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On Fibonacci and Lucas Numbers of the Form cx^2

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Abstract

In this paper, by using some congruences concerning with Fibonacci and Lucas numbers, we completely solve the Diophantine equations $L_n = 2L_m x^2$, $F_n = 2F_m x^2$, $L_n = 6L_m x^2$, $F_n = 3F_m x^2$, and $F_n = 6F_m x^2$.

1 Introduction

Fibonacci and Lucas sequences are defined as follows; $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ and $L_0 = 2, L_1 = 1$, $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$, respectively. F_n is called the n-th Fibonacci number and L_n is called the n-th Lucas number. Fibonacci and Lucas numbers for negative subscripts are given by $F_{-n} = (-1)^{n+1} F_n$ for $n \geq 1$ and $L_{-n} = (-1)^n L_n$ for $n \geq 1$. It can be seen that $L_n = F_{n-1} + F_{n+1}$ and $L_{n-1} + L_{n+1} = 5F_n$ for every $n \in \mathbb{Z}$. For more information about Fibonacci and Lucas sequences, one can consult [9], [18].

Let α and β denote the roots of the equation $x^2 - x - 1 = 0$. Then $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$. It can be seen that $\alpha\beta = -1$ and $\alpha + \beta = 1$. Moreover it is well known and easy to show that

$$F_n = (\alpha^n - \beta^n)/\sqrt{5}$$

and

$$L_n = \alpha^n + \beta^n$$

for every $n \in \mathbb{Z}$.

In the following section, we will give some congruences concerning with Fibonacci and Lucas numbers. By using these congruences, we may prove many properties known before.

2 Preliminaries

The problem of characterizing the square Fibonacci numbers was first introduced in the book by Ogilvy [12, p. 100]. In 1963, both Moser and Carlitz [10], and Rollet [17] proposed this problem. In 1964, the square conjecture was proved by Cohn [4] and independently by Wyler [19]. Later the problem of characterizing the square Lucas numbers was solved by Cohn [6] and by Alfred [1]. Moreover in 1965, Cohn solved the Diophantine equations $F_n = 2x^2$ and $L_n = 2x^2$ in [6].

We give the following theorem from [5].

Theorem 1. *If $F_n = x^2$, then $n = 1, 2, 12$. If $F_n = 2x^2$, then $n = 3$, 6. If $L_n = x^2$, then $n = 1, 3$ and if $L_n = 2x^2$, then $n = 6$.*

The proofs of the following two theorems are given in [8].

Theorem 2. *Let $n \in \mathbb{N} \cup \{0\}$ and $k, m \in \mathbb{Z}$. Then*

$$F_{2mn+k} \equiv (-1)^{mn} F_k \,(\mathrm{mod}\ F_m) \tag{1}$$

and

$$L_{2mn+k} \equiv (-1)^{mn} L_k \,(\mathrm{mod}\ F_m)\,. \tag{2}$$

Theorem 3. *Let $n \in \mathbb{N} \cup \{0\}$ and $k, m \in \mathbb{Z}$. Then*

$$L_{2mn+k} \equiv (-1)^{(m+1)n} L_k \,(\mathrm{mod}\ L_m) \tag{3}$$

and

$$F_{2mn+k} \equiv (-1)^{(m+1)n} F_k \,(\mathrm{mod}\ L_m)\,. \tag{4}$$

From the identity (2), it follows that $8 \nmid L_n$ for any natural number n.

Now we give two lemmas and a corollary, which will be needed later. The proofs of the lemmas can be achieved by induction. For the proof of the corollary, one can consult [2] or [11].

Lemma 4. $L_{2^k} \equiv 3 \pmod 4$ *for the all positive integers k with $k \geq 1$.*

Lemma 5. *If* $r \geq 3$, *then* $L_{2^r} \equiv 2 \pmod 3$.

Corollary 6. *If $k \geq 1$, then there is no integer x such that $x^2 \equiv -1 \pmod{L_{2^k}}$.*

The following lemma can be proved by induction.

