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Integer Sequences, Functions of Slow Increase, and the Bell Numbers

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In memory of my sister Fedra Marina Jakimczuk (1970–2010)

Abstract

In this article we first prove a general theorem on integer sequences A_n such that the following asymptotic formula holds,

$$\frac{A_n}{A_{n-1}} \sim Cn^{\alpha} f(n)^{\beta},$$

where $f(x)$ is a function of slow increase, $C > 0$, $\alpha > 0$ and β is a real number.

We also obtain some results on the Bell numbers B_n using well-known formulae. We compare the Bell numbers with a^n $(a > 0)$ and $(n!)^h$ $(0 < h \leq 1)$.

Finally, applying the general statements proved in the article we obtain the formula

$$B_{n+1} \sim e \ (B_n)^{1+\frac{1}{n}}.$$

1 Integer Sequences. A General Theorem.

We shall need the following well-known lemmas [12, pp. 332, 294].

Lemma 1. *If s_n is a sequence of positive numbers with limit s then the sequence*

$$\sqrt[n]{s_1 s_2 \cdots s_n}$$

has also limit s.

Lemma 2. *The following limit holds,*

$$\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}.$$

We recall the definition of function of slow increase [7, Definition 1].

Definition 3. Let $f(x)$ be a function defined on interval $[a, \infty)$ such that $f(x) > 0$, $\lim_{x\to\infty} f(x) = \infty$ and with continuous derivative $f'(x) > 0$. The function $f(x)$ is of slow increase if and only if the following condition holds

$$\lim_{x\to\infty} \frac{f'(x)}{\frac{f(x)}{x}} = \lim_{x\to\infty} \frac{xf'(x)}{f(x)} = 0. \tag{1}$$

Typical functions of slow increase are $f(x) = \log x$, $f(x) = \log^2 x$ and $f(x) = \log\log x$.

Lemma 4. *If $f(x)$ is a function of slow increase on the interval $[b, \infty)$ then the following asymptotic formula holds*

$$\sqrt[n]{f(b)f(b+1)\cdots f(n)} \sim f(n), \tag{2}$$

where b is a positive integer.

Proof. Note that we always can suppose that $f(x) > 1$ on the interval $[b, \infty)$.

Since $\log f(x)$ is increasing and positive in the interval $[b, \infty)$ we find that

$$\begin{aligned}\sum_{i=b}^{n} \log f(i) = \sum_{i=b}^{n} (1 \cdot \log f(i)) &= \int_b^n \log f(x)\ dx + O(\log f(n)) = n \log f(n) \\ &+ \int_b^n \frac{xf'(x)}{f(x)}\ dx + O(\log f(n)).\end{aligned} \tag{3}$$

Note that the second equation in (3) is a sum of areas of rectangles of height $\log f(i)$ and base 1. Consequently the third equation in (3) is immediate.

L'Hôpital's rule gives (see (1))

$$\lim_{x\to\infty} \frac{\log f(x)}{x} = \lim_{x\to\infty} \frac{f'(x)}{f(x)} = 0.$$

Therefore

$$O(\log f(n)) = o(n). \tag{4}$$

If the integral $\int_b^x \frac{tf'(t)}{f(t)}\ dt$ converges we obtain

$$\lim_{x\to\infty} \frac{\int_b^x \frac{tf'(t)}{f(t)}\ dt}{x} = 0.$$

On the other hand, if the integral $\int_b^x \frac{tf'(t)}{f(t)}\ dt$ diverges we obtain from L'Hôpital's rule and (1) that

$$\lim_{x\to\infty} \frac{\int_b^x \frac{tf'(t)}{f(t)}\ dt}{x} = 0.$$

Therefore
$$\int_b^n \frac{xf'(x)}{f(x)}\,dx = o(n). \tag{5}$$
Equations (3), (4) and (5) give
$$\sum_{i=b}^{n} \log f(i) = n\log f(n) + o(n). \tag{6}$$
That is,
$$\frac{1}{n}\sum_{i=b}^{n} \log f(i) = \log f(n) + o(1).$$
That is (2). □

Theorem 5. *Let A_n $(n \geq 0)$ be a sequence of positive numbers (in particular integers) such that*
$$\frac{A_n}{A_{n-1}} \sim Cn^{\alpha} f(n)^{\beta}, \tag{7}$$
where $f(x)$ is a function of slow increase on the interval $[b,\infty)$, $C > 0$, $\alpha > 0$ and β is a real number. If $1 \leq n < b$ we put $f(n) = 1$.

