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Full Subsets of $\mathbb{N}$

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Abstract

Let A be a subset of $\mathbb{N}$. We say that A is m-full if $\sum A = [m]$ for a positive integer m, where $\sum A$ is the set of all positive integers which are a sum of distinct elements of A and $[m] = \{1, \ldots, m\}$. In this paper, we show that a set $A = \{a_1, \ldots, a_k\}$ with $a_1 < \cdots < a_k$ is full if and only if $a_1 = 1$ and $a_i \leq a_1 + \cdots + a_{i-1} + 1$ for each $i, 2 \leq i \leq k$. We also prove that for each positive integer $m \notin \{2, 4, 5, 8, 9\}$ there is an m-full set. We determine the numbers $\alpha(m) = \min\{|A| : \sum A = [m]\}, \beta(m) = \max\{|A| : \sum A = [m]\}, L(m) = \min\{\max A : \sum A = [m]\}$ and $U(m) = \max\{\max A : \sum A = [m]\}$ in terms of m. We also give a formula for $F(m)$, the number of m-full sets.

1 Introduction

Let n be a positive integer and denote by $D(n)$ and $\sigma(n)$ the set of its positive divisors and the sum of its positive divisors, respectively. A positive integer n is called perfect if $\sigma(n) = 2n$. Euclid proved that the formula $2^{p-1}(2^p - 1)$ gives an even perfect number whenever $2^p - 1$ is prime. It has been proved for the first time by Euler that if an even positive integer n is perfect then $n = 2^{p-1}q$, where p and $q = 2^p - 1$ are primes. In this case

$$D(n) = \{1, 2, 2^2, \ldots, 2^{p-1}, q, 2q, 2^2q, \ldots, 2^{p-1}q\}.$$

A simple argument shows that each $1 \le \ell \le 2n = 2^p q$ can be written as a sum of distinct elements of $D(n)$. To see this, note that each $\ell, 1 \le \ell \le 2^p - 1$, can be written as a sum $2^{\ell_1} + \cdots + 2^{\ell_r}$, where $r \ge 1$ and $0 \le \ell_1 < \cdots < \ell_r \le p-1$. Now if $2^p \le \ell \le 2^p q$, then we can write $\ell = \alpha q + \beta$, where $0 \le \beta \le 2^p - 1$ and $1 \le \alpha \le 2^p - 1$. Thus we can write $\alpha = 2^{\alpha_1} + \cdots + 2^{\alpha_i}$ and $\beta = 2^{\beta_1} + \cdots + 2^{\beta_j}$, where $i \ge 1, j \ge 0, 0 \le \alpha_1 < \cdots < \alpha_i \le p-1$ and $0 \le \beta_1 < \cdots < \beta_j \le p-1$. Hence

$$\ell = 2^{\alpha_1}q + \cdots + 2^{\alpha_i}q + 2^{\beta_1} + \cdots + 2^{\beta_j}$$

is a sum of distinct elements of $D(n)$.

These considerations motivate us to find all positive integers n having the property that each $1 \le \ell \le \sigma(n)$ can be written as a sum of distinct elements of $D(n)$. This leads us to the following problem:

Let $A = \{a_1, \ldots, a_k\}$ be a subset of $\mathbb{N}$. Define the *sum set* of A, denoted by $\sum A$, by

$$\sum A = \{a_{i_1} + \cdots + a_{i_r} : a_{i_1} < \cdots < a_{i_r}, 1 \le r \le k\}.$$

For what positive integer m does there exist a set A with $\sum A = [m]$, where $[m] = \{1, \ldots, m\}$?

We show that each positive integer $m \notin \{2, 4, 5, 8, 9\}$ has this property and determine the numbers

$$\begin{aligned}
\alpha(m) &= \min\{|A| : \textstyle\sum A = [m]\}, \\
\beta(m) &= \max\{|A| : \textstyle\sum A = [m]\}, \\
L(m) &= \min\{\max A : \textstyle\sum A = [m]\}, \\
U(m) &= \max\{\max A : \textstyle\sum A = [m]\}.
\end{aligned}$$

2 The Results

Definition 1. Let m be a positive integer. A subset A of $\mathbb{N}$ is called *m-full* if $\sum A = [m]$. A is called full if it is m-full for some positive integer m.

