certain conditions, the double iterated series equals the difference of two single series. The organization of the paper is as follows: in Section 2 we give the first two main results of the paper, which are about the evaluation of the double iterated series (4), and in Section 3 we give the closed form evaluation of multiple series of the form $\sum_{n_1=1}^{\infty}\cdots\sum_{n_k=1}^{\infty}a_{n_1+n_2+\cdots+n_k}$.

2 Double iterated series

The first main result of this section is the following theorem.

Theorem 1. *Suppose that both series*

$$\sum_{k=1}^{\infty}a_k \qquad \text{and} \qquad \sum_{k=1}^{\infty}ka_k \tag{5}$$

converge and let σ and $\widetilde{\sigma}$ denote their sums, respectively. Then the iterated series (4) converges and its sum s equals $\widetilde{\sigma}-\sigma$.

Proof. For positive integers ν and n we let

$$A_\nu=\sum_{m=1}^{\infty}a_{\nu+m} \qquad \text{and} \qquad s_n=\sum_{\nu=1}^{n}A_\nu.$$

Likewise, for every positive integer n we let

$$\sigma_n = \sum_{k=1}^{n} a_k \qquad \text{and} \qquad \widetilde{\sigma}_n = \sum_{k=1}^{n} k a_k.$$

Since $A_\nu = \sigma - \sigma_\nu$, it follows that

$$s_n = n\sigma - \sum_{\nu=1}^{n} \sigma_\nu = n\sigma - \sum_{k=1}^{n}(n+1-k)a_k = n\sigma + \widetilde{\sigma}_n - (n+1)\sigma_n = \frac{u_n}{v_n}, \tag{6}$$

where

$$u_n = \sigma - \sigma_n + \frac{\widetilde{\sigma}_n - \sigma_n}{n} \qquad \text{and} \qquad v_n = \frac{1}{n}\,.$$

On the other hand, it is straight-forward to show that

$$\frac{u_{n+1} - u_n}{v_{n+1} - v_n} = \widetilde{\sigma}_{n+1} - \sigma_{n+1}.$$

Since

$$\lim_{n\to\infty} \frac{u_{n+1} - u_n}{v_{n+1} - v_n} = \widetilde{\sigma} - \sigma,$$

an application of Cesáro-Stolz theorem, the $\left(\frac{0}{0}\right)$ case, implies that

$$s = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{u_n}{v_n} = \widetilde{\sigma} - \sigma,$$

and the theorem is proved. □

Solution to Problem 1. Let $a_k = (-1)^k / \left(\lfloor\sqrt{k}\rfloor\right)^3$, and note that both series (5)

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty}(-1)^k \frac{1}{\left(\lfloor\sqrt{k}\rfloor\right)^3} \qquad \text{and} \qquad \sum_{k=1}^{\infty} k a_k = \sum_{k=1}^{\infty}(-1)^k \frac{k}{\left(\lfloor\sqrt{k}\rfloor\right)^3},$$

converge based on the Leibniz test. Moreover,

$$\begin{aligned}
\widetilde{\sigma} &= \sum_{k=1}^{\infty} k a_k = \sum_{k=1}^{\infty} \frac{(-1)^k k}{\left(\lfloor\sqrt{k}\rfloor\right)^3} = \sum_{N=1}^{\infty}\left(\sum_{k=N^2}^{N^2+2N} \frac{(-1)^k k}{\left(\lfloor\sqrt{k}\rfloor\right)^3}\right) = \sum_{N=1}^{\infty} \frac{1}{N^3}\left(\sum_{k=N^2}^{N^2+2N} (-1)^k k\right) \\
&= \sum_{N=1}^{\infty} \frac{(-1)^N (N^2+N)}{N^3} = \sum_{N=1}^{\infty} \frac{(-1)^N}{N} + \sum_{N=1}^{\infty} \frac{(-1)^N}{N^2} \\
&= -\ln 2 - \frac{\pi^2}{12}.
\end{aligned}$$