Lemma 7. *If* $r \geq 2$, *then* $L_{2^r} \equiv 7 \pmod 8$.

The proofs of the following theorems can be found in [3], [18] or [8].

Theorem 8. *Let $m,n \in \mathbb{N}$ and $m \geq 2$. Then $L_m | L_n$ if and only if $m|n$ and $\dfrac{n}{m}$ is an odd integer.*

Theorem 9. *Let $m,n \in \mathbb{N}$ and $m \geq 3$. Then $F_m | F_n$ if and only if $m|n$.*

Theorem 10. *Let $m,n \in \mathbb{N}$ and $m \geq 2$. Then $L_m | F_n$ if and only if $m|n$ and $\dfrac{n}{m}$ is an even integer.*

Also we give some identities about Fibonacci and Lucas numbers which will be needed in the sequel:

$$L_{2n} = L_n^2 - 2(-1)^n \tag{5}$$

$$L_{3n} = L_n(L_n^2 - 3(-1)^n) \tag{6}$$

$$F_{2n} = F_n L_n \tag{7}$$

$$F_{3n} = F_n(5F_n^2 + 3(-1)^n) \tag{8}$$

$$L_n^2 - 5F_n^2 = 4(-1)^n \tag{9}$$

$$2|F_n \ \Leftrightarrow 2|L_n \Leftrightarrow 3|n \tag{10}$$

$$(F_n, L_n) = 1 \text{ or } (F_n, L_n) = 2 \tag{11}$$

Let $\left(\frac{a}{p}\right)$ represent the Legendre symbol. Then we have

$$\left(\frac{2}{p}\right) = 1 \text{ if and only if } p \equiv \pm 1 \pmod 8 \tag{12}$$

and

$$\left(\frac{-2}{p}\right) = 1 \text{ if and only if } p \equiv 1,\ 3 \pmod 8 . \tag{13}$$

For the proof of (12) and (13), one can consult [2] or [11].

3 Main Theorems

Many authors investigated Fibonacci and Lucas numbers of the form cx^2. In [5], Cohn solved $F_n = cx^2$ and $L_n = cx^2$ for $c = 1, 2$. In [14], Robbins considered Fibonacci numbers of the form px^2. Robbins solved the equation $F_n = px^2$ for all p such that $p \equiv 3 \pmod 4$ or $p < 10000$. Later, in [15] Robbins considered Fibonacci numbers of the form cx^2. The author obtained all solutions of $F_n = cx^2$ for composite values of $c \leq 1000$. After that, in [16], the same author solved $L_n = px^2$, where p is an odd prime and $p < 1000$. Moreover, in [20], Zhou dealt with Lucas numbers of the form $L_n = px^2$, where p is a prime number, and he gave solutions for $1000 < p < 60000$. In this section, we consider the equations $L_n = 2L_m x^2$, $F_n = 2F_m x^2$, $L_n = 6L_m x^2$, $F_n = 3F_m x^2$, and $F_n = 6F_m x^2$.

In [13], Ribenboim considers square-classes of Fibonacci numbers. F_m, F_n are in the same square-class if there exist non-zero integers x, y such that $F_m x^2 = F_n y^2$; or equivalently, when $F_m F_n$ is a square. In a similar way, he considers square-classes of Lucas numbers. A square-class will be called trivial if it consists of only one number. Ribenboim showed that the square-class of L_m is trivial when $m \neq 0,\ 1,\ 3,$ and 6. He also showed that the square-class of F_m is trivial when $m \neq 1,\ 2,\ 3,\ 6,\ 12$. Now, we can give following two theorems, which can be obtained from Proposition 1 and Proposition 2 given in [13].

From now on, we will assume that n and m are positive integers.

Theorem 11. *Let $m > 3$ be an integer and $F_n = F_m x^2$ for some $x \in \mathbb{Z}$. Then $n = m$.*

Theorem 12. *Let $m \geq 2$ be an integer and $L_n = L_m x^2$ for some $x \in \mathbb{Z}$. Then $n = m$.*

The proofs of the following two theorems can be obtained from Theorem 6 and Theorem 12 given in [7], but we will give a different proof.

