Journal of Integer Sequences, Vol. 14 (2011),
Article 11.3.1

Touchard Polynomials, Partial Bell Polynomials and Polynomials of Binomial Type

Miloud Mihoubi and Mohammed Said Maamra[1]
University of Science and Technology Houari Boumediene (USTHB)
Faculty of Mathematics
PB 32
El Alia, 16111, Algiers
Algeria
miloudmihoubi@gmail.com
mmihoubi@usthb.dz
mmaamra@yahoo.fr

Abstract

Touchard generalized the Bell polynomials in order to give some combinatorial interpretation on permutations. Chrysaphinou introduced and studied a class of polynomials related to Touchard's generalization. In the present paper, we establish some relations between Touchard polynomials, Bell polynomials and the polynomials of binomial type. Several identities and relations with Stirling numbers are obtained.

1 Introduction

Among the partition polynomials, the partial Bell polynomials, introduced by Bell [3], play an important role in different application frameworks. Several properties and identities are given, see [5, 7, 9, 10]. Another partition polynomials, called Touchard polynomials, introduced by Touchard [11], present an extension of the partial Bell polynomials. Some algebraic, combinatorial and probabilistic properties of these polynomials are studied by Touchard [11], Chrysaphinou [6], Charalambides [5], Kuzmin and Leonova [8]. In this paper,

[1]Research supported by LAID3 Laboratory of USTHB University.

we give some relations between Touchard polynomials and partial Bell polynomials. We exploit these relations and the polynomials of binomial type to derive some identities for these polynomials.

In this context, let $(x_i; i \geq 1)$ and $(y_i; i \geq 1)$ be two sequences of real numbers. The Touchard polynomials

$$T_{n,k}(x_j; y_j) := T_{n,k}(x_1, \ldots, x_n; y_1, \ldots, y_n), \ n \geq k \geq 0,$$

are defined by their bivariate generating function

$$\sum_{n\geq 0}\sum_{k=0}^{n} T_{n,k}(x_j; y_j) u^k \frac{t^n}{n!} = \exp\left(u \sum_{i\geq 1} x_i \frac{t^i}{i!} + \sum_{i\geq 1} y_i \frac{t^i}{i!} \right).$$

The vertical generating function of Touchard polynomials, for fixed k, is given by

$$\sum_{n\geq k} T_{n,k}(x_j; y_j) \frac{t^n}{n!} = \frac{1}{k!} \left(\sum_{i\geq 1} x_i \frac{t^i}{i!} \right)^k \exp\left(\sum_{i\geq 1} y_i \frac{t^i}{i!} \right), \ k = 0, 1, \ldots.$$

The partial Bell polynomials are given by

$$B_{n,k}(x_j) := B_{n,k}(x_1, \ldots, x_n) = T_{n,k}(x_1, \ldots, x_n; 0, \ldots, 0), \ n \geq k \geq 0.$$

The polynomials of binomial type $(f_n(x))$ are defined by

$$\sum_{n\geq 0} f_n(x) \frac{u^n}{n!} = \left(\sum_{n\geq 0} f_n(1) \frac{u^n}{n!} \right)^x, \quad \text{with } f_0(x) = 1 \ \text{ and } \ f_1(x) \neq 0 \ \text{ for } x \neq 0.$$

For a real number a we consider in the following the sequence $(f_n(x; a))$ defined by

$$f_n(x; a) := \frac{x}{an + x} f_n(an + x).$$

The sequence $(f_n(x; a))$ is also of binomial type, see [9], and for more details on sequences of binomial type see [1].

We use the following notation and hypothesis:

$D \equiv \frac{d}{dx}$, $D^k \equiv \frac{d^k}{dx^k}$, $D^k_{x=x_0} \equiv \frac{d^k}{dx^k}|_{x=x_0}$.

For $n < 0$, we set $f_n(x) = 0$, $T_{n,k}(x_j;\ y_j) = 0$ and $B_{n,k}(x_j) = 0$.

