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Sums Involving Moments of Reciprocals of Binomial Coefficients

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Abstract

We investigate sums of the form $\sum_{0\leq k\leq n} k^m \binom{n}{k}^{-1}$. We establish a recurrence relation and compute its ordinary generating function. As application we give the asymptotic expansion. The results extend the earlier works by various authors. In the last section, we establish that $\sum_{0\leq k\leq n} \frac{k^m}{n^m}\binom{n}{k}^{-1}$ tends to 1 as $n\to\infty$ and that $\sum_{0\leq k\leq n-m} k^m \binom{n}{k}^{-1}$ tends to $m!$ as $n\to\infty$.

1 Introduction

For all nonnegative integers n, m, let

$$S_n^{(m)} := \sum_{k=0}^{n} k^m \binom{n}{k}^{-1}. \tag{1}$$

There are several papers in the literature dealing with sums involving inverses of binomial coefficients [2, 9, 11, 12, 15, 16, 17, 19]. The cases $m = 0$ and $m = 1$ were intensively studied.

In 1947, Staver [13], using the identity

$$\sum_{k=0}^{n} k \binom{n}{k}^{-1} = \sum_{k=0}^{n} (n-k) \binom{n}{k}^{-1},$$

obtained a relation between $S_n^{(1)}$ and $S_n^{(0)}$:

$$S_n^{(1)} = \frac{n}{2} S_n^{(0)}, \tag{2}$$

and established a recurrence relation for $S_n^{(0)}$

$$S_{n+1}^{(0)} = \frac{n+2}{2(n+1)} S_n^{(0)} + 1. \tag{3}$$

Staver also proved for the first time the well-known formula

$$S_n^{(0)} = \frac{n+1}{2^{n+1}} \sum_{k=1}^{n+1} \frac{2^k}{k}. \tag{4}$$

For applications of (4), see Nedemeyer and Smorodinsky [5], Mansour and West [7], and for a probabilistic application, see Letac [4, p. 14]. Finally, for the asymptotic expansion, see Comtet [1, p. 294] and Yang and Zhao [18].

In 1981, using induction and the relation

$$\binom{n}{k}^{-1} = \binom{n-1}{k-1}^{-1} - \frac{n-k}{n-k+1} \binom{n}{k-1}^{-1}$$

Rockett [9] proved (3) and (4).

In 1993, Sury [14] connected the inverse binomial coefficients to the beta function as follows

$$\binom{n}{k}^{-1} = (n+1) \int_0^1 x^k (1-x)^{n-k} \, dx$$

and proved the relation (4) (see also [16]). Some years later, Mansour [6], generalized the idea of Sury [14], and gave an approach based on calculus to obtain the generating function for related combinatorial identities.

Theorem 1 (Mansour [6]). *Let $r, n \geq k$ be any nonnegative integer numbers, and let $f(n,k)$ be given by*

$$f(n,k) = \frac{(n+r)!}{n!} \int_{u_1}^{u_2} p^k(t)\, q^{n-k}(t)\, dt,$$

where $p(t)$ and $q(t)$ are two functions defined on $[u_1, u_2]$. Let $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ be any two sequences, and let $A(x), B(x)$ be the corresponding ordinary generating functions. Then

$$\sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} f(n,k)\, a_k b_{n-k} \right] x^n = \frac{d^r}{dx^r} \left[x^r \int_{u_1}^{u_2} A(xp(t))\, B(xq(t))\, dt \right]. \tag{5}$$

In particular,

$$\sum_{n\geq 0} S_n^{(0)} x^n = \frac{2}{(x-1)(x-2)} - \frac{2\ln(1-x)}{(x-2)^2},$$

and

$$\sum_{n\geq 0} S_n^{(1)} x^n = -\frac{x(3x-4)}{(x-1)^2(x-2)^2} + \frac{2x\ln(1-x)}{(x-2)^3}.$$

We shall use the following well-known basic tools [3]:

(i) The Stirling numbers of the second kind $\left\{ {n \atop k} \right\}$ (A008277), can be defined by the generating function

$$\prod_{j=1}^{k} \frac{x}{1-jx} = \sum_{n\geq k} \left\{ {n \atop k} \right\} x^n.$$