Theorem 13. *There is no integer x such that $L_n = 2L_m x^2$ for $m > 1$.*

Proof. Assume that $L_n = 2L_m x^2$. Then $L_m | L_n$ and therefore $n = mk$ for some odd natural number k by Theorem 8. Firstly assume that m is an odd integer. Since $2|L_n$, we get $3|n$ by (10). Thus we see that $3 \nmid m$. For if $3|m$, then $L_3 | L_m$, i.e., $4|L_m$ by Theorem 8. This implies that $8|L_n$, which is impossible. Since $3 \nmid m$, it follows that $3|k$. That is, $k = 3t$ for some odd positive integer t. Thus $n = mk = 3mt$ and mt is an odd integer. Therefore, since $3|n$, it follows that $L_3 | L_n$, i.e., $4|2L_m x^2$ by Theorem 8. Since $3 \nmid m$, L_m is an odd integer. Therefore $2|x^2$, i.e., x is an even integer. This implies that $8|L_n$, which is impossible.

Now assume that m is an even integer. If x is an even integer, then we see that $8|L_n$, which is impossible. Therefore x is an odd integer. Assume that $3|m$. Then L_m is an even integer. Therefore $L_3 | L_n$ by Theorem 8. It follows that $n = 3b$ for some odd integer b by Theorem 8. That is, n is an odd integer. But this is impossible. Because since m is an even integer, n is also an even integer. Assume that $3 \nmid m$. Then since $n = mk$ and $3|n$, we get $3|k$, i.e., $k = 3t$ for some odd integer t. Since t is an odd integer, $t = 4q \pm 1$ for some nonnegative integer t. Thus $n = mk = 3m(4q \pm 1) = 2 \cdot 6mq \pm 3m$. Then

$$L_n = L_{2 \cdot 6mq \pm 3m} \equiv L_{\pm 3m} \pmod{F_6}$$

and therefore

$$2L_m x^2 \equiv L_{3m} \pmod 8$$

by (2). Since $x^2 \equiv 1 \pmod 8$ and m is even integer, we get

$$2L_m \equiv L_m(L_m^2 - 3) \pmod 8$$

by (6). Moreover, since $3 \nmid m$, L_m is odd integer. Therefore we get

$$2 \equiv L_m^2 - 3 \pmod 8.$$

Thus

$$2 \equiv -2 \pmod 8,$$

which is impossible. This completes the proof. $\square$

In [5], it is shown that, for $m = 1, 2$, the equation $F_n = 2F_m x^2 = 2x^2$ has solution only for $n = 3, 6$. More generally, we can give the following theorem.

Theorem 14. *If $F_n = 2F_m x^2$ and $m \geq 3$, then $m = 3$, $x^2 = 36$, and $n = 12$ or $m = 6$, $x^2 = 9$, and $n = 12$.*

Proof. If $m = 3$, then $F_n = 2F_3 x^2 = (2x)^2$. Thus it can be seen that $n = 12$, $x^2 = 36$ by Theorem 1. Assume that $m > 3$ and $F_n = 2F_m x^2$. Then $F_m | F_n$ and therefore $n = mk$ for some natural number k by Theorem 9.