For $x \in \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers, we set

$$\binom{x}{k} := \frac{x(x-1)\cdots(x-k+1)}{k!} \text{ for } k = 1, 2, \ldots, \quad \binom{x}{0} = 1 \text{ and } \binom{x}{k} = 0 \ \text{ otherwise.}$$

Also, for all nnnegative integers n, m we put

$$1_{(m|n)} = \begin{cases} 1, & \text{if } m \text{ divides } n; \\ 0, & \text{otherwise;} \end{cases} \qquad \text{and} \qquad 1_{(n\geq m)} = \begin{cases} 1, & \text{if } n \geq m; \\ 0, & \text{otherwise.} \end{cases}$$

2 The main results

In this section, we establish some relations between the Touchard polynomials, partial Bell polynomials and polynomials of binomial type. Furthermore, we use these relations to develop several identities for Touchard polynomials. We start with the following theorem:

Theorem 1. *Let n, k, m be integers such that $n \geq k \geq 1, m \geq 1;$ a be a real number and (x_n) be a sequence of real numbers. Then*

$$T_{n,k}\left(x_j;\ -m(j-1)!a^{j/m}1_{(m|j)}\right) = B_{n,k}(x_j) - am!\binom{n}{m}B_{n-m,k}(x_j).$$

Proof. Let $y_n = -m(n-1)!a^{n/m}1_{(m|n)}$.
For $a = 0$ the theorem is trivial, otherwise, for $|t| < |a|^{-1/m}$ we have

$$\exp\left(\sum_{i\geq 1} y_i \frac{t^i}{i!}\right) = \exp\left(-\sum_{j\geq 1} a^j \frac{t^{mj}}{j}\right) = \exp(\ln(1-at^m)) = 1 - at^m.$$

Then

$$\begin{aligned}
\sum_{n\geq k} T_{n,k}(x_j; y_j)\frac{t^n}{n!} &= \frac{1}{k!}\left(\sum_{i\geq 1} x_i \frac{t^i}{i!}\right)^k \exp\left(\sum_{i\geq 1} y_i \frac{t^i}{i!}\right) \\
&= \frac{1}{k!}\left(\sum_{i\geq 1} x_i \frac{t^i}{i!}\right)^k (1 - at^m) \\
&= \sum_{n\geq k}\left(B_{n,k}(x_j) - a\frac{n!}{(n-m)!}B_{n-m,k}(x_j)\right)\frac{t^n}{n!}.
\end{aligned}$$

Hence the theorem is proved. □

If we set $x_n = nf_{n-1}(x; b)$ in Theorem 1 and use Proposition 1 of [9], we obtain:

Corollary 2. *Let $(f_n(x))$ be a sequence of binomial type of polynomials. We have*

$$T_{n,k}\left(jf_{j-1}(x;b);\ -m(j-1)!a^{j/m}1_{(m|j)}\right) = \frac{n!}{k!}\left(\frac{f_{n-k}(kx;b)}{(n-k)!} - a\frac{f_{n-m-k}(kx;b)}{(n-m-k)!}\right).$$

Example 3. For $f_n(x) = x^n$ in Corollary 2 we get

$$\begin{aligned}
&T_{n,k}\left(jx(b(j-1)+x)^{j-2};\ -m(j-1)!a^{j/m}1_{(m|j)}\right) \\
&= x\frac{n!}{(k-1)!}\left(\frac{(b(n-k)+kx)^{n-k-1}}{(n-k)!} - a\frac{(b(n-m-k)+kx)^{n-m-k-1}}{(n-m-k)!}1_{(n\geq m+k)}\right),
\end{aligned}$$

and for $f_n(x) = n!\binom{x}{n}$ in Corollary 2 we get

$$
\begin{aligned}
&T_{n,k}\left(xj!\frac{\binom{b(j-1)+x}{j-1}}{b(j-1)+x};\ -m(j-1)!a^{j/m}1_{(m|j)} \right) \\
&= x\frac{n!}{(k-1)!}\left(\frac{\binom{b(n-k)+kx}{n-k}}{b(n-k)+kx} - a\frac{\binom{b(n-m-k)+kx}{n-m-k}}{b(n-m-k)+kx}1_{(n\geq m+k)} \right).
\end{aligned}
$$

As above, for particular cases of Touchard polynomials, the following proposition gives another expression in term of polynomials of binomial type.

Proposition 4. *Let b, α be two real numbers and $(f_n(x))$ be a sequence of binomial type of polynomials. We have*

$$
T_{n,k}\left(jf_{j-1}(x);\ \alpha Df_j(0) \right) = \binom{n}{k} f_{n-k}(kx+\alpha),
$$

or more generally

$$
\begin{aligned}
&T_{n,k}\left(jx\frac{f_{j-1}(b(j-1)+x)}{b(j-1)+x};\ \alpha\frac{f_j(bj)}{bj} \right) \\
&= \binom{n}{k}(kx+\alpha)\frac{f_{n-k}(b(n-k)+kx+\alpha)}{b(n-k)+kx+\alpha},\ b \neq 0.
\end{aligned}
$$