Theorem 2. *A subset $A = \{a_1, \ldots, a_k\}$ of $\mathbb{N}$ with $a_1 < \cdots < a_k$ is full if and only if $a_1 = 1$ and $a_i \le a_1 + \cdots + a_{i-1} + 1$ for each $i, 2 \le i \le k$.*

Proof. Let A be full and $\sum A = [m]$ for a positive integer m. Clearly $a_1 = 1$. If $a_j > a_1 + \cdots + a_{j-1} + 1$ for some $j, 2 \le j \le k$, then $a_1 + \cdots + a_{j-1} + 1$ is not a sum of distinct elements of A. But $1 \le a_1 + \cdots + a_{j-1} + 1 \le a_1 + \cdots + a_k = m$. This contradicts to the fact that $\sum A = [m]$.

Conversely, suppose that $a_1 = 1$ and $a_i \leq a_1 + \cdots + a_{i-1} + 1$ for each $i, 2 \leq i \leq k$. We claim that $\sum A = [a_1 + \cdots + a_k]$. We prove this by induction on k. For $k = 1$ the result is obvious. Suppose that the result is true for $k-1$. Then $\sum A \setminus \{a_k\} = [a_1 + \cdots + a_{k-1}]$. Now suppose that $a_1 + \cdots + a_{k-1} + 1 \leq \ell \leq a_1 + \cdots + a_k$ and write $\ell = a_k + a$. Then a belongs to $\{0, 1, \ldots, a_1 + \cdots + a_{k-1}\}$ since $a_k \leq a_1 + \cdots + a_{k-1} + 1$. If $a = 0$ then $\ell = a_k \in \sum A$ and if $a \neq 0$ then $a \in [a_1 + \cdots + a_{k-1}] = \sum A \setminus \{a_k\}$ can be written as $a_{i_1} + \cdots + a_{i_r}$. Thus $\ell = a_{i_1} + \cdots + a_{i_r} + a_k \in \sum A$. □

Proposition 3. *Let $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, with $p_1 < \cdots < p_r$ primes, be a positive integer. Then $D(n) = \{d : d|n\}$ is full if and only if $p_1 = 2$ and $p_i \leq \sigma(p_1^{\alpha_1} \cdots p_{i-1}^{\alpha_{i-1}}) + 1$ for each $i, 2 \leq i \leq r$.*

Proof. If $D(n)$ is m-full then $m = \sigma(n)$. Since $p_1^{\alpha_1} \cdots p_{i-1}^{\alpha_{i-1}} | n$ and $p_1^{\alpha_1} \cdots p_{i-1}^{\alpha_{i-1}} \neq n$, we have $\sigma(p_1^{\alpha_1} \cdots p_{i-1}^{\alpha_{i-1}}) < \sigma(n)$. Hence $\sigma(p_1^{\alpha_1} \cdots p_{i-1}^{\alpha_{i-1}}) + 1$ is a member of $[\sigma(n)]$. Thus if $p_i > \sigma(p_1^{\alpha_1} \cdots p_{i-1}^{\alpha_{i-1}}) + 1$ for some i, then the number $\sigma(p_1^{\alpha_1} \cdots p_{i-1}^{\alpha_{i-1}}) + 1$ is a member of $[\sigma(n)]$ which is not a sum of distinct elements of $D(n)$. On the other hand, if the condition $p_i \leq \sigma(p_1^{\alpha_1} \cdots p_{i-1}^{\alpha_{i-1}}) + 1$ for each $i, 2 \leq i \leq r$, is satisfied, then using an argument similar to the one used in Theorem 2, we can inductively prove that each element of $[\sigma(n)]$ can be written as a sum of distinct elements of $D(n)$. □

In the next theorem, we characterize all m for which there is an A with $\sum A = [m]$.

Theorem 4. *Let m be a positive integer. There is a set A such that $\sum A = [m]$ if and only if $m \notin \{2, 4, 5, 8, 9\}$.*

Proof. By simple inspection, there is no A with $\sum A = [m]$ for $m = 2, 4, 5, 8, 9$. Conversely, for $m = 1, 3, 6, 7, 10$ note that $A = \{1\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 3, 4\}$ are m-full. The recursive construction below shows us that for each $m \geq 10$ there is an A with $\sum A = [m]$.