The following formulae hold,
$$\frac{\sqrt[n]{\frac{A_1}{A_0}\frac{A_2}{A_1}\cdots\frac{A_n}{A_{n-1}}}}{\frac{A_n}{A_{n-1}}} \to \frac{1}{e^{\alpha}}, \tag{8}$$
$$A_{n+1} \sim e^{\alpha} A_n^{1+\frac{1}{n}}, \tag{9}$$
$$\log A_n = \alpha n \log n + \beta n \log f(n) + (-\alpha + \log C)n + o(n), \tag{10}$$
$$\log A_n \sim \alpha n \log n, \tag{11}$$
$$A_n = \frac{\left(Cn^{\alpha} f(n)^{\beta}\right)^n}{e^{(\alpha+o(1))n}}. \tag{12}$$
Proof. We have (see (7))
$$\frac{\frac{A_n}{A_{n-1}}}{Cn^{\alpha} f(n)^{\beta}} \to 1. \tag{13}$$
Consequently (13) and Lemma 1 give
$$\sqrt[n]{\prod_{k=1}^{n} \frac{\frac{A_k}{A_{k-1}}}{Ck^{\alpha} f(k)^{\beta}}} = \frac{\sqrt[n]{\prod_{k=1}^{n} \frac{A_k}{A_{k-1}}}}{\sqrt[n]{\prod_{k=1}^{n} Ck^{\alpha} f(k)^{\beta}}} \to 1.$$

That is

$$\sqrt[n]{A_n} \sim \sqrt[n]{\frac{A_1}{A_0}\frac{A_2}{A_1}\cdots\frac{A_n}{A_{n-1}}} \sim \sqrt[n]{\prod_{k=1}^{n} Ck^\alpha f(k)^\beta}. \tag{14}$$

Lemma 2 and Lemma 4 give

$$\sqrt[n]{\prod_{k=1}^{n} Ck^\alpha f(k)^\beta} = C\left(\sqrt[n]{n!}\right)^\alpha \left(\sqrt[n]{f(1)f(2)\cdots f(n)}\right)^\beta \sim C\frac{n^\alpha}{e^\alpha}f(n)^\beta. \tag{15}$$

Equations (14), (15) and (7) give

$$\sqrt[n]{A_n} \sim \sqrt[n]{\frac{A_1}{A_0}\frac{A_2}{A_1}\cdots\frac{A_n}{A_{n-1}}} \sim C\frac{n^\alpha}{e^\alpha}f(n)^\beta \sim \frac{1}{e^\alpha}\frac{A_n}{A_{n-1}}. \tag{16}$$

Equation (16) gives (8). Equation (16) and [7, Theorem 8] give

$$A_n^{\frac{1}{n}} \sim A_{n-1}^{\frac{1}{n-1}}. \tag{17}$$

Equations (17) and (16) give

$$A_n \sim e^\alpha A_{n-1}^{1+\frac{1}{n-1}}.$$

That is (9). Equation (16) gives

$$\frac{1}{n}\log A_n = \log\left(C\frac{n^\alpha}{e^\alpha}f(n)^\beta\right) + o(1).$$

That is (10). Equation (10) gives (11), since (from L'Hôpital's rule and (1))

$$\lim_{x\to\infty}\frac{\log f(x)}{\log x} = \lim_{x\to\infty}\frac{xf'(x)}{f(x)} = 0.$$

Finally, equation (12) is an immediate consequence of equation (10). □

Remark 6. Note that: (i) The following limit holds $Cn^\alpha f(n)^\beta \to \infty$ (see(7)).