Proof. Let

$$
1 + \sum_{n\geq 1} f_n(x)\frac{t^n}{n!} = \exp\left(x\sum_{i\geq 1} y_i\frac{t^i}{i!} \right)
$$

be the exponential generating function of the sequence $(f_n(x))$ and $x_n = nf_{n-1}(x)$. Necessarily $y_n = Df_n(0)$. We have

$$
\begin{aligned}
\sum_{n\geq k} T_{n,k}\left(x_j;\ \alpha y_j \right)\frac{t^n}{n!} &= \frac{1}{k!}\left(\sum_{i\geq 1} x_i\frac{t^i}{i!} \right)^k \exp\left(\alpha\sum_{i\geq 1} y_i\frac{t^i}{i!} \right) \\
&= \frac{t^k}{k!}\exp\left((kx+\alpha)\sum_{i\geq 1} y_i\frac{t^i}{i!} \right) \\
&= \frac{t^k}{k!}\sum_{n\geq 0} f_n(kx+\alpha)\frac{t^n}{n!} \\
&= \sum_{n\geq k}\binom{n}{k} f_{n-k}(kx+\alpha)\frac{t^{n+k}}{n!}.
\end{aligned}
$$

Then, we obtain

$$
T_{n,k}\left(jf_{j-1}(x); \alpha Df_j(0) \right) = \binom{n}{k} f_{n-k}(kx+\alpha).
$$

To finish the proof, replace $f_n(x)$ by $f_n(x;b)$ in the last identity. ∎

Example 5. For $f_n(x) = x^n$ in Proposition 4 we get

$$T_{n,k}\Big(xj(b(j-1)+x)^{j-2};\ \alpha(j-1)(bj)^{j-2}\Big) = \binom{n}{k}(kx+\alpha)(b(n-k)+kx+\alpha)^{n-k-1},$$

and for $f_n(x) = n!\binom{x}{n}$ in Proposition 4 we get

$$\begin{aligned}&T_{n,k}\left(\frac{j!x}{b(j-1)+x}\binom{b(j-1)+x}{j-1};\ \alpha(-1)^{j-1}(j-1)!\right)\\ &= \frac{n!}{k!}\frac{kx+\alpha}{n(n-k)+kx+\alpha}\binom{n(n-k)+kx+\alpha}{n-k}.\end{aligned}$$

Hence we may state the following:

Corollary 6. *Let r, s, p be a nonnegative integers, $r \geq 1$, and (x_n) be a sequence of real numbers with $x_1 = 1$. We have*

$$\begin{aligned}&T_{n,k}\left(\frac{js}{(r(j-1)+s)}\frac{B_{(r+1)(j-1)+s,\ r(j-1)+s}(x_i)}{\binom{(r+1)(j-1)+s}{r(j-1)+s}};\ \frac{p}{rj}\frac{B_{(r+1)j,\ rj}(x_i)}{\binom{(r+1)j}{rj}}\right)\\ &= \binom{n}{k}\frac{ks+p}{r(n-k)+ks+p}\frac{B_{(r+1)(n-k)+ks+p,\ r(n-k)+ks+p}(x_i)}{\binom{(r+1)(n-k)+ks+p}{r(n-k)+ks+p}}.\end{aligned}$$

Proof. For $x_2 \neq 0$, let $\{f_n(x)\}$ be a sequence of binomial type such that $f_n(1) = \frac{x_{n+1}}{n+1}$. From the known identity $B_{n,k}(jf_{j-1}(1)) = \binom{n}{k}f_{n-k}(k)$ we get

$$f_n(k) = \binom{n+k}{k}^{-1}B_{n+k,k}(x_i),\quad n \geq 0, k \geq 1.$$

Take $b = r$, $x = s$ and $\alpha = p$ in Proposition 4 to get

$$T_{n,k}\left(js\frac{f_{j-1}(r(j-1)+s)}{r(j-1)+s};\ p\frac{f_j(rj)}{rj}\right) = \binom{n}{k}(ks+p)\frac{f_{n-k}(r(n-k)+ks+p)}{r(n-k)+ks+p}.$$

Therefore, it suffices to use the identity $f_n(k) = \binom{n+k}{k}^{-1}B_{n+k,k}(x_i)$ to express in the last identity $f_{i-1}(r(i-1)+s)$ and $f_{n-k}(r(n-k)+ks+p)$ by the partial Bell polynomials. For the case $x_2 = 0$ the corollary remains true by continuity. □