The most basic recurrence relation is

$$\left\{ {n+1 \atop k} \right\} = \left\{ {n \atop k-1} \right\} + k \left\{ {n \atop k} \right\},$$

with $\left\{ {n \atop 1} \right\} = \left\{ {n \atop n} \right\} = 1$. An important relation involving $\left\{ {n \atop k} \right\}$ is

$$x^n = \sum_{k=0}^{n} (-1)^{n+k} \left\{ {n \atop k} \right\} x(x+1)\cdots(x+k-1). \tag{6}$$

(ii) The Eulerian numbers $\left\langle {n \atop k} \right\rangle$ (A008292) are defined by

$$\left\langle {n \atop k} \right\rangle = \sum_{i=0}^{k} (-1)^i (k-i)^n \binom{n+1}{i}, \quad n \geq k \geq 1.$$

Which also satisfy the recursive relation

$$\left\langle {n \atop k} \right\rangle = k \left\langle {n-1 \atop k} \right\rangle + (n-k+1) \left\langle {n-1 \atop k-1} \right\rangle,$$

with $\left\langle {1 \atop 1} \right\rangle = 1$.

(iii) The Worpitzky numbers $W_{n,k}$ (A028246), are defined by

$$W_{n,k} = \sum_{i=0}^{k} (-1)^{i+k} (i+1)^n \binom{k}{i}.$$

They can also be expressed through the Stirling numbers of the second kind as follows

$$W_{n,k} = k! \left\{ {n+1 \atop k+1} \right\}. \tag{7}$$

The Worpitzky numbers satisfy the recursive relation

$$W_{n,k} = (k+1)\, W_{n-1,k} + kW_{n-1,k-1} \quad (n \geq 1, k \geq 1). \tag{8}$$

Some simple properties related to these three remarkable sequences are

$$\sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle x^k = \sum_{k=0}^{n} (x-1)^{n-k}\, kW_{n-1,k-1}, \tag{9}$$

$$\sum_{k=0}^{n} \binom{n}{k} \left\{ {k \atop t} \right\} = \left\{ {n+1 \atop t+1} \right\}, \tag{10}$$

and

$$\sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle \binom{k+1}{t} = W_{n,n-t}. \tag{11}$$

2 Some results on $S_n^{(2)}$

In this section, we give some results concerned $S_n^{(2)}$. Applying Theorem 1, for $a_n = n^2$ and $b_n = 1$, we obtain, for $|x| < 1$,

$$A(x) = \frac{x(x+1)}{(1-x)^3},$$

and

$$B(x) = \frac{1}{1-x},$$

from (5) we get the generating function for $S_n^{(2)}$

$$\begin{aligned} \sum_{n \geq 0} S_n^{(2)} x^n &= \frac{d}{dx} \left[x \int_0^1 \frac{xt\,(xt+1)}{(1-xt)^3\,(1-x+xt)} dt \right] \\ &= \frac{2x\,(2x^3 - 3x^2 - 3x + 5)}{(x-2)^3 (x-1)^3} - 2\left(x^2 + 2x - 2\right) \frac{\ln(1-x)}{(x-2)^4}. \end{aligned} \tag{12}$$

In addition, we have a relation between $S_n^{(2)}$ and $S_n^{(0)}$ given by the following theorem.

Theorem 2. *If n is a nonnegative integer, then $S_n^{(2)}$ satisfies the recursion relation*

$$S_{n+1}^{(2)} = \frac{(n-1)(n+2)^2}{2(n-2)(n+1)^2} S_n^{(2)} + \frac{(n+2)(n^2-2n-2)}{2(n-2)}, \tag{13}$$

and

$$S_n^{(2)} = \frac{1}{4}(n+1)(n-2) S_n^{(0)} + \frac{1}{2}(n+1)^2. \tag{14}$$

Proof. We use the WZ method [8]. Denote the summand in $S_n^{(2)}$ by $L(n,k) := k^2 \binom{n}{k}^{-1}$, by the Zeilberger's Maple package EKHAD, we construct the function

$$G(n,k) = \left(n^2k^2 + n + 3nk - nk^2 + 2 + 2k - 2k^2\right)(n+1-k)\binom{n}{k}^{-1}$$

