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A Note on a Problem of Motzkin Regarding Density of Integral Sets with Missing Differences

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Abstract

For a given set M of positive integers, a problem of Motzkin asks to determine the maximal density $\mu(M)$ among sets of nonnegative integers in which no two elements differ by an element of M. The problem is completely settled when $|M| \leq 2$, and some partial results are known for several families of M when $|M| \geq 3$. In 1985 Rabinowitz & Proulx provided a lower bound for $\mu(\{a, b, a+b\})$ and conjectured that their bound was sharp. Liu & Zhu proved this conjecture in 2004. For each $n \geq 1$, we determine $\kappa(\{a, b, n(a+b)\})$, which is a lower bound for $\mu(\{a, b, n(a+b)\})$, and conjecture this to be the exact value of $\mu(\{a, b, n(a+b)\})$.

1 Introduction

For $x \in \mathbb{R}$ and a set S of nonnegative integers, let $S(x)$ denote the number of elements $n \in S$ such that $n \leq x$. The upper and lower densities of S, denoted by $\overline{\delta}(S)$ and $\underline{\delta}(S)$ respectively, are given by

$$\overline{\delta}(S) := \limsup_{x\to\infty} \frac{S(x)}{x}, \quad \underline{\delta}(S) := \liminf_{x\to\infty} \frac{S(x)}{x}.$$

If $\overline{\delta}(S) = \underline{\delta}(S)$, we denote the common value by $\delta(S)$, and say that S has density $\delta(S)$. Given a set of positive integers M, S is said to be an M-set if $a \in S$, $b \in S$ imply $a - b \notin M$. Motzkin [15] asked to determine $\mu(M)$ given by

$$\mu(M) := \sup_S \overline{\delta}(S)$$

where S varies over all M-sets. Cantor & Gordon [3] showed the existence of $\mu(M)$ for any M, determined $\mu(M)$ when $|M| \leq 2$, and gave the following lower bound for $\mu(M)$:

$$\mu(M) \geq \kappa(M) := \sup_{\gcd(c,m)=1} \frac{1}{m} \min_i |cm_i|_m, \tag{1}$$

where m_i are the elements of M, and $|x|_m$ denotes the absolute value of the absolutely least remainder of x modulo m. We use the following equivalent form for the lower bound for $\mu(M)$, due to Haralambis [11]:

$$\kappa(M) = \max_{\substack{m=m_i+m_j \\ 1\leq k\leq \frac{m}{2}}} \frac{1}{m} \min_{m_i\in M} |km_i|_m, \tag{2}$$

where m_i, m_j represent distinct elements of M. The following useful upper bound for $\mu(M)$ is due to Haralambis [11]:

$$\mu(M) \leq \alpha \tag{3}$$

provided there exists a positive integer k such that $S(k) \leq (k+1)\alpha$ for every M-set S with $0 \in S$ and for some $\alpha \in [0, 1]$. The problem of Motzkin has a rich and diverse history but little progress towards the general problem has been made so far. Exact results for $\mu(M)$ have been few, and computation of $\mu(M)$ has only been completely possible when $|M| \leq 2$. Cantor & Gordon [3] showed that

$$\mu(\{m\}) = \frac{1}{2}, \quad \mu(\{m_1, m_2\}) = \frac{\lfloor (m_1 + m_2)/2 \rfloor}{m_1 + m_2}.$$

There have, however, been a number of results that give the exact value or bounds for $\mu(M)$ in other cases; see [13] for an exhaustive bibliography.

Connections with coloring problems in graph theory have been found useful in solving the Motzkin problem. One such connection, introduced by Hale [10] and shown to be equivalent to the Motzkin problem by Griggs & Liu [9], is the *T-coloring problem*. Another connection with colorings of graphs involves the *fractional chromatic number* of *distance graphs*.

The lower bound for $\mu(M)$, denoted by $\kappa(M)$ in (1), is itself at the heart of a longstanding conjecture. The *Lonely Runner Conjecture* (LRC) stated independently by Wills [17] in the context of diophantine approximations and by Cusick [7] while studying view obstructions problems in n-dimensional geometry, was actually given this apt name by Bienia *et al* [1].

Chen [6] characterized 3-sets M for which $1/\kappa(M)$ is an integer and also obtained general bounds.

We consider the problem of determining $\mu(M)$ for the family $M = \{a, b, n(a+b)\}$, $n \geq 1$. By a result of Cantor & Gordon [3], we know that $\mu(kM) = \mu(M)$. Thus, it is no loss of generality to assume that $\gcd(a, b) = 1$, and that $a < b$. We determine the value of $\kappa(M)$, which is the lower bound for $\mu(M)$ and conjecturally equal to it. This extends a result of Liu & Zhu [[13], Theorem 5.1], wherein they determined $\mu(M)$ in the case $M = \{a, b, a+b\}$. Rabinowitz & Proulx [16] provided a lower bound for $\mu(M)$ in this case and conjectured that their bound was sharp. An extensive list of work related to the Motzkin problem may be found in [13].

2 Main Result

For the set $M = \{a, b, a + b\}$, Rabinowitz & Proulx [16] conjectured the exact value of $\mu(M)$ in 1985, and Liu & Zhu [13] proved this conjecture in 2004. We determine $\kappa(M)$ for $M = \{a, b, n(a + b)\}$ where n is a fixed positive integer. If the conjecture of Haralambis [11] is true, then $\kappa(M) = \mu(M)$ in this case, thus providing a generalization of the result of Liu & Zhu.