Example 7. By Corollary 6 and the identity $B_{n,k}(1!, 2!, \ldots, (q+1)!, 0, \ldots) = \frac{n!}{k!}\binom{k}{n-k}_q$, see [2], we get

$$\begin{aligned}&T_{n,k}\left(\frac{j!s}{r(j-1)+s}\binom{r(j-1)+s}{j-1}_q;\ \frac{(j-1)!p}{r}\binom{rj}{j}_q\right)\\ &= \frac{n!}{k!}\frac{ks+p}{r(n-k)+ks+p}\binom{r(n-k)+ks+p}{n-k}_q,\end{aligned}$$

where $\binom{k}{n}_q$ is the coefficients defined by $(1+x+x^2+\cdots+x^q)^k = \sum_{n\geq 0}\binom{k}{n}_q x^n$.
Also, for $x_n = 1$, $x_n = (-1)^{n-1}(n-1)!$, $x_n = (n-1)!$ or $x_n = n!$ we get identities related Touchard polynomials to Stirling numbers of the first and second kind.

The following two propositions give relations between Touchard polynomials and the successive derivatives of polynomials of binomial type.

Proposition 8. *Let b be a real number, $b \neq 0$, and $(f_n(x))$ be a sequence of binomial type of polynomials. We have*

$$T_{n,k}\Big(Df_j(0);\ xDf_j(0)\Big) = \frac{1}{k!}D^k f_n(x),$$

or more generally

$$T_{n,k}\left(\frac{f_j(bj)}{bj};\ x\frac{f_j(bj)}{bj}\right) = \frac{1}{k!}D^k\left(\frac{x}{bn+x}f_n(bn+x)\right).$$

Proof. Let $(f_n(x))$ be a sequence of binomial type defined as in the proof of Proposition 4 and $x_n = nf_{n-1}(x)$. We have $y_n = Df_n(0)$ and

$$\sum_{n\geq k} D^k f_n(x)\frac{t^n}{n!} = \left(\sum_{i\geq 1} y_i \frac{t^i}{i!}\right)^k \exp\left(x\sum_{i\geq 1} y_i\frac{t^i}{i!}\right) = k!\sum_{n\geq k} T_{n,k}(y_j;\ xy_j)\frac{t^n}{n!}.$$

Then

$$T_{n,k}\Big(y_j;\ xy_j\Big) = T_{n,k}\Big(Df_j(0);\ xDf_j(0)\Big) = \frac{1}{k!}D^k f_n(x).$$

After that, replace $f_n(x)$ by $f_n(x;b)$ in the last identity. $\square$

Example 9. For $f_n(x) = x^n$ in Proposition 8 we get

$$T_{n,k}\Big((bj)^{j-1};\ x(bj)^{j-1}\Big) = \frac{n!}{(k!)^2}(bn+x)^{n-k-1}(b(n-1)+x),$$

and for $f_n(x) = n!\binom{x}{n}$ in Proposition 8 we get

$$T_{n,k}\left(\frac{j!}{bj}\binom{bj}{j};\ x\frac{j!}{bj}\binom{bj}{j}\right) = \frac{n!}{k!}D^k\left(\frac{x}{bn+x}\binom{bn+x}{n}\right).$$

Proposition 10. *Let b, α, β be real numbers, r be a positive integer and $(f_n(x))$ be a sequence of binomial type of polynomials. We have*

$$\begin{aligned} &T_{n,k}\Big(jD^r_{z=0}(e^{\beta z}f_{j-1}(x+z));\ \alpha Df_j(0)\Big) \\ &= \frac{(kr)!}{k!}T_{n,kr}\Big(\beta 1_{(j=1)} + jDf_{j-1}(0)1_{(j\geq 2)};\ (kx+\alpha)Df_j(0)\Big), \end{aligned}$$

or more generally,

$$\begin{aligned} &T_{n,k}\left(jD^r_{z=0}\Big(\frac{(x+z)f_{j-1}(b(j-1)+x+z)}{b(j-1)+x+z}e^{\beta z}\Big);\ \alpha\frac{f_j(bj)}{bj}\right) \\ &= \frac{(kr)!}{k!}T_{n,kr}\left(\beta 1_{(j=1)} + j\frac{f_{j-1}(b(j-1))}{b(j-1)}1_{(j\geq 2)};\ (kx+\alpha)\frac{f_j(bj)}{bj}\right). \end{aligned}$$