such that

$$(n-1)(n+2)^2 L(n,k) - 2(n-2)(n+1)^2 L(n+1,k) = G(n,k+1) - G(n,k).$$

By summing the above telescoping equation over k from 0 to $n-1$, we obtain the following recurrence relation

$$(n-1)(n+2)^2 \sum_{k=0}^{n-1} L(n,k) - 2(n-2)(n+1)^2 \sum_{k=0}^{n-1} L(n+1,k) = G(n,n) - G(n,0),$$

that we can rewrite

$$(n-1)(n+2)^2 S_n^{(2)} - (n-1)(n+2)^2 n^2 - 2(n-2)(n+1)^2 S_{n+1}^{(2)} \\ + 2(n-2)(n+1)^2 \left(\frac{n^2}{n+1} + (n+1)^2\right) = n^3(n-1),$$

as desired. We prove the relation (14) by induction on n, the result clearly holds for $n=0$, we now show that the formula for $n+1$ follows from (13) and induction hypothesis

$$S_{n+1}^{(2)} = \frac{(n-1)(n+2)^2}{2(n-2)(n+1)^2}\left(\frac{1}{4}(n+1)(n-2) S_n^{(0)} + \frac{1}{2}(n+1)^2\right) \\ + \frac{(n+2)(n^2-2n-2)}{2(n-2)}.$$

Applying (3), we obtain

$$S_{n+1}^{(2)} = \frac{1}{4}(n+2)(n-1) S_{n+1}^{(0)} + \frac{1}{2}(n+2)^2.$$

This completes the proof. □

Using the relation $\sum_{k=0}^{n} k^3 \binom{n}{k}^{-1} = \sum_{k=0}^{n} (n-k)^3 \binom{n}{k}^{-1}$ and relations (2) and (14), we easily obtain the following identity between $S_n^{(3)}$ and $S_n^{(0)}$.

Corollary 3. *For any nonnegative integers n, we have*

$$S_n^{(3)} = \frac{1}{8} n \left(n^2 - 3n - 6\right) S_n^{(0)} + \frac{3}{4} n (n+1)^2.$$

3 Generalization

Before stating the main result of this section, we need a lemma.

Lemma 4.

$$S_{n+j}^{(0)} = (n+j+1)\left(\frac{1}{2^j\,(n+1)}S_n^{(0)} + \sum_{r=0}^{j-1}\frac{1}{2^r\,(n+j-r+1)}\right), \tag{15}$$

Proof. We proceed by induction on j. For $j=0$ the identity (15) holds. Now suppose that (15) holds for some j and replace n by $n+1$ in (15), then using relation (3) the result follows. ◻

Theorem 5. *For any nonnegative integers m and n, we have*

$$S_n^{(m)} = \sum_{j=0}^{m}(-1)^{m+j}\binom{n+j}{n}W_{m,j}\left(S_{n+j}^{(0)} - \frac{1}{(n+j)!}\sum_{r=0}^{j-1}r!\,(n+j-r)!\right), \tag{16}$$

with the usual convention that the empty sum is 0.

Proof. We can write $S_n^{(m)}$ as follows

$$\begin{aligned}
S_n^{(m)} &= \sum_{k=0}^{n}((k+1)-1)^m\binom{n}{k}^{-1}, \\
&= \sum_{k=0}^{n}\binom{n}{k}^{-1}\sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i}(k+1)^i,
\end{aligned}$$

and with (6), we obtain

$$\begin{aligned}
S_n^{(m)} &= \sum_{k=0}^{n}\binom{n}{k}^{-1}\sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i}\sum_{j=0}^{i}(-1)^{i+j}\left\{{i \atop j}\right\}(k+1)\cdots(k+j), \\
&= \sum_{k=0}^{n}\sum_{i=0}^{m}\sum_{j=0}^{i}\frac{1}{n!}(-1)^{m+j}\binom{m}{i}\left\{{i \atop j}\right\}k!\,(k+1)\cdots(k+j)\,(n-k)!.
\end{aligned}$$