Firstly, assume that k is an even integer. Then $k = 2t$ for some integer t. Therefore $n = mk = 2mt$. Thus

$$F_n = F_{2mt} = F_{mt} L_{mt} = 2F_m x^2$$

by (7). This shows that $(F_{mt}/F_m) L_{mt} = 2x^2$. It can be easily seen that if $(F_{mt}/F_m, L_{mt}) = d$, then $d = 1$ or $d = 2$ by (11). Thus we have the following equations:

$$\frac{F_{mt}}{F_m} = u^2, \; L_{mt} = 2v^2, \tag{14}$$

$$\frac{F_{mt}}{F_m} = 2u^2, \; L_{mt} = v^2, \tag{15}$$

$$\frac{F_{mt}}{F_m} = 2u^2, \; L_{mt} = (2v)^2, \tag{16}$$

and

$$\frac{F_{mt}}{F_m} = (2u)^2, \; L_{mt} = 2v^2. \tag{17}$$

Assume that (14) is satisfied. Then $mt = m$, i.e., $t = 1$ by Theorem 11. Therefore $L_m = 2v^2$ and this implies that $m = 6$ by Theorem 1. Thus we get $m = 6$, $x^2 = 9$, and $n = 12$. By using Theorem 1 and Theorem 11, it can be seen that the other three cases are impossible.

Secondly, assume that k is an odd integer. Suppose that m is an even integer, i.e., $m = 2r$ for some natural number r. Then we can write $n = mk = 2kr$. Thus

$$F_n = F_{2kr} = F_{kr} L_{kr} = 2F_r L_r x^2$$

by (7). This shows that $(F_{kr}/F_r)(L_{kr}/L_r) = 2x^2$. A similar argument shows that the equation $(F_{kr}/F_r)(L_{kr}/L_r) = 2x^2$ has no solution. Now assume that m is an odd integer. Firstly, suppose that $3 \nmid k$. Since k is an odd integer, we can write $k = 6q \pm 1$ for some nonnegative integer q. Therefore $n = mk = m(6q \pm 1) = 2 \cdot 3mq \pm m$. Thus we get

$$F_n = F_{2 \cdot 3mq \pm m} \equiv F_{\pm m} \pmod{L_3},$$

i.e.,

$$F_n \equiv F_m \pmod 4$$

by (4). Since F_n is even integer, F_m is also an even integer. Thus $3|m$, and therefore $m = 3a$ for some integer a by (10). On the other hand, since F_m is even integer, $4|F_n$, and thus $6|n$ by Theorem 9. Since $n = mk = 3ak$, we get $6|3ak$, i.e., $2|ak$. Moreover, since k is odd integer, it is seen that $2|a$. This implies that $2|m$, which is impossible. Because m is an odd integer. Assume that $3|k$. Then $k = 3s$ for some odd integer s. Therefore $n = mk = 3ms$. Thus since ms is odd integer, we get

$$F_n = F_{3ms} = F_{ms}(5F_{ms}^2 - 3) = 2F_m x^2$$

by (8). This shows that $(F_{ms}/F_m)(5F_{ms}^2 - 3) = 2x^2$. It can be easily seen that if $d = (F_{ms}/F_m, 5F_{ms}^2 - 3)$, then $d = 1$ or $d = 3$. Assume that $d = 3$. Then $3|F_{ms}$, and thus $4|ms$ by Theorem 9. But this is impossible, since ms is odd integer. Therefore $d = 1$. Then we get

$$\frac{F_{ms}}{F_m} = u^2,\ 5F_{ms}^2 - 3 = 2v^2 \tag{18}$$

or

$$\frac{F_{ms}}{F_m} = 2u^2,\ 5F_{ms}^2 - 3 = v^2 \tag{19}$$

for some integers u and v. Assume that (18) is satisfied. Then $ms = m$, i.e., $s = 1$ by Theorem 11. Therefore $5F_m^2 - 3 = 2v^2$ and this shows that $2v^2 = 5F_m^2 - 3 = L_m^2 + 1 = L_{2m} - 1$ by (9) and (5). This implies that $L_{2m} = 2v^2 + 1$. Since $L_{2m} = 2v^2 + 1$, we get $3 \nmid m$. Thus we can write $m = 6q \pm 1 = 3 \cdot 2^{r+1}b \pm 1$, where $q = 2^r b$ for some odd integer b with $r \geq 0$. This shows that