If $\beta \geq 0$ the proof is trivial. If $\beta < 0$ use [7, Theorem 2 and Theorem 4]. Consequently we have

$$\lim_{n\to\infty}\frac{A_{n+1}}{A_n} = \infty.$$

(ii) This last limit implies the following formula $(A_{n+1} - A_n) \sim A_{n+1}$.

(iii) Equation (7) implies the more general relation,

$$\frac{A_n}{A_{n-m}} = \frac{A_n}{A_{n-1}}\frac{A_{n-1}}{A_{n-2}}\cdots\frac{A_{n-m+1}}{A_{n-m}} \sim C^m \prod_{k=n-m+1}^{n} k^\alpha f(k)^\beta.$$

2 Introduction to Bell Numbers.

The n-th Bell number B_n is the number of partitions of a set of n elements in disjoint subsets.

The Bell numbers satisfy the following recurrence relation [1, p. 216].

$$B_0 = 1,$$

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k. \tag{18}$$

The first Bell numbers are $B_0 = 1$, $B_1 = 1$, $B_2 = 2$, $B_3 = 5$, $B_4 = 15$, $B_5 = 52$, $B_6 = 203$, $B_7 = 877$, $B_8 = 4140$, $B_9 = 21147$, $B_{10} = 115975$.

N. G. de Bruijn [6, pp. 102–109] proved the following asymptotic formula,

$$\log B_n = n \log n - n \log \log n - n + o(n). \tag{19}$$

L. Lovász [10, Ex. 9(b), p. 17] proved the following asymptotic formula

$$B_n \sim n^{-\frac{1}{2}} (\lambda(n))^{n+\frac{1}{2}} e^{\lambda(n)-n-1}, \tag{20}$$

where

$$\lambda(n) = \frac{n}{W(n)}. \tag{21}$$

The function $x = W(y)$ is the inverse function of $y = xe^x$ on the interval $(0, \infty)$. The function $x = W(y)$ is called Lambert W-function.

The following results are well-known [5]. We establish these results in the next lemma. For sake of completeness we give a proof of the lemma.

Lemma 7. *The function $x = W(y)$ is positive, strictly increasing on the interval $(0, \infty)$ and $\lim_{y\to\infty} W(y) = \infty$.*

The following formulae hold.

$$W(y) \sim \log y, \tag{22}$$

$$W'(y) = \frac{W(y)}{y(1 + W(y))}. \tag{23}$$

Proof. The first statement is trivial.

We have (definition of $x = W(y)$)

$$y = W(y)e^{W(y)}. \tag{24}$$

Consequently

$$1 = W'(y)e^{W(y)} + W(y)e^{W(y)}W'(y).$$

That is

$$W'(y) = \frac{1}{e^{W(y)}(1 + W(y))} = \frac{W(y)}{y(1 + W(y))}.$$

On the other hand (24) gives

$$\log y = \log W(y) + W(y).$$

Therefore

$$\frac{W(y)}{\log y} = 1 - \frac{\log W(y)}{\log y}. \tag{25}$$

Also (from L'Hôpital's rule and (23))

$$\lim_{y\to\infty} \frac{\log W(y)}{\log y} = \lim_{y\to\infty} \frac{yW'(y)}{W(y)} = \lim_{y\to\infty} \frac{1}{1+W(y)} = 0. \tag{26}$$

Finally, equations (25) and (26) give (22). □

Remark 8. Note that the Lambert W-function $W(y)$ is a function of slow increase since (see (1) and (23))

$$\lim_{y\to\infty} \frac{yW'(y)}{W(y)} = \lim_{y\to\infty} \frac{1}{1+W(y)} = 0.$$

3 Some Results on Bell Numbers.

The limit

$$\lim_{n\to\infty} \frac{B_n}{n!} = 0.$$

is well-known [9, p. 64]. In the following Theorem we include it for sake of completeness.