After some rearrangement,

$$\begin{aligned}
S_n^{(m)} &= \sum_{k=0}^{n}\sum_{i=0}^{m}\sum_{j=0}^{i}(-1)^{m+j}\binom{m}{i}\left\{{i \atop j}\right\}(n+1)\cdots(n+j)\binom{n+j}{k+j}^{-1}, \\
&= \sum_{i=0}^{m}\sum_{j=0}^{i}(-1)^{m+j}\binom{m}{i}\left\{{i \atop j}\right\}j!\frac{(n+j)!}{n!j!}\sum_{r=j}^{n+j}\binom{n+j}{r}^{-1}.
\end{aligned}$$

Now, from (10) and (11), the result holds. ◻

Setting $m = 4$ in (16), we have the following

Corollary 6. *If n is a nonnegative integer, then*

$$S_n^{(4)} = \frac{1}{16}(n+1)\left(n^3 - 7n^2 - 2n + 16\right) S_n^{(0)} + \frac{1}{8}(7n-8)(n+1)^3 .$$

Theorem 7. *For any nonnegative integers m and n*

$$S_{n+1}^{(m)} = \delta_{0m} + \frac{1}{n+1}\sum_{j=0}^{m+1}\binom{m+1}{j} S_n^{(j)}, \tag{17}$$

where δ_{ij} is the Kronecker symbol.

Proof. Recall that $\binom{n+1}{k} = \frac{n+1}{k}\binom{n}{k-1}$, we have

$$\begin{aligned}
\sum_{k=0}^{n+1}\frac{k^m}{\binom{n+1}{k}} &= \delta_{0m} + \sum_{k=1}^{n+1} k^m \binom{n+1}{k}^{-1} \\
&= \delta_{0m} + \frac{1}{n+1}\sum_{k=1}^{n+1} k^{m+1}\binom{n}{k-1}^{-1} \\
&= \delta_{0m} + \frac{1}{n+1}\sum_{k=0}^{n}(k+1)^{m+1}\binom{n}{k}^{-1} \\
&= \delta_{0m} + \frac{1}{n+1}\sum_{j=0}^{m+1}\binom{m+1}{j}\sum_{k=0}^{n} k^j \binom{n}{k}^{-1} .
\end{aligned}$$

This proves the theorem. ∎

Setting $m = 1$ in (17) and using (14), we have the following

Corollary 8. *If n is a nonnegative integer, then*

$$S_{n+1}^{(1)} = \frac{n+2}{2n} S_n^{(1)} + \frac{1}{2}(n+1) .$$

4 Ordinary generating function

We apply Theorem 1, for $a_n = n^m$ $(m \geq 1)$, $b_n = 1$, and for $|x| < 1$ we have

$$\begin{aligned}
A(x) &= \frac{1}{(1-x)^{m+1}}\sum_{k=0}^{m}\left\langle {m \atop k} \right\rangle x^{k+1} = \sum_{k=0}^{m}\frac{(-1)^{m+k}}{(1-x)^{k+1}} W_{m,k}, \\
B(x) &= \sum_{n\geq 0} x^n = \frac{1}{1-x}.
\end{aligned}$$

From (5), we get

$$\sum_{n\geq 0} S_n^{(m)} x^n = \frac{d}{dx}\left[x\int_0^1 \frac{\sum_{k=0}^{m}\left\langle {m \atop k}\right\rangle (xt)^{k+1}}{(1-xt)^{m+1}(1-x+xt)}dt\right]. \tag{18}$$

Making the substitution $xt = y$ in the right-hand side of (18), we obtain

$$\sum_{n\geq 0} S_n^{(m)} x^n = \frac{d}{dx}\left[\int_0^x \frac{\sum_{k=0}^{m}\left\langle {m \atop k}\right\rangle y^{k+1}}{(1-y)^{m+1}(1-x+y)}dy\right].$$

Since the degree of the denominator is at least one higher than that of the numerator, this fraction decomposes into partial fractions of the form

$$\frac{\sum_{k=0}^{m}\left\langle {m \atop k}\right\rangle y^{k+1}}{(1-y)^{m+1}(1-x+y)} = \frac{\alpha^{(m)}(x)}{1-x+y} + \sum_{s=0}^{m}\frac{\alpha_s^{(m)}(x)}{(1-y)^{m-s+1}}. \tag{19}$$