$$L_{2m} = L_{2 \cdot 2^{r+1} 3b \pm 2} \equiv -L_{\pm 2} \pmod{L_{2^{r+1}}}$$

and therefore

$$2v^2 + 1 \equiv -3 \pmod{L_{2^{r+1}}},$$

i.e.,

$$2v^2 \equiv -4 \pmod{L_{2^{r+1}}}$$

by (3). Since $L_{2^{r+1}}$ is an odd integer, we get

$$v^2 \equiv -2 \pmod{L_{2^{r+1}}}.$$

This shows that $\left(\dfrac{-2}{p}\right) = 1$ for every prime divisor of $L_{2^{r+1}}$. Then it follows that

$$p \equiv 1, 3 \pmod 8$$

by (13) and therefore

$$L_{2^{r+1}} \equiv 1, 3 \pmod 8.$$

This shows that $r = 0$ by Lemma 7. Consequently, q is an odd integer. Therefore it can be easily seen that $m = 12c + 5$ or $m = 12c + 7$ for some integer c. Thus we get

$$L_m \equiv 3 \pmod 8$$

or

$$L_m \equiv 5 \pmod 8$$

by (2). On the other hand, since

$$2v^2 = L_m^2 + 1,$$

we get

$$2v^2 \equiv 1 \pmod{L_m},$$

and therefore

$$(2v)^2 \equiv 2 \pmod{L_m}.$$

This shows that $\left(\dfrac{2}{p}\right) = 1$ for every prime divisor p of L_m. Then it follows that

$$p \equiv \pm 1 \pmod 8$$

by (12) and therefore

$$L_m \equiv \pm 1 \pmod 8.$$

But this contradicts with the fact that $L_m \equiv 3, 5 \pmod 8$. Assume that (19) is satisfied. Then we get $v^2 = 5F_{ms}^2 - 3 = L_{ms}^2 + 1$ by (9). This implies that $L_{ms} = 0$, which is impossible. This completes the proof. $\square$

Theorem 15. *If $L_n = 6L_m x^2$ and $m \geq 1$, then $m = 2$, $x^2 = 1$, and $n = 6$.*

Proof. Assume that $L_n = 6L_m x^2$ for some integer x. Then $3|L_n$ and therefore $n = 2k_0$ for some odd integer k_0 by Theorem 8. Moreover, since $2|L_n$, we get $3|n$ by (10). This shows that $3|k_0$ and then $k_0 = 3t$ for some odd integer t. Thus $n = 6t = 6(2u + 1) = 12u + 6$. Therefore

$$L_n = L_{12u+6} \equiv L_6 \pmod 8$$

by (2). That is,

$$L_n \equiv 2 \pmod 8.$$

Since $8 \nmid L_n$, it can be seen that x is an odd integer. Therefore

$$x^2 \equiv 1 \pmod 8,$$

which implies that

$$6L_m x^2 \equiv 6L_m \pmod 8.$$

This shows that

$$6L_m \equiv 2 \pmod 8,$$

which implies that $m \neq 1$. Now assume that $m > 2$. Since $L_m|L_n$, there exists an odd integer k such that $n = mk$ by Theorem 8. On the other hand, since $2|n$, it is seen that $2|m$. Therefore $m = 2r$ for some odd integer r. If $r = 6q+3$, then $m = 2r = 12q+6$ and therefore

$$L_m \equiv L_6 \pmod 8$$

by (2). That is,

$$L_m \equiv 2 \pmod 8,$$

which is impossible since

$$6L_m \equiv 2 \pmod 8.$$

Therefore $3 \nmid r$. Since $n = mk$, $m = 2r$ and $3 \nmid r$, it follows that $3|k$ and thus $k = 3s$ for some odd integer s. Then

$$L_n = L_{mk} = L_{3ms} = L_{ms}(L_{ms}^2 - 3) = 6L_m x^2$$

by (6). It can be seen that $(L_{ms}, L_{ms}^2 - 3) = 3$. Thus $\left(L_{ms}, \dfrac{L_{ms}^2 - 3}{3}\right) = 1$. Then we get