On the other hand,

$$\sigma = \sum_{k=1}^{\infty} a_k = \sum_{N=1}^{\infty}\left(\sum_{k=N^2}^{N^2+2N} \frac{(-1)^k}{\left(\lfloor\sqrt{k}\rfloor\right)^3}\right) = \sum_{N=1}^{\infty} \frac{1}{N^3}\left(\sum_{k=N^2}^{N^2+2N} (-1)^k\right) = \sum_{N=1}^{\infty} \frac{(-1)^N}{N^3} = -\frac{3}{4}\,\zeta(3),$$

where $\zeta(3)=\sum_{k=1}^{\infty} 1/k^3$ denotes the Apéry constant. It follows, based on Theorem 1, that

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^{n+m}}{\left(\lfloor\sqrt{n+m}\rfloor\right)^3}=\frac{3}{4}\,\zeta(3)-\ln 2-\frac{\pi^2}{12},$$

and the problem is solved.

Remark 2. It is worth mentioning that if the series $\sum_{k=2}^{\infty}(k-1)a_k$, would be absolutely convergent, the conclusion of Theorem 1 would follow by rearranging the terms of the iterated series (4) as follows

$$\sum_{k=2}^{\infty}(k-1)a_k=a_{1+1}+\left(a_{1+2}+a_{2+1}\right)+\cdots+\left(a_{1+(k-1)}+a_{2+(k-2)}+\cdots+a_{(k-1)+1}\right)+\cdots.$$

Unfortunately, when $a_k=(-1)^k/\lfloor\sqrt{k}\rfloor^3$, the absolute convergence of the series $\sum_{k=2}^{\infty}(k-1)a_k$ fails to hold. Also, the convergence of series (5) is sufficient but not necessary for the convergence of the iterated series (4). This is shown by the series (1), (2), and (3). Clearly, Theorem 1 does not apply to these series. However, the next theorem, which can be applied for summing the series (1), (2) and (3), is the second new result of this section.

Theorem 3. *With the same notation as in the statement and the proof of Theorem 1, let us suppose that the first series in (5) converges, that $\widetilde{\sigma}_{2n}=o(n)$, and that the limit*

$$\ell=\lim_{n\to\infty}\left(na_{2n}+\widetilde{\sigma}_{2n-1}\right) \tag{7}$$

exists in $\mathbb{R}$. Then the iterated series (4) converges and its sum s is given by $s=\ell-\sigma$.

Proof. Let $w_n=na_{2n}+\widetilde{\sigma}_{2n-1}$. With the same notation as in the statement and the proof of Theorem 1, we have, based on (6), that

$$\sigma+s_{2n}=(2n+1)(\sigma-\sigma_{2n})+\widetilde{\sigma}_{2n}=\frac{x_n}{y_n}\,,$$

where $x_n=\sigma-\sigma_{2n}+\frac{1}{2n+1}\,\widetilde{\sigma}_{2n}$ and $y_n=\frac{1}{2n+1}$. On the other hand,

$$\frac{x_{n+1}-x_n}{y_{n+1}-y_n}=\frac{2n+1}{2}\,a_{2n+2}+\widetilde{\sigma}_{2n+1}=w_{n+1}-\frac{1}{2}\,a_{2n+2}\,.$$

Since $a_n\to 0$ as $n\to\infty$, we have, based on the Cesáro-Stolz theorem, the $\left(\frac{0}{0}\right)$ case, and (7), that $\lim_{n\to\infty}(\sigma+s_{2n})=\ell$, and it follows that $\lim_{n\to\infty}s_{2n}=\ell-\sigma$. On the other hand, since

$$s_{2n+1}=s_{2n}+A_{2n+1}=s_{2n}+\sigma-\sigma_{2n+1},$$

we get that $\lim_{n\to\infty}s_{2n+1}=\lim_{n\to\infty}s_{2n}=\ell-\sigma$. Thus, $\lim_{n\to\infty}s_n=\ell-\sigma$, and the theorem is proved. $\square$