We note in passing that (19) is equivalent to

$$\begin{aligned}\sum_{k=0}^{m}\left\langle {m \atop k}\right\rangle y^{k+1} &= (1-y)^{m+1}\alpha^{(m)}(x) + (1-x+y)\sum_{s=0}^{m}(1-y)^s\alpha_s^{(m)}(x) \qquad (20)\\ &= \sum_{k=0}^{m}(-1)^{m+k} y(1-y)^{m-k} W_{m-1,k-1}.\end{aligned}$$

For $y = 1$ and using the fact that $W_{p,p} = p!$ for $p \geq 0$, we immediately obtain the well-known identity

$$\sum_{k=0}^{m}\left\langle {m \atop k}\right\rangle = m!.$$

Next, if we set $y = 0$ for $|x| < 1$, we obtain the following relation between $\alpha^{(m)}(x)$ and $\alpha_s^{(m)}(x)$

$$\sum_{s=0}^{m}\alpha_s^{(m)}(x) = \frac{\alpha^{(m)}(x)}{x-1}. \tag{21}$$

Proposition 9. *For $m \geq 1$, we have*

$$\begin{aligned}\alpha_s^{(m)}(x) &= \sum_{k=0}^{m}\sum_{i=0}^{s}\frac{(-1)^{i+s}}{(2-x)^{i+1}}\left\langle {m \atop k}\right\rangle\binom{k+1}{s-i} \qquad (22)\\ &= \sum_{j=m-s}^{m}\frac{(-1)^{m+j}}{(2-x)^{s-m+1+j}}W_{m,j}.\end{aligned}$$

and

$$\begin{aligned}\alpha^{(m)}(x) &= \frac{1}{(2-x)^{m+1}}\sum_{k=0}^{m}\left\langle {m \atop k} \right\rangle (x-1)^{k+1} \\ &= \sum_{j=0}^{m}\frac{(-1)^{m+j}}{(2-x)^{j+1}}W_{m,j}, \\ &= \alpha_m^{(m)}(x).\end{aligned} \tag{23}$$

Proof. We verify that (22) and (23) satisfy (20). Denote the right-hand side of (20) by $R^{(m)}(y)$

$$R^{(m)}(y) = (1-y)^{m+1}\alpha^{(m)}(x) + (1-x+y)\sum_{s=0}^{m}(1-y)^s\alpha_s^{(m)}(x).$$

After some rearrangement, we get

$$\begin{aligned}R^{(m)}(y) &= \frac{(1-y)^{m+1}}{(2-x)^{m+1}}\sum_{k=0}^{m}\left\langle {m \atop k} \right\rangle (x-1)^{k+1} + \\ &\qquad (1-x+y)\sum_{s=0}^{m}(1-y)^s\sum_{k=0}^{m}\sum_{i=0}^{s}\frac{(-1)^{i+s}}{(2-x)^{i+1}}\left\langle {m \atop k} \right\rangle\binom{k+1}{s-i}\end{aligned}$$

$$\begin{aligned}&= \sum_{k=0}^{m}\left\langle {m \atop k} \right\rangle\Bigg[\frac{(1-y)^{m+1}}{(2-x)^{m+1}}(x-1)^{k+1} + \\ &\qquad \frac{1-x+y}{2-x}\sum_{s=0}^{m}\frac{(1-y)^s}{(2-x)^s}\sum_{j=0}^{s}\binom{k+1}{j}(-1)^j(2-x)^j\Bigg],\end{aligned}$$

using binomial formula, we obtain

$$\begin{aligned}R^{(m)}(y) = \sum_{k=0}^{m}\left\langle {m \atop k} \right\rangle\Bigg[&\frac{(1-y)^{m+1}}{(2-x)^{m+1}}\sum_{j=0}^{k+1}\binom{k+1}{j}(2-x)^j(-1)^j + \\ &\sum_{s=0}^{m}\frac{(1-y)^s}{(2-x)^s}\sum_{j=0}^{s}\binom{k+1}{j}(-1)^j(2-x)^j - \\ &\sum_{s=0}^{m}\frac{(1-y)^{s+1}}{(2-x)^{s+1}}\sum_{j=0}^{s}\binom{k+1}{j}(-1)^j(2-x)^j\Bigg].\end{aligned}$$