$$\frac{L_{ms}}{L_m}\left(\frac{L_{ms}^2 - 3}{3}\right) = 2x^2.$$

This shows that

$$\frac{L_{ms}}{L_m} = 2u^2 \text{ and } \frac{L_{ms}^2 - 3}{3} = v^2 \tag{20}$$

or

$$\frac{L_{ms}}{L_m} = u^2 \text{ and } \frac{L_{ms}^2 - 3}{3} = 2v^2 \tag{21}$$

for some integers u and v. Assume that (20) is satisfied. Then $3\left(\dfrac{L_{ms}}{3}\right)^2 - 1 = v^2$ and therefore

$$v^2 \equiv -1 \pmod 3,$$

which is a contradiction. Now assume that (21) is satisfied. Then $L_{ms} = L_m u^2$, which implies that $ms = m$ by Theorem 12. That is, $s = 1$. Thus $L_m^2 - 3 = 6v^2$. Since $L_m^2 = L_{2m} + 2$ by (5), we see that $L_{2m} - 1 = 6v^2$. Moreover, since $m = 2r$, it follows that $L_{4r} - 1 = 6v^2$. On the other hand, we can write $4r$ as $4r = 4(4u \pm 1) = 16u \pm 4 = 2 \cdot 2^k a \pm 4$ for some odd integer a with $k \geq 3$. This shows that

$$L_{4r} = L_{2 \cdot 2^k a \pm 4} \equiv -L_{\pm 4} \pmod{L_{2^k}}$$

by (3) and therefore

$$1 + 6v^2 \equiv -7 \pmod{L_{2^k}}.$$

Then we get

$$6v^2 \equiv -8 \pmod{L_{2^k}}.$$

That is,

$$3v^2 \equiv -4 \pmod{L_{2^k}}.$$

Thus

$$(3v)^2 \equiv -12 \pmod{L_{2^k}}.$$

This shows that $\left(\dfrac{-12}{p}\right) = 1$ for every prime divisor p of L_{2^k}. Then it follows that

$$p \equiv 1 \pmod 3$$

and therefore

$$L_{2^k} \equiv 1 \pmod 3.$$

But this contradicts with Lemma 5. This completes the proof. □

In [8], the authors showed that $L_n = 3L_m x^2$ has no solution if $m > 1$. Now we give a similar result for Fibonacci numbers.

Theorem 16. *Let $m \geq 3$ be an integer and $F_n = 3F_m x^2$ for some integer x. Then $m = 4$, $x^2 = 16$, and $n = 12$.*

Proof. Assume that $m \geq 3$ and $F_n = 3F_m x^2$ for some integer x. Then $F_m | F_n$ and therefore $n = mk$ for some integer k by Theorem 9.

Firstly, assume that k is an even integer. Then $k = 2s$ for some $s \in \mathbb{N}$. Therefore $n = mk = 2ms$. Thus

$$F_n = F_{2ms} = F_{ms} L_{ms} = 3F_m x^2$$

by (7). This shows that

$$(F_{ms}/F_m) L_{ms} = 3x^2.$$

By using Theorem 1, Theorem 12, and Theorem 15, it can be shown that the equation $(F_{ms}/F_m)L_{ms} = 3x^2$ has no solution.

Now assume that k is an odd integer. Since $F_n = 3F_m x^2$, we get $4|n$ by Theorem 9. Moreover, since $n = mk$ and k is odd, we get $4|m$. Assume that x is an even integer. Then $4|F_n$. Thus $L_3|F_n$ and $3|n$ by Theorem 10. Therefore since $4|n$ and $3|n$, we get $12|n$. That is, $n = 12t$ for some $t \in \mathbb{N}$. On the other hand since $4|m$, we get $m = 4r$ for some $r \in \mathbb{N}$. Therefore $12t = n = mk = 4rk$. Then it follows that $3t = rk$. Thus