Solution to Problem 2. Let $a_k = (-1)^k/k$ and we note that

$$\sigma = \sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(-1)^k}{k} = -\ln 2.$$

On the other hand, $\widetilde{\sigma}_{2n} = \sum_{k=1}^{2n} (-1)^k = 0$, and

$$w_n = na_{2n} + \widetilde{\sigma}_{2n-1} = \frac{1}{2} + \sum_{k=1}^{2n-1} k \cdot \frac{(-1)^k}{k} = -\frac{1}{2}, \quad \text{for all } n \in \mathbb{N}.$$

Thus, $\ell = -\frac{1}{2}$. An application of Theorem 3 shows that

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^{n+m}}{n+m} = \ln 2 - \frac{1}{2}\,,$$

and the problem is solved.

Before we give the solution to Problem 3 we need to introduce a notation. Recall that, the double factorial of a positive integer n is defined by the following formula

$$n!! = \begin{cases} n(n-2)\cdots 5 \cdot 3 \cdot 1, & \text{if } n \text{ is odd;} \\ n(n-2)\cdots 6 \cdot 4 \cdot 2, & \text{if } n \text{ is even.} \end{cases}$$

Solution to Problem 3. Let $a_k = (-1)^k \frac{\ln k}{k}$. It can be proved, see [9], that

$$\sigma = \sum_{k=1}^{\infty} (-1)^k \frac{\ln k}{k} = \gamma \ln 2 - \frac{1}{2} \ln^2 2,$$

where γ denotes the Euler-Mascheroni constant. On the other hand,

$$\widetilde{\sigma}_{2n} = \sum_{k=1}^{2n} (-1)^k \ln k = \ln \frac{(2n)!!}{(2n-1)!!} = \frac{1}{2} \ln \frac{\big((2n)!!\big)^2}{\big((2n-1)!!\big)^2 (2n+1)} + \frac{\ln(2n+1)}{2}\,,$$

and

$$\begin{aligned} w_n &= na_{2n} + \widetilde{\sigma}_{2n-1} = \frac{1}{2} \ln(2n) + \sum_{k=1}^{2n-1} (-1)^k \ln k = \frac{1}{2} \ln(2n) + \ln \frac{(2n-2)!!}{(2n-1)!!} \\ &= \frac{1}{2} \ln \frac{\big((2n)!!\big)^2}{\big((2n-1)!!\big)^2 (2n+1)} + \frac{1}{2} \ln \frac{2n+1}{2n}\,. \end{aligned}$$

An application of the Wallis formula

$$\lim_{n\to\infty} \frac{\big((2n)!!\big)^2}{\big((2n-1)!!\big)^2 (2n+1)} = \frac{\pi}{2}\,,$$

shows that $\widetilde{\sigma}_{2n} = o(n)$ and $\ell = \lim_{n\to\infty} w_n = \frac{1}{2}\ln\frac{\pi}{2}$. Theorem 3 implies that

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)^{n+m}\frac{\ln(n+m)}{n+m} = \frac{1}{2}\ln\frac{\pi}{2} - \sigma,$$

and hence

$$\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}(-1)^{n+m}\frac{\ln(n+m)}{n+m} = \sigma + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)^{n+m}\frac{\ln(n+m)}{n+m} = \frac{1}{2}\ln\frac{\pi}{2},$$

and the problem is solved.

Before we give the solution to Problem 4 we need to mention the properties of a special function. Recall that, the dilogarithm function, (see [11, 12]), denoted by $\mathrm{Li}_2(z)$, is the special function defined by

$$\mathrm{Li}_2(z) = \sum_{n=1}^{\infty}\frac{z^n}{n^2} = -\int_0^z \frac{\ln(1-t)}{t}dt, \qquad |z| \le 1.$$

A special Landen identity involving this function, [12, p. 107], is given by

$$\mathrm{Li}_2(z) + \mathrm{Li}_2(1-z) = \frac{\pi^2}{6} - \ln(z)\ln(1-z).$$

This implies, when $z = 1/2$, that

$$\mathrm{Li}_2\left(\frac{1}{2}\right) = \frac{\pi^2}{12} - \frac{\ln^2 2}{2}. \tag{8}$$

Now we are ready to give the solution to Problem 4.