Now, for $k \le m$

$$R^{(m)}(y) = \sum_{k=0}^{m} \left\langle {m \atop k} \right\rangle \Bigg[\sum_{s=m+1}^{m+1} \frac{(1-y)^s}{(2-x)^s} \sum_{j=0}^{m+1} \binom{k+1}{j} (2-x)^j (-1)^j + \sum_{s=0}^{m} \frac{(1-y)^s}{(2-x)^s} \sum_{j=0}^{s} \binom{k+1}{j} (-1)^j (2-x)^j - \sum_{s=0}^{m} \frac{(1-y)^{s+1}}{(2-x)^{s+1}} \sum_{j=0}^{s} \binom{k+1}{j} (-1)^j (2-x)^j \Bigg]$$

$$= \sum_{k=0}^{m} \left\langle {m \atop k} \right\rangle \Bigg[\sum_{s=0}^{m+1} \frac{(1-y)^s}{(2-x)^s} \sum_{j=0}^{s} \binom{k+1}{j} (-1)^j (2-x)^j - \sum_{s=1}^{m+1} \frac{(1-y)^s}{(2-x)^s} \sum_{j=0}^{s-1} \binom{k+1}{j} (-1)^j (2-x)^j \Bigg]$$

$$= \sum_{k=0}^{m} \left\langle {m \atop k} \right\rangle \Bigg[1 + \sum_{s=1}^{m+1} \frac{(1-y)^s}{(2-x)^s} \Bigg(\sum_{j=0}^{s} \binom{k+1}{j} (-1)^j (2-x)^j - \sum_{j=0}^{s-1} \binom{k+1}{j} (-1)^j (2-x)^j \Bigg) \Bigg].$$

Finally,

$$R^{(m)}(y) = \sum_{k=0}^{m} \left\langle {m \atop k} \right\rangle \left(1 + \sum_{s=1}^{k+1} \binom{k+1}{s} (y-1)^s \right) = \sum_{k=0}^{m} \left\langle {m \atop k} \right\rangle y^{k+1}.$$

Now, according to (11) and (7), we have

$$\begin{aligned}
\alpha_s^{(m)}(x) &= \sum_{i=0}^{s} \frac{(-1)^{i+s}}{(2-x)^{i+1}} \sum_{k=0}^{m} \left\langle {m \atop k} \right\rangle \binom{k+1}{s-i} \\
&= \sum_{i=0}^{s} \frac{(-1)^{i+s} (m-s+i)!}{(2-x)^{i+1}} \left\{ {m+1 \atop m-s+i+1} \right\} \\
&= \sum_{i=0}^{s} \frac{(-1)^{i+s}}{(2-x)^{i+1}} W_{m,m-s+i} \\
&= \sum_{j=m-s}^{m} \frac{(-1)^{m+j}}{(2-x)^{s-m+1+j}} W_{m,j},
\end{aligned}$$

on the other side, it follows from (9) that

$$\begin{aligned}\alpha^{(m)}(x) &= \sum_{j=0}^{m-1} (-1)^{m+j+1} \frac{(j+1)(x-1)}{(2-x)^{j+2}} W_{m-1,j} \\ &= \sum_{j=0}^{m-1} \frac{(-1)^{m+j}}{(2-x)^{j+1}} (j+1) W_{m-1,j} - \sum_{j=0}^{m-1} \frac{(-1)^{m+j}}{(2-x)^{j+2}} (j+1) W_{m-1,j} \\ &= \sum_{j=0}^{m} \frac{(-1)^{m+j}}{(2-x)^{j+1}} (j+1) W_{m-1,j} + \sum_{j=0}^{m} \frac{(-1)^{m+j}}{(2-x)^{j+1}} j W_{m-1,j-1}.\end{aligned}$$

Using (8), we get $\alpha^{(m)}(x)$ as desired. This completes the proof. $\square$

Now, Integrating the right-hand side of (19) over y, we obtain

$$\int_0^x \frac{\sum_{k=0}^{m} \left\langle {m \atop k} \right\rangle y^{k+1}}{(1-y)^{m+1}(1-x+y)} dy = \\ -2\alpha^{(m)}(x)\ln(1-x) + \sum_{s=0}^{m-1} \frac{\alpha_s^{(m)}(x)}{s-m}\left(1-(1-x)^{s-m}\right). \quad (24)$$