Theorem 9. *The following limits hold.*

$$\lim_{n\to\infty} \frac{B_n}{a^n} = \infty \qquad (a > 0), \tag{27}$$

$$\lim_{n\to\infty} \frac{B_n}{(n!)^h} = \infty \qquad (0 < h < 1), \tag{28}$$

$$\lim_{n\to\infty} \frac{B_n}{n!} = 0.$$

Proof. Equation (19) gives

$$\log\left(\frac{B_n}{a^n}\right) = \log B_n - n\log a = n\log n - n\log a + o(n\log n).$$

Therefore

$$\lim_{n\to\infty} \log\left(\frac{B_n}{a^n}\right) = \infty,$$

and consequently

$$\lim_{n\to\infty} \frac{B_n}{a^n} = \infty.$$

That is (27).

The well-known Stirling formula is

$$n! \sim \sqrt{2\pi}\frac{n^n\sqrt{n}}{e^n}.$$

Therefore

$$\log n! = n\log n - n + \frac{1}{2}\log n + \log\sqrt{2\pi} + o(1) = n\log n - n + o(n),$$

and

$$\log(n!)^h = hn\log n - hn + o(n).$$

Consequently (see (19))

$$\log\left(\frac{B_n}{(n!)^h}\right) = (1-h)n\log n - n\log\log n - n + hn + o(n). \tag{29}$$

If $0 < h < 1$ equation (29) gives

$$\lim_{n\to\infty}\log\left(\frac{B_n}{(n!)^h}\right) = \infty.$$

That is,

$$\lim_{n\to\infty}\frac{B_n}{(n!)^h} = \infty.$$

On the other hand if $h = 1$ equation (29) gives

$$\lim_{n\to\infty}\log\left(\frac{B_n}{n!}\right) = -\infty.$$

That is,

$$\lim_{n\to\infty}\frac{B_n}{n!} = 0.$$

□

M. Klazar [8, Proposition 2.6] and D. E. Knuth [9, eq. (30), p. 69] proved the following asymptotic formula

$$\frac{B_{n+1}}{B_n} \sim \frac{n}{\log n}.$$

This formula is derived as a consequence of the asymptotic formula obtained in the classical paper [11].

In the following Theorem we derive this formula from the Lovász's formula (20). We also use the well-known properties of the Lambert W-function established in Lemma 7.

Theorem 10. *The following asymptotic formula holds,*

$$\frac{B_{n+1}}{B_n} \sim \frac{n}{\log n}. \tag{30}$$

Proof. Substituting (21) into (20) we obtain

$$B_n \sim \frac{n^n}{W(n)^{n+\frac{1}{2}}} e^{\frac{n}{W(n)}-n-1}.$$

Consequently

$$\frac{B_{n+1}}{B_n} \sim \frac{n+1}{W(n+1)} \sqrt{\frac{W(n)}{W(n+1)}} \left(\frac{W(n)}{W(n+1)}\right)^n e^{\left(\frac{n+1}{W(n+1)}-\frac{n}{W(n)}\right)}. \tag{31}$$

Equation (22) gives

$$W(n+1) \sim W(n). \tag{32}$$

Equations (32) and (22) give

$$\frac{n+1}{W(n+1)} \sim \frac{n}{\log n}. \tag{33}$$

Equation (32) gives

$$\sqrt{\frac{W(n)}{W(n+1)}} \to 1. \tag{34}$$

Let us consider the function $\frac{y}{W(y)}$. The derivative of $\frac{y}{W(y)}$ is (see (23))

$$\frac{W(y) - yW'(y)}{W(y)^2} = \frac{W(y) - \frac{W(y)}{1+W(y)}}{W(y)^2} = \frac{1}{1+W(y)}.$$

Consequently we have (Lagrange's Theorem)

$$\frac{n+1}{W(n+1)} - \frac{n}{W(n)} = \frac{1}{1+W(n+\epsilon(n))} \to 0, \tag{35}$$

where $0 < \epsilon(n) < 1$.