Proof. Let $(f_n(x))$ be a sequence of binomial type defined as in the proof of Proposition 4 and $x_n = nD^r_{z=0}(e^{\beta z} f_{n-1}(x+z))$. We have $y_n = Df_n(x)(0)$ and

$$\begin{aligned}
\frac{1}{k!}\left(\sum_{i\geq 1} x_i \frac{t^i}{i!}\right)^k &= \left(\sum_{i\geq 1} iD^r_{z=0}\left(e^{\beta z} f_{i-1}(x+z)\right)\frac{t^i}{i!}\right)^k \\
&= \frac{t^k}{k!}F(t)^{kx}(D^r_{z=0}(e^{\beta z}F(t)^z))^k \\
&= \frac{t^k}{k!}F(t)^{kx}(\beta + \ln F(t))^{kr} \\
&= \frac{t^k}{k!}D^{kr}_{z=0}(e^{\beta z}F(t)^{kx+z}) \\
&= \frac{1}{k!}\left(\beta t + \sum_{i\geq 2} iy_{i-1}\frac{t^i}{i!}\right)^{kr} \exp\left(kx\sum_{i\geq 1} y_i \frac{t^i}{i!}\right).
\end{aligned}$$

Then

$$\begin{aligned}
\sum_{n\geq k} T_{n,k}\left(x_j;\ \alpha y_j\right)\frac{t^n}{n!} &= \frac{1}{k!}\left(\sum_{i\geq 1} x_i\frac{t^i}{i!}\right)^k \exp\left(\alpha \sum_{i\geq 1} y_i\frac{t^i}{i!}\right) \\
&= \frac{1}{k!}\left(\beta t + \sum_{i\geq 2} iy_{i-1}\frac{t^i}{i!}\right)^{kr} \exp\left((kx+\alpha)\sum_{i\geq 1} y_i\frac{t^i}{i!}\right) \\
&= \frac{(kr)!}{k!}\sum_{n\geq k} T_{n,kr}\left(\beta 1_{(j=1)} + jy_{j-1}1_{(j\geq 2)};\ (kx+\alpha)y_j\right)\frac{t^n}{n!}.
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
&T_{n,k}\left(jD^r_{z=0}(e^{\beta z}f_{j-1}(x+z)); \alpha Df_j(0)\right) \\
&= \frac{(kr)!}{k!}T_{n,kr}\left(\beta 1_{(j=1)} + jy_{j-1}1_{(j\geq 2)};\ (kx+\alpha)Df_j(0)\right).
\end{aligned}$$

To finish the proof, replace $f_n(x)$ by $f_n(x;b)$ in the last identity. $\square$

Example 11. For $f_n(x) = x^n$ in Proposition 10 we get

$$\begin{aligned}
&T_{n,k}\left(jD^r_{z=0}((x+z)(b(j-1)+x+z)^{j-2}e^{\beta z}); \alpha(bj)^{j-1}\right) \\
&= \frac{(kr)!}{k!}T_{n,kr}\left(\beta 1_{(j=1)} + j(b(j-1))^{j-2}1_{(j\geq 2)};\ (kx+\alpha)(bj)^{j-1}\right).
\end{aligned}$$

Proposition 12. *Let (x_n) and (y_n) be two sequences of real numbers and r be a positive integer. We have*

$$B_{n,k}\left(\frac{r}{\binom{j+r-1}{r-1}}T_{j+r-1,r}(x_i;y_i)\right) = \frac{\binom{kr}{k}}{\binom{n+(r-1)k}{(r-1)k}}T_{n+(r-1)k,kr}(x_i;ky_i)$$

Proof. Starting with the vertical generating function of Touchard polynomials to get

$$\left(\sum_{n\geq r} T_{n,r}(x_i;y_i)\frac{t^n}{n!}\right)^k = \frac{1}{(r!)^k}\left(\sum_{i\geq 1} x_i\frac{t^i}{i!}\right)^{kr} \exp\left(k\sum_{i\geq 1} y_i\frac{t^i}{i!}\right)$$
$$= \frac{(kr)!}{(r!)^k}\sum_{n\geq kr} T_{n,kr}(x_i;ky_i)\frac{t^n}{n!}.$$

Then

$$T_{n,kr}(x_i;ky_i) = \frac{(r!)^k k!}{(kr)!}B_{n,k}\Big(\underbrace{0,\ldots,0}_{r-1}, T_{r,r}(x_i;y_i), T_{r+1,r}(x_i;y_i),\ldots\Big),$$

and from [5, p. 450] we have

$$B_{n,k}\Big(0,\ldots,0,a_r,a_{r+1},\ldots\Big) = \frac{n!}{(n-(r-1)k)!}B_{n-(r-1)k,k}\left(\frac{i!a_{i+r-1}}{(i+r-1)!}\right).$$