By differentiating (24) we get the ordinary generating function of $S_n^{(m)}$

$$\sum_{n\geq 0} S_n^{(m)} x^n = -2\frac{d}{dx}\alpha^{(m)}(x)\ln(1-x) + \frac{\alpha^{(m)}(x)}{1-x} + \\ \sum_{s=0}^{m-1} \frac{\frac{d}{dx}\alpha_s^{(m)}(x)}{s-m}\left(1-(1-x)^{s-m}\right) + \sum_{s=0}^{m} \alpha_s^{(m)}(x)(1-x)^{s-m-1} \quad (25)$$

with

$$\frac{d}{dx}\alpha_s^{(m)}(x) = \sum_{j=m-s}^{m} \frac{(s-m+j+1)(-1)^{m+j}}{(2-x)^{s-m+j+2}} W_{m,j}.$$

With Proposition 9, we can now rewrite (25) as follows

Theorem 10. *For any real number x such that $|x| < 1$ and for all positive integers $m \geq 1$, we have*

$$\sum_{n\geq 0} S_n^{(m)} x^n = \left(\sum_{j=0}^{m} \frac{2(j+1)(-1)^{m+j+1}}{(2-x)^{j+2}} W_{m,j}\right)\ln(1-x) \\ + \sum_{0\leq j\leq s\leq m-1} \frac{(-1)^j W_{m,m-j}}{(2-x)^{s-j+1}}\left(\frac{(s-j+1)\left(1-(1-x)^{s-m}\right)}{(s-m)(2-x)} + (1-x)^{s-m-1}\right) \\ + \frac{1}{1-x}\sum_{j=0}^{m} \frac{2(-1)^{j+m}}{(2-x)^{j+1}} W_{m,j}.$$

5 Asymptotic expansion

Yang and Zhao [18] proved recently the asymptotic expansions for $S_n^{(0)}$ and $S_n^{(1)}$ of the following type:

$$S_n^{(0)} \sim 2 + \frac{2}{n-1} - \frac{1}{2^{n-1}}, n \to \infty, \tag{26}$$

and

$$S_n^{(1)} \sim \frac{n}{2}\left(2 + \frac{2}{n-1} - \frac{1}{2^{n-1}}\right), n \to \infty.$$

However, there are finer asymptotics which we proceed to discuss now. We note that the following claim is consistent with Theorem 7.

$$S_n^{(m)} \sim n^m, \quad n \to \infty.$$

It is not difficult to observe that $T_n^{(m)} := \sum_{k=0}^{n-m} k^m \binom{n}{k}^{-1}$ also converges as $n \to \infty$ and, we prove alongside the above asymptotic formula that this limit is $m!$.

Theorem 11. *For $m > 0$, we have*

$$\lim_{n\to\infty} T_n^{(m)} = m!,$$

and

$$\lim_{n\to\infty} \frac{S_n^{(m)}}{n^m} = 1.$$

Proof. Let us denote the sum $S_n^{(0)}$ by S_n for simplicity just in this proof. One has the recurrence relation $S_n = \frac{n+1}{2n} S_{n-1} + 1$ (see [14], for instance). Using this, it can be shown that

$$S_n \to 2,$$

$$n(S_n - 2) \to 2,$$

$$n(n(S_n - 2) - 2) = n^2 S_n - 2n^2 - 2n \to 4,$$

and so on. More generally, there are constants $a_0, a_1, a_2, \ldots$ such that, recursively

$$u_n^{(k)} = n u_n^{(k-1)} - a_{k-1},$$

$$u_n^{(k)} \to 0 \textit{ as } n \to \infty,$$

where $u_n^{(0)} = S_n - 2$.
In other words, for all $k \geq 1$,

$$n^k S_n - (2n^k + a_0 n^{k-1} + \cdots + a_{k-2} n) \to a_k,$$

as $n \to \infty$, for some a_k.
In fact, the sequence $a_0, a_1, \ldots$ is $2, 4, 16, 88, 616, 5224, \ldots$ is described by the generating function

$$\sum_{n\geq 0} a_n \frac{x^n}{n!} = \frac{2}{(2-e^x)^2}.$$

The constants can also be defined as

$$a_k = \sum_{r=0}^{k} (r+2)\, W_{k,r} - (r+1)\, W_{k-1,r},$$

or recursively as

$$a_k = 4 + \sum_{r=1}^{k} \left(\binom{k+1}{r} + \binom{k}{r} \right) (-1)^r (2 - a_0 + a_1 - \cdots \pm a_{r-1}) + (-1)^{k+1} (2 - a_0 + \cdots \pm a_{k-1}).$$