$$F_n = F_{12t} = F_{6t} L_{6t} = 3F_{2r} L_{2r} x^2$$

by (7). Since $(6t/2r) = k$ and k is odd, we can write

$$\frac{F_{6t}}{F_{2r}} . \frac{L_{6t}}{L_{2r}} = 3x^2.$$

Assume that $3|r$. Then, it can be seen that $\left(\dfrac{F_{6t}}{F_{2r}}, \dfrac{L_{6t}}{L_{2r}}\right) = 1$. Therefore

$$\frac{F_{6t}}{F_{2r}} = u^2, \ \frac{L_{6t}}{L_{2r}} = 3v^2 \tag{22}$$

or

$$\frac{F_{6t}}{F_{2r}} = 3u^2, \ \frac{L_{6t}}{L_{2r}} = v^2 \tag{23}$$

for some integers u and v. A similar argument shows that (22) and (23) are impossible. Now assume that $3 \nmid r$. Then since $3t = rk$, it follows that $3|k$. Thus $k = 3s$ for some $s \in \mathbb{N}$. Then $3t = rk = 3rs$ and therefore $t = rs$. Also since $3 \nmid r$, it can be seen that $\left(\frac{F_{6t}}{F_{2r}}, \frac{L_{6t}}{L_{2r}}\right) = 2$. Therefore

$$\frac{F_{6t}}{F_{2r}} = 2u^2, \ \frac{L_{6t}}{L_{2r}} = 6v^2 \tag{24}$$

or

$$\frac{F_{6t}}{F_{2r}} = 6u^2, \ \frac{L_{6t}}{L_{2r}} = 2v^2 \tag{25}$$

for some integers u and v. Assume that (24) is satisfied. Then $2r = 2$ by Theorem 15. This shows that $r = 1$ and $t = s$. Thus $L_{6t} = 6L_2v^2 = L_6v^2$ and this implies that $6t = 6$, i.e., $t = 1$ by Theorem 12. Therefore $k = 3s = 3t = 3$ and $m = 4r = 4$. Therefore $n = 12$ and $x^2 = 16$.

Now assume that (25) is satisfied. Then it follows that

$$L_{6t} = 2L_{2r}v^2,$$

which is impossible by Theorem 13 and Theorem 15.

Now assume that x is an odd integer. Then

$$F_n \equiv 3F_m \pmod 8.$$

Since $4|m$, it follows that $m = 12q$ or $m = 12q \pm 4$ for some integer q. If $m = 12q \pm 4$, then

$$F_m \equiv F_{12q\pm4} \equiv F_{\pm4} \equiv \pm3 \pmod 8$$

by (1). Therefore

$$F_n \equiv \pm1 \pmod 8,$$

which is impossible since $4|n$. Because if $4|n$, then $n = 12r \pm 4$ or $n = 12r$ for some integer r, and therefore $F_n \equiv \pm3, 0 \pmod 8$ by (1). If $m = 12q$, then $n = mk = 12qk$. This shows that $6qk/6q$ is an odd integer. Then from the identity

$$F_n = F_{12qk} = F_{6qk}L_{6qk} = 3F_mx^2 = 3F_{6q}L_{6q}x^2,$$

it follows that

$$\frac{F_{6qk}}{F_{6q}} . \frac{L_{6qk}}{L_{6q}} = 3x^2.$$

Since $\left(\frac{F_{6qk}}{F_{6q}}, \frac{L_{6qk}}{L_{6q}}\right) = 1$, we get

$$\frac{F_{6qk}}{F_{6q}} = u^2, \; \frac{L_{6qk}}{L_{6q}} = 3v^2 \tag{26}$$

or

$$\frac{F_{6qk}}{F_{6q}} = 3u^2, \; \frac{L_{6qk}}{L_{6q}} = v^2 \tag{27}$$

for some integers u and v. Similarly, it can be seen that (26) and (27) are impossible. This completes the proof. □

Lastly, we can give the following theorem without proof since its proof is similar to that of Theorem 16.

Theorem 17. *There is no integer x such that $F_n = 6F_m x^2$.*

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