We have

$$\left(\frac{W(n+1)}{W(n)}\right)^n = \exp\left(n\left(\log W(n+1) - \log W(n)\right)\right). \tag{36}$$

Let us consider the function $\log W(y)$. The derivative of $\log W(y)$ is (see (23))

$$\frac{W'(y)}{W(y)} = \frac{1}{y(1+W(y))}.$$

Consequently we have (Lagrange's Theorem)

$$\begin{aligned} & n\left(\log W(n+1) - \log W(n)\right) = n\frac{1}{(n+\epsilon(n))(1+W(n+\epsilon(n)))} \\ = \;& \frac{1}{(1+\frac{\epsilon(n)}{n})(1+W(n+\epsilon(n)))} \to 0, \end{aligned} \tag{37}$$

where $0 < \epsilon(n) < 1$.

Equations (36) and (37) give

$$\left(\frac{W(n)}{W(n+1)}\right)^n \to 1. \tag{38}$$

Finally, equations (31), (33), (34), (35) and (38) give (30). $\square$

The asymptotic formula (30) implies that the Bell numbers satisfy condition (7). In this case $C=1$, $\alpha=1$, $\beta=-1$ and $f(n)=\log n$. Consequently we have the following Corollary.

Corollary 11. *The following formulae hold,*

$$\lim_{n\to\infty}\frac{B_{n+1}}{B_n}=\infty,$$

$$(B_{n+1}-B_n)\sim B_{n+1},$$

$$\frac{\sqrt[n]{\frac{B_1}{B_0}\frac{B_2}{B_1}\frac{B_3}{B_2}\cdots\frac{B_n}{B_{n-1}}}}{\frac{B_n}{B_{n-1}}}=\frac{\sqrt[n]{B_n}}{\frac{B_n}{B_{n-1}}}\to\frac{1}{e},$$

$$B_{n+1}\sim e\ \left(B_n\right)^{1+\frac{1}{n}}.$$

Proof. It is an immediate consequence of Theorem 5 and Remark 6. $\square$

The following Theorem is well-known [4, Ex. 1(2), p. 291] [2, Corollary 5]. We give a short proof using equation (18).

Theorem 12. *The sequence $B_{n+1}-B_n$ is strictly increasing.*

Proof. We have (see (18))

$$B_{n+2}-B_{n+1}=\sum_{k=0}^{n}\binom{n+1}{k}B_k,\qquad B_{n+1}-B_n=\sum_{k=0}^{n-1}\binom{n}{k}B_k.$$

Consequently

$$(B_{n+2}-B_{n+1})-(B_{n+1}-B_n)=\sum_{k=1}^{n-1}\left(\binom{n+1}{k}-\binom{n}{k}\right)B_k+(n+1)B_n>0,$$

since

$$\binom{n+1}{k}>\binom{n}{k}\qquad(k=1,\ldots,n).$$

$\square$

In closing the article we give one more property of the Bell numbers but before prove the following general statement.

Theorem 13. *Let F_n be a strictly increasing sequence of positive integers such that*

$$\log F_n \sim Cn \log n \qquad (C > 0).$$

Let $\omega(x)$ be the number of F_n that do not exceed x. The following asymptotic formula holds.

$$\omega(x) \sim \frac{\log x}{C \log \log x}.$$

Proof. Let α_n be a strictly increasing sequence of positive numbers and let $\alpha(x)$ be the number of α_n that do not exceed x. It is well-known [3, p. 129] that

$$\alpha_n \sim Cn \log n \Leftrightarrow \alpha(x) \sim \frac{x}{C \log x}.$$

Now,

$$F_n \leq x \Leftrightarrow \alpha_n = \log F_n \leq \log x.$$

Consequently

$$\omega(x) = \alpha(\log x) \sim \frac{\log x}{C \log \log x}.$$

□

Example 14. If $F_n = B_n$ is the n-th Bell number and $\omega(x)$ is the number of Bell numbers that do not exceed x then (see (19))

$$\omega(x) \sim \frac{\log x}{\log \log x}.$$

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