Suppose there is an $A = \{a_1, \ldots, a_k\}$ with $a_1 < \cdots < a_k$ such that $\sum A = [m]$, for some $m \geq 10$. If $a_k < a_1 + \cdots + a_{k-1} + 1$ then $\sum (A \setminus \{a_k\}) \cup \{a_k + 1\} = [m+1]$, and there remains nothing to prove. If $a_i < a_1 + \cdots + a_{i-1} + 1$ for some i, $3 \leq i < k$, then choose j as the greatest i with this property. In this case we have

$$a_j < a_1 + \cdots + a_{j-1} + 1, \quad a_{j+1} = a_1 + \cdots + a_{j-1} + a_j + 1.$$

Notice that $j \geq 3$ since $a_1 = 1, a_2 = a_1 + 1 = 2$ and $a_3 \in \{3, 4\}$. Hence $a_j + 1 \neq a_{j+1}$ and if we omit a_j from A and add $a_j + 1$ to it, then the resulting set is still full and its sum set is $[m+1]$. Otherwise, there is no i with the property $a_i < a_1 + \cdots + a_{i-1} + 1$ and so $a_i = a_1 + \cdots + a_{i-1} + 1$ for each $i, 2 \leq i \leq k$. We can therefore deduce that $A = \{1, 2, 4, \ldots, 2^{k-1}\}$. Whence $m = 2^k - 1$ and we must give a class of sets B with $\sum B = [2^k]$, $k \geq 4$, to complete the proof.

The class of sets B is defined by a recursive construction. For $k = 4$ choose the set $B = \{1, 2, 3, 4, 6\}$. If $\sum B = [2^k]$ for $B = \{b_1, \ldots, b_s\}$ then the set $B' = \{1, 2b_1, \ldots, 2b_{s-1}, 2b_s - 1\}$ is clearly full and since $1 + 2b_1 + \cdots + 2b_{s-1} + 2b_s - 1 = 2(b_1 + \cdots + b_s) = 2 \cdot 2^k = 2^{k+1}$, we have $\sum B' = [2^{k+1}]$ and the proof is complete. □

A natural question is to determine the number of m-full sets for a given positive integer m. We denote this number by $F(m)$. As an example, a straightforward argument shows that $F(12) = 2$ and the two 12-full sets are $\{1, 2, 3, 6\}$ and $\{1, 2, 4, 5\}$. We give a formula for $F(m)$, but prior to that we discuss some related problems.

Proposition 5. *Let $m \notin \{2,4,5,8,9\}$ be a positive integer and*

$$\begin{aligned}\alpha(m) &= \min\{|A| : \sum A = [m]\},\\ \beta(m) &= \max\{|A| : \sum A = [m]\}.\end{aligned}$$

Then

$$\begin{aligned}\alpha(m) &= \min\{\ell : m \leq 2^\ell - 1\} = \lceil \log_2(m+1) \rceil\\ \beta(m) &= \max\{\ell : \frac{\ell(\ell+1)}{2} \leq m\}.\end{aligned}$$

Proof. Let $A = \{a_1, \ldots, a_k\}$ be an arbitrary m-full set. Then, by Theorem 2,

$$m = a_1 + \cdots + a_k \leq 1 + 2 + 4 + \cdots + 2^{k-1} = 2^k - 1.$$

We claim that if $n = \min\{\ell : m \leq 2^\ell - 1\}$ then $k \geq n$. In contrary, suppose that $k < n$. Then $2^k - 1 < m$. Thus $m = a_1 + \cdots + a_k \leq 2^k - 1 < m$ which is a contradiction. Hence $k \geq n$ and, since A was arbitrary, we therefore have $\alpha(m) \geq n$.

On the other hand, we show that the minimum is attained. At first we show this when m is a power of 2. For $m = 16$ we can choose $A = \{1,2,3,4,6\}$. Suppose that for $m = 2^{n-1}$ there is an m-full set $A = \{a_1, a_2, \ldots, a_n\}$. Then $A' = \{1, 2a_1, 2a_2, \ldots, 2a_{n-1}, 2a_n - 1\}$ is a 2^n-full set with $n+1$ elements.