Solution to Problem 4. Let $a_k = (-1)^k\frac{H_k}{k}$. To calculate

$$\sigma = \sum_{k=1}^{\infty}(-1)^k\frac{H_k}{k},$$

we will be using a general result established by Kantor [8]: given a sequence $(\alpha_k)_{k\ge1}$ of real numbers, we consider the power series

$$f(x) = \sum_{k=1}^{\infty}\alpha_k x^k \quad \text{and} \quad g(x) = \sum_{k=1}^{\infty}\alpha_k H_k x^k. \tag{9}$$

By the ratio test, the two power series have the same radius of convergence. Kantor [8] proved that for every real number x for which the first series in (9) converges one has

$$g(x) = \int_0^1 \frac{f(x) - f(tx)}{1-t}\,dt.$$

Let $\alpha_k = (-1)^k/k$, and note that

$$f(x) = \sum_{k=1}^{\infty} (-1)^k \frac{x^k}{k} = -\ln(1+x), \qquad x \in (-1,1].$$

It follows that

$$\sigma = g(1) = \int_0^1 \frac{\ln(1+t) - \ln 2}{1-t}\, dt.$$

The substitution $x = 1 - t$ implies that

$$\sigma = \int_0^1 \frac{1}{x} \ln\left(1 - \frac{x}{2}\right) dx = \int_0^{1/2} \frac{\ln(1-y)}{y}\, dy = -\mathrm{Li}_2\left(\frac{1}{2}\right) = \frac{\ln^2 2}{2} - \frac{\pi^2}{12}.$$

On the other hand,

$$\widetilde{\sigma}_{2n} = \sum_{k=1}^{2n} (-1)^k H_k = \sum_{k=1}^{n} \big(H_{2k} - H_{2k-1}\big) = \sum_{k=1}^{n} \frac{1}{2k} = \frac{1}{2} H_n,$$

and

$$w_n = n a_{2n} + \widetilde{\sigma}_{2n-1} = \frac{1}{2} H_{2n} + \widetilde{\sigma}_{2n} - H_{2n} = \frac{1}{2}\big(H_n - H_{2n}\big).$$

Since $H_n = \ln n + \gamma + o(1)$, we have that all the hypotheses of Theorem 3 are satisfied and $\ell = \lim_{n\to\infty} w_n = -\frac{1}{2}\ln 2$. Thus,

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (-1)^{n+m} \frac{H_{n+m}}{n+m} = l - \sigma = -\frac{\ln 2}{2} - \frac{\ln^2 2}{2} + \frac{\pi^2}{12},$$

and the problem is solved.

3 Multiple series of a special form

In this section we study the multiple series of the form

$$\sum_{n_1,n_2,\ldots,n_k=1}^{\infty} a_{n_1+n_2+\cdots+n_k}, \tag{10}$$

where $(a_n)_{n\in\mathbb{N}}$ is a sequence of real numbers that verifies a special condition. Before we give the main theorem of this section we collect some results we need in our analysis. Recall that, the Stirling numbers of the first kind, denoted by $s(n,k)$, are the special numbers defined by the generating function

$$z(z-1)(z-2)\cdots(z-n+1) = \sum_{k=0}^{n} s(n,k) z^k.$$

For recurrence relations as well as interesting properties satisfied by these numbers the reader is referred to the book by Srivastava and Choi [12]. The main result of this section is the following theorem.