Then

$$T_{n,kr}(x_j;ky_j) = \frac{(r!)^k n!k!}{(kr)!(n-(r-1)k)!}B_{n-(r-1)k,k}\left(\frac{j!}{(r-1+j)!}T_{r-1+j,r}(x_i;y_i)\right).$$

Change n by $n+(r-1)k$ to finish the proof. $\square$

Example 13. Let r,s,p,q be nonnegative integers with $q\geq 1$.
Use Application 7 in Proposition 12 to obtain

$$B_{n,k}\left(\frac{j!(rs+p)}{r(j-1)+rs+p}\binom{r(j-1)+rs+p}{j-1}_q\right)$$
$$= \frac{n!}{k!}\frac{k(rs+p)}{r(n-k)+k(rs+p)}\binom{r(n-k)+k(rs+p)}{n-k}_q.$$

From the identity $T_{n,r}(i!;(i-1)!s) = \frac{n!}{r!}\binom{n+s-1}{r+s-1}$ given in [5, p. 453] we obtain

$$B_{n,k}\left(j!\binom{j+r+s-2}{r+s-1}\right) = \frac{n!}{k!}\binom{n+k(r+s-1)-1}{k(r+s)-1}.$$

From Theorems 15 and 16 given in [4] we have

$$T_{n,r}((i-1)!;(i-1)!s) = \begin{bmatrix} n+s \\ r+s \end{bmatrix}_s \quad \text{and} \quad T_{n,r}(1;s1_{(i=1)}) = \begin{Bmatrix} n+s \\ r+s \end{Bmatrix}_s.$$

These identities and Proposition 12 give

$$B_{n,k}\left(\frac{r}{\binom{j+r-1}{r-1}}\begin{bmatrix} j+r+s-1 \\ r+s \end{bmatrix}_s\right) = \frac{\binom{kr}{k}}{\binom{n+(r-1)k}{(r-1)k}}\begin{bmatrix} n+(r+s-1)k \\ (r+s)k \end{bmatrix}_{ks},$$
$$B_{n,k}\left(\frac{r}{\binom{j+r-1}{r-1}}\begin{Bmatrix} j+r+s-1 \\ r+s \end{Bmatrix}_s\right) = \frac{\binom{kr}{k}}{\binom{n+(r-1)k}{(r-1)k}}\begin{Bmatrix} n+(r+s-1)k \\ (r+s)k \end{Bmatrix}_{ks}.$$

References

[1] M. Aigner, *Combinatorial Theory*. Springer, 1979.

[2] H. Belbachir, S. Bouroubi, and A. Khelladi, Connection between ordinary multinomials, generalized Fibonacci numbers, partial Bell partition polynomials and convolution powers of discrete uniform distribution. *Ann. Math. Inform.* **35** (2008), 21–30.

[3] E. T. Bell, Exponential polynomials. *Ann. Math.* **35** (1934), 258–277.

[4] A. Z. Broder, The r-Stirling numbers. *Discrete Math.*, **49** (1984), 241–259.

[5] C. A. Charalambides, *Enumerative Combinatorics.* Chapman & Hall/CRC, 2001.

[6] O. Chrysaphinou, On Touchard polynomials. *Discrete Math.* **54** (1985), 143–152.

[7] L. Comtet, *Advanced Combinatorics.* D. Reidel Publishing Company, Dordrecht-Holland, 1974.

[8] O. V. Kuzmin and O. V. Leonova, Touchard polynomials and their applications. *Disc. Math. Appl.* **10** (2000), 391–402.

[9] M. Mihoubi, Bell polynomials and binomial type sequences. *Discrete Math.* **308** (2008), 2450–2459.

[10] M. Mihoubi, The role of binomial type sequences in determination identities for Bell polynomials. To appear, *Ars Combin.* Preprint available at http://arxiv.org/abs/0806.3468v1.

[11] J. Touchard, Sur les cycles des substitutions, *Acta Math.* **70** (1939), 243–279.

2010 *Mathematics Subject Classification*: Primary 05A10; Secondary 05A99, 11B73, 11B75.
Keywords: Touchard polynomials; partial Bell polynomials; polynomials of binomial type.

Received October 21 2010; revised version received February 13 2011. Published in *Journal of Integer Sequences*, March 25 2011.

Return to Journal of Integer Sequences home page.