We now suppose that $m = 2^{n-1} + r$, where r belongs to $\{0, 1, 2, \ldots, 2^{n-1} - 1\}$ and we prove the existence of an m-full set with n elements by induction on r. This is proved for $r = 0$. Suppose that this is true for $r - 1$; namely there is a $(2^{n-1} + r - 1)$-full set $A = \{a_1, a_2, \ldots, a_n\}$. If $m = 2^{n-1} + r$ is a power of 2 then there is nothing to prove. We may thus assume that there is an i for which $a_i < a_1 + \cdots + a_{i-1} + 1$. Now if there is the least j with $a_{i+j} \neq a_i + j$ then the set $\{a_1, \ldots, a_{i+j-2}, a_{i+j-1} + 1, a_{i+j}, \ldots, a_n\}$ is an m-full set with n elements. Otherwise, the set $\{a_1, \ldots, a_{n-1}, a_n + 1\}$ is an m-full set with n elements.

To prove the other assertion, let $n' = \max\{\ell : \frac{\ell(\ell+1)}{2} \leq m\}$. We claim that if $A = \{a_1, \ldots, a_k\}$ is an arbitrary m-full set, then $k \leq n'$. On the contrary, suppose that $k > n'$. Then $\frac{k(k+1)}{2} > m$. Hence $m = a_1 + a_2 + \cdots + a_k \geq 1 + 2 + \cdots + k = \frac{k(k+1)}{2} > m$ which is a contradiction. Thus $k \leq n'$ and, since A was arbitrary, we therefore have $\beta(m) \leq n'$.

On the other hand, we show that the maximum is attained. Let $m = \frac{n'(n'+1)}{2} + r$, where r belongs to $\{0, 1, \ldots, n'\}$. Then $A = \{1, 2, 3, \ldots, n' - 1, n' + r\}$ is an m-full set with n' elements. $\square$

Remark 6. The direct problem for subset sums is to find lower bounds for $|\sum A|$ in terms of $|A|$. The inverse problem for subset sums is to determine the structure of the extremal sets A of integers for which $|\sum A|$ is minimal. M. B. Nathanson gives a complete solution for the direct and the inverse problem for subset sums in [1] and proves that if A is a set of positive integers with $|A| \geq 2$ then $|\sum A| \geq \binom{|A|+1}{2}$. This immediately implies the last part of Proposition 5.

Theorem 7. *Let $m \notin \{2,4,5,8,9,14\}$ be a positive integer and denote* $\min\{\max A : \sum A = [m]\}$ *by $L(m)$. If $m = \frac{n(n+1)}{2} + r$, where $r = 0, 1, \ldots, n$ then*

$$L(m) = \begin{cases} n, & \text{if } r = 0; \\ n+1, & \text{if } 1 \leq r \leq n-2; \\ n+2, & \text{if } 1 \leq r = n-1 \text{ or } n. \end{cases}$$

Proof. Let $m = \frac{n(n+1)}{2} + r$, where $r = 0, 1, \ldots, n$. Then there are 3 cases:

Case 1: $r = 0$. Let $A = \{1, \ldots, n\}$. Since $\sum A = [\frac{n(n+1)}{2}]$ and $\max A$ has the minimum possible value, $L(m) = n$.

Case 2: $r = 1, \ldots, n-2$. We can consider the set

$$A = \{1, \ldots, n, n+1\} \setminus \{n+1-r\}$$

which is m-full, since $n+1-r \geq 3$ and $1 + \cdots + n + (n+1) - (n+1-r) = \frac{n(n+1)}{2} + r = m$. Hence $L(m) = n+1$.

Case 3: $r = n-1$ or n. In this case we cannot omit $n+1-r$ from the set $\{1, \ldots, n, n+1\}$, because $n+1-r$ is 1 or 2 and the resulting set is not full. Thus we have to add $n+2$ and omit some other element. In fact for $r = n-1$ the suitable set with minimum possible value for its maximum is $\{1, \ldots, n, n+2\} \setminus \{3\}$, and for $r = n$ the desired set is $\{1, \ldots, n-1, n+1, n+2\} \setminus \{3\}$ (note that in this case we have $n \neq 4$ since $m \neq 14$ and we can therefore deduce that the latter set is full). We can therefore deduce that $L(m) = n+2$. $\square$

For the remaining case $m = 14$, it can be easily verified that $L(14) = 7$. The first few values of $L(m)$, Sloane's OEIS [2, A188429], are

m	1	3	6	7	10	11	12	13	14	15	16	17	18	19	20
$L(m)$	1	2	3	4	4	5	5	6	7	5	6	6	6	7	7

Theorem 8. *Let $m \geq 20$ be a positive integer and denote* $\max\{\max A : \sum A = [m]\}$ *by $U(m)$. Then*

$$U(m) = \lceil \frac{m}{2} \rceil.$$

Proof. Let $A = \{a_1, \ldots, a_k\}$ with $a_1 < \cdots < a_k$ be m-full. By Theorem 2, $a_k \leq a_1 + \cdots + a_{k-1}$, which is equivalent to $a_k \leq m - a_k$ since A is m-full. Therefore $a_k \leq \lceil \frac{m}{2} \rceil$ holds.