Theorem 4. *Let $k \geq 1$ be a natural number and let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that $\sum_{n=1}^{\infty} a_n n^{k-1}$ converges absolutely. Then the multiple series (10) converges and*

$$\sum_{n_1,n_2,\ldots,n_k=1}^{\infty} a_{n_1+n_2+\cdots+n_k} = \frac{1}{(k-1)!}\sum_{i=1}^{k} s(k,i)\left(\sum_{p=k}^{\infty} a_p p^{i-1}\right),$$

where $s(k,i)$ are the Stirling numbers of the first kind.

Proof. We have

$$\begin{aligned}
\sum_{n_1,n_2,\ldots,n_k=1}^{\infty} a_{n_1+n_2+\cdots+n_k} &= \sum_{p=k}^{\infty}\left(\sum_{n_1+n_2+\cdots+n_k=p} a_{n_1+n_2+\cdots+n_k}\right)\\
&= \sum_{p=k}^{\infty} a_p\left(\sum_{n_1+n_2+\cdots+n_k=p} 1\right)\\
&= \sum_{p=k}^{\infty} a_p\binom{p-1}{k-1}\\
&= \frac{1}{(k-1)!}\sum_{p=k}^{\infty}\frac{a_p}{p}\left(p(p-1)(p-2)\cdots(p-k+1)\right)\\
&= \frac{1}{(k-1)!}\sum_{p=k}^{\infty}\frac{a_p}{p}\left(\sum_{i=0}^{k} s(k,i)p^i\right)\\
&= \frac{1}{(k-1)!}\sum_{i=0}^{k} s(k,i)\left(\sum_{p=k}^{\infty} a_p p^{i-1}\right)\\
&= \frac{1}{(k-1)!}\sum_{i=1}^{k} s(k,i)\left(\sum_{p=k}^{\infty} a_p p^{i-1}\right),
\end{aligned}$$

since $s(k,0)=0$ for $k\in\mathbb{N}$. The theorem is proved. □

The next result, which is a consequence of Theorem 4, is about the calculation of another multiple series of a special form.

Corollary 5. *Let $1 \leq i \leq k$ be fixed natural numbers and let $(a_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers such that $\sum_{n=1}^{\infty} a_n n^k$ converges. Then*

$$\sum_{n_1,n_2,\ldots,n_k=1}^{\infty} n_i\cdot a_{n_1+n_2+\cdots+n_k} = \frac{1}{k!}\sum_{i=1}^{k} s(k,i)\left(\sum_{p=k}^{\infty} a_p p^{i}\right),$$

where $s(k,i)$ are the Stirling numbers of the first kind.

Proof. We have, based on symmetry reasons, that for all $i, j = 1, \ldots, k$, one has that

$$\sum_{n_1,n_2,\ldots,n_k=1}^{\infty} n_i \cdot a_{n_1+n_2+\cdots+n_k} = \sum_{n_1,n_2,\ldots,n_k=1}^{\infty} n_j \cdot a_{n_1+n_2+\cdots+n_k},$$

and hence

$$\sum_{n_1,n_2,\ldots,n_k=1}^{\infty} n_i \cdot a_{n_1+n_2+\cdots+n_k} = \frac{1}{k} \sum_{n_1,n_2,\ldots,n_k=1}^{\infty} (n_1+n_2+\cdots+n_k) \cdot a_{n_1+n_2+\cdots+n_k},$$

and the result follows based on Theorem 4 applied to the sequence $(na_n)_{n\in\mathbb{N}}$. $\square$

Corollary 6. *a) Let k be a fixed positive integer and let $m > k$. Then*

$$\sum_{n_1,n_2,\ldots,n_k=1}^{\infty} \frac{1}{(n_1+n_2+\cdots+n_k)^m}$$
$$= \frac{1}{(k-1)!} \sum_{i=1}^{k} s(k,i) \left(\zeta(m+1-i) - 1 - \frac{1}{2^{m+1-i}} - \cdots - \frac{1}{(k-1)^{m+1-i}} \right),$$

where the parenthesis contains only the term $\zeta(m+1-i)$ when $k = 1$.

b) Let k and i be fixed positive integers such that $1 \le i \le k$, and let m be such that $m - k > 1$. Then

$$\sum_{n_1,n_2,\ldots,n_k=1}^{\infty} \frac{n_i}{(n_1+n_2+\cdots+n_k)^m}$$
$$= \frac{1}{k!} \sum_{i=1}^{k} s(k,i) \left(\zeta(m-i) - 1 - \frac{1}{2^{m-i}} - \cdots - \frac{1}{(k-1)^{m-i}} \right),$$

where the parenthesis contains only the term $\zeta(m-i)$ when $k = 1$.