This is seen from

$$\begin{aligned} 2n^{k+1} S_n &= n^k (2n S_n) \\ &= n^k ((n+1) S_{n-1} + 2n) \\ &= (n^{k+1} + n^k) S_{n-1} + 2n^{k+1} \\ &= \sum_{r=1}^{k} \left(\binom{k+1}{r} + \binom{k}{r} \right) (n-1)^r S_{n-1} + 2n^{k+1}. \end{aligned}$$

Write

$$X^m = c_0 + c_1 (X+1) + c_2 (X+1)(X+2) + \cdots + c_m (X+1)(X+2) \cdots (X+m), \tag{27}$$

where c_i's depend on m ($c_m = 1$). We have

$$\begin{aligned} T_n^{(m)} &= \sum_{k=0}^{n-m} (c_0 + c_1(k+1) + c_2(k+1)(k+2) + \cdots + c_m(k+1)(k+2)\cdots(k+m)) \binom{n}{k}^{-1} \\ &= c_0 \sum_{k=0}^{n-m} \binom{n}{k}^{-1} + c_1(n+1) \sum_{k=0}^{n-m} \binom{n+1}{k+1}^{-1} + \cdots + c_m(n+1)\cdots(n+m) \sum_{k=0}^{n-m} \binom{n+m}{k+m}^{-1} \\ &= c_0 \left(S_n - \sum_{k=n-m+1}^{n} \binom{n}{k}^{-1} \right) + c_1(n+1) \left(S_{n+1} - \sum_{k=n-m+2}^{n+1} \binom{n+1}{k}^{-1} - 1 \right) + \cdots \\ &\quad \cdots + c_m(n+1)\cdots(n+m) \left(S_{n+m} - \sum_{k=n+1}^{n+m} \binom{n+m}{k}^{-1} - \sum_{k=0}^{m-1} \binom{n+m}{k}^{-1} \right). \end{aligned}$$

A computation using the above result

$$n^k S_n - (2n^k + a_0 n^{k-1} + \cdots + a_{k-1} n + a_k) \to 0,$$

shows

$$\lim_{n\to\infty} T_n^{(m)} = 0!c_0 + 1!c_1 + 2!c_2 + \cdots + (m-1)!c_{m-1} + 2(m!)c_m.$$

This can be shown by induction on m. The proof is finished by observing that $c_m = 1$ and

$$0!c_0 + 1!c_1 + 2!c_2 + \cdots + m!c_m = 0,$$

which follows by looking at the constant term of (27). This finishes the proof of the first assertion. For the second one, we proceed similarly to the above discussion. We have

$$\begin{aligned} S_n^{(m)} &= \sum_{k=0}^{n} \left(c_0 + c_1(k+1) + c_2(k+1)(k+2) + \cdots + (k+1)(k+2)\cdots(k+m)\right) \binom{n}{k}^{-1} \\ &= c_0 S_n + c_1(n+1)\left(S_{n+1} - 2\right) + c_2(n+1)(n+2)\left(S_{n+2} - 2\sum_{k=0}^{1}\binom{n+2}{k}^{-1}\right) + \cdots \\ &\cdots + (n+1)\cdots(n+m)\left(S_{n+m} - 2\sum_{k=0}^{m-1}\binom{n+m}{k}^{-1}\right). \end{aligned}$$

Thus, looking at the largest degree factor, we obtain

$$S_n^{(m)} \sim n^m.$$

□

A corollary of the above proof is that the sum

$$S_n^{(0)} = 2 + \frac{2}{n} + \frac{4}{n^2} + \frac{16}{n^3} + \frac{88}{n^4} + \frac{616}{n^5} + \frac{5224}{n^6} + \cdots,$$

as $n \to \infty$.

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