Now put $b = \lceil \frac{m}{2} \rceil$. Since $m - \lceil \frac{m}{2} \rceil \geq 10$, by Theorem 4 we can find a B' with $\sum B' = [m - \lceil \frac{m}{2} \rceil]$. Whence $B = B' \cup \{b\}$ satisfies the properties $\sum B = [m]$ and $\max B = \lceil \frac{m}{2} \rceil$. $\square$

For the remaining cases, it can be easily verified that the values of $U(m)$, Sloane's OEIS [2, A188430], are

m	1	3	6	7	10	11	12	13	14	15	16	17	18	19
$U(m)$	1	2	3	4	4	5	6	7	7	8	6	7	8	9

We can now determine the value of $F(m)$ for each positive integer m. We assume that $L(m) = U(m) = 0$ whenever $m \in \{2, 4, 5, 8, 9\}$.

Lemma 9. *Let m be a positive integer and $F(m, i)$ denote the number of m-full sets A with $\max A = i$, where $L(m) \le i \le U(m)$. Then*

$$F(m, i) = \sum_{j=L(m-i)}^{\min\{U(m-i), i-1\}} F(m - i, j).$$

Proof. Let $A = \{a_1, \ldots, a_k\}$ with $a_1 < \cdots < a_k = i$ be an m-full set. Then $A' = A \setminus \{a_k\}$ is an $(m - i)$-full set such that $j = \max A' < i$. Thus $L(m - i) \le j \le \min\{U(m - i), i - 1\}$ and the result follows. $\square$

Theorem 10. *Let m be a positive integer and denote the number of m-full sets by $F(m)$. Then $F(m) = \sum_{i=L(m)}^{U(m)} F(m, i)$.*

Proof. Let $A = \{a_1, \ldots, a_k\}$ with $a_1 < \cdots < a_k$ be an m-full set. Then $L(m) \le a_k \le U(m)$ and the result is now obvious. $\square$

The first few values of $F(m)$, Sloane's OEIS [2, A188431], are

m	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$F(m)$	1	0	1	0	0	1	1	0	0	1	1	2	2	1	2	1	2	3	4	5

Example 11. We evaluate $F(21)$. Using Theorem 7 and Theorem 8, $L(21) = 6$ and $U(21) = 12$. Thus

$$F(21) = F(21, 6) + F(21, 7) + F(21, 8) + F(21, 9) + F(21, 10) + F(21, 11) + F(21, 12).$$

We need to compute $L(m)$ and $U(m)$ for $m = 15, 14, 13, 12, 11, 10, 9$. We have

m	9	10	11	12	13	14	15
$L(m)$	0	4	5	5	6	7	5
$U(m)$	0	4	5	6	7	7	8

Noting the facts that $L(6) = U(6) = 3$ and $L(7) = U(7) = 4$ we therefore have

$$\begin{aligned} F(21) &= F(15, 5) + F(13, 6) + F(13, 7) + F(12, 5) + F(12, 6) + F(11, 5) + F(10, 4) \\ &= F(10, 4) + F(7, 4) + F(6, 3) + F(7, 4) + F(6, 3) + F(6, 3) + F(6, 3) \\ &= F(6, 3) + F(7, 4) + F(6, 3) + F(7, 4) + F(6, 3) + F(6, 3) + F(6, 3) \\ &= 7. \end{aligned}$$

The seven 21-full sets are

$$\{1, 2, 3, 4, 5, 6\}, \{1, 2, 4, 6, 8\}, \{1, 2, 3, 7, 8\}, \{1, 2, 4, 5, 9\},$$
$$\{1, 2, 3, 6, 9\}, \{1, 2, 3, 5, 10\}, \{1, 2, 3, 4, 11\}.$$

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