The next result specializes to the case when the sequence $a_n = 1/n!$. Recall that, if $k \in \mathbb{N}$, the series $S_k = \sum_{n=1}^{\infty} \frac{n^k}{n!}$, also known as the Wolstenholme series, equals an integral multiple of e, i.e., $S_k = B_k e$, (see [1, p. 197]). The integer B_k is known in the mathematical literature as the kth Bell number [2] and the equality $S_k = B_k e$ is known as Dobinski's formula [3]. For example, $B_1 = 1$, $B_2 = 2$, $B_3 = 5$, $B_4 = 15$, $B_5 = 52$. One can prove that the sequence $(B_k)_{k\in\mathbb{N}}$ verifies the recurrence formula

$$B_k = \sum_{j=1}^{k-1} \binom{k-1}{j} B_j + 1.$$

The next corollary gives a recurrence relation involving the Stirling numbers of the first kind and the Bell numbers. The formula is obtained by calculating, by two different methods, a multiple series involving a factorial term.

Corollary 7. *Let $k \geq 2$ and let i be fixed positive integers such that $1 \leq i \leq k$. Then*

$$a) \qquad \sum_{n_1,n_2,\ldots,n_k=1}^{\infty} \frac{n_i}{(n_1+n_2+\cdots+n_k)!} = \frac{e}{k!}.$$

$$b) \qquad \sum_{i=1}^{k} s(k,i) \left(B_i e - \sum_{p=1}^{k-1} \frac{p^i}{p!} \right) = e.$$

Proof. We have, based on Corollary 5, that

$$\sum_{n_1,n_2,\ldots,n_k=1}^{\infty} \frac{n_i}{(n_1+n_2+\cdots+n_k)!} = \frac{1}{k!} \sum_{i=1}^{k} s(k,i) \left(B_i e - \sum_{p=1}^{k-1} \frac{p^i}{p!} \right).$$

On the other hand,

$$\begin{aligned}
\sum_{n_1,n_2,\ldots,n_k=1}^{\infty} \frac{n_i}{(n_1+n_2+\cdots+n_k)!} &= \frac{1}{k} \sum_{n_1,n_2,\ldots,n_k=1}^{\infty} \frac{1}{(n_1+n_2+\cdots+n_k-1)!} \\
&= \sum_{p=k}^{\infty} \left(\sum_{n_1+\cdots+n_k=p} \frac{1}{(n_1+n_2+\cdots+n_k-1)!} \right) \\
&= \frac{1}{k} \sum_{p=k}^{\infty} \frac{1}{(p-1)!} \binom{p-1}{k-1} \\
&= \frac{1}{k!} \sum_{p=k}^{\infty} \frac{1}{(p-k)!} \\
&= \frac{e}{k!},
\end{aligned}$$

and the corollary is proved. ☐

The next result refers to the calculation of multiple Wolstenholme series.

Corollary 8. *Let m and k be natural numbers. Then*

$$\sum_{n_1,\ldots,n_k=1}^{\infty} \frac{(n_1+n_2+\cdots+n_k)^m}{(n_1+n_2+\cdots+n_k)!} = \frac{1}{(k-1)!} \sum_{i=1}^{k} s(k,i) \left(B_{m+i-1} e - \sum_{p=1}^{k-1} \frac{p^{m+i-1}}{p!} \right),$$

where the parenthesis contains only the term $B_{m+i-1}e$ when $k=1$.

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