Theorem 1. *Let $M = \{a, b, n(a + b)\}$, where $a < b$, $\gcd(a, b) = 1$ and $n \geq 1$. Then*

$$\kappa(M) = \begin{cases} \frac{n(a+b-\lambda)}{2\left(a+n(a+b)\right)}, & \text{if } \lambda \equiv a + b \pmod 2; \\ \frac{n(a+b+\lambda-1)}{2\left(b+n(a+b)\right)}, & \text{if } \lambda \not\equiv a + b \pmod 2, \end{cases}$$

where $\lambda = 1 + \left\lfloor \frac{b-a}{2n+1} \right\rfloor$.

Proof. We use (2) to compute $\kappa(M)$. The choice $m = a+b$ trivially yields $\min\left\{|ax|_m, |bx|_m, |n(a+b)x|_m\right\} = 0$ for each x. There are two choices remaining for m, and we determine $\kappa(M)$ by comparing the two rational numbers corresponding to these cases. In each of the two cases we need to compute $\kappa(M)$ with $m = a + n(a + b)$ and $m = b + n(a + b)$, and compare the two. Note that the definition of λ implies

$$b - a < (2n + 1)\lambda \leq b - a + 2n + 1.$$

Case I: $\big(\lambda \equiv a + b \ (\mathbf{mod}\ 2)\big)$

Subcase (i): $\big(m = a + n(a + b)\big)$ Choose x such that

$$(a + b)x \equiv \tfrac{a+b-\lambda}{2} \pmod m.$$

Then

$$-ax \equiv n(a + b)x \equiv n\tfrac{a+b-\lambda}{2} \equiv \tfrac{m-(a+n\lambda)}{2} \pmod m,$$

and

$$bx = (a + b)x - ax \equiv \tfrac{m+\{b-(n+1)\lambda\}}{2} \pmod m.$$

Therefore
$$\min\left\{|ax|_m, |bx|_m, |n(a+b)x|_m\right\} = \tfrac{m-(a+n\lambda)}{2} \tag{4}$$
since $b-(n+1)\lambda < a+n\lambda$ if and only if $(2n+1)\lambda > b-a$.

Write $\ell := a+n\lambda$. We show that
$$\min\left\{|ay|_m, |by|_m, |n(a+b)y|_m\right\} \le \tfrac{m-(a+n\lambda)}{2} = \tfrac{m}{2} - \tfrac{\ell}{2}$$
for each y, $1 \le y \le \frac{m}{2}$. Let $\mathscr{I} := \left(\frac{m}{2}-\frac{\ell}{2}, \frac{m}{2}+\frac{\ell}{2}\right)$. We show that $-ay \bmod m \in \mathscr{I}$ and $by \bmod m \in \mathscr{I}$ is simultaneously impossible for $1 \le y \le \frac{m}{2}$. Suppose
$$(a+b)y \equiv \tfrac{a+b-\lambda}{2} + i \pmod m,$$
with $1 \le i \le m-1$. Then
$$\begin{aligned} -ay \equiv n(a+b)y &\equiv \tfrac{m}{2} - \tfrac{\ell}{2} + ni \pmod m, \\ by \equiv (a+b)y - ay &\equiv \tfrac{m}{2} - \tfrac{\ell}{2} + \tfrac{a+b-\lambda}{2} + (n+1)i \pmod m \end{aligned} \tag{5}$$
Thus $-ay \bmod m \in \mathscr{I}$ if and only if
$$km + \tfrac{m}{2} - \tfrac{\ell}{2} < \tfrac{m}{2} - \tfrac{\ell}{2} + ni < km + \tfrac{m}{2} + \tfrac{\ell}{2}$$
for some integer k, with $0 \le k \le n-1$. This is equivalent to
$$k\tfrac{m}{n} < i < k\tfrac{m}{n} + \tfrac{\ell}{n},$$
so that
$$k(a+b)+1 \le i \le k(a+b)+a+\lambda-1. \tag{6}$$
For $k(a+b)+1 \le i \le k(a+b)+a+\lambda-1$, we show that
$$km + \tfrac{m}{2} + \tfrac{\ell}{2} < \tfrac{m}{2} - \tfrac{\ell}{2} + \tfrac{a+b-\lambda}{2} + (n+1)\left\{k(a+b)+1\right\}$$
and
$$\tfrac{m}{2} - \tfrac{\ell}{2} + \tfrac{a+b-\lambda}{2} + (n+1)\left\{k(a+b)+a+\lambda-1\right\} < (k+1)m + \tfrac{m}{2} - \tfrac{\ell}{2}.$$
This will prove that
$$km + \tfrac{m}{2} + \tfrac{\ell}{2} < by < (k+1)m + \tfrac{m}{2} - \tfrac{\ell}{2},$$
so that $by \bmod m \notin \mathscr{I}$, as claimed. Each of the above two inequalities is easy to prove. Using the fact that $(n+1)(a+b) = m+b$, each inequality can be shown to hold if $(2n+1)\lambda < (b-a)+2(n+1)$, which is true by the definition of λ. This completes the subcase when $m = a+n(a+b)$.

Subcase (ii): $(m = b+n(a+b))$ The argument in this subcase is similar to the one in subcase (i). We omit the calculation and state only the significant parts. Choose x such that
$$(a+b)x \equiv \tfrac{a+b+\lambda}{2} \pmod m.$$

Then

$$-bx \equiv n(a+b)x \equiv \tfrac{m-(b-n\lambda)}{2} \pmod{m},$$

and

$$ax = (a+b)x - bx \equiv -\tfrac{m-\{a+(n+1)\lambda\}}{2} \pmod{m}.$$

Therefore

$$\min\left\{|ax|_m, |bx|_m, |n(a+b)x|_m\right\} = \tfrac{m-\{(a+(n+1)\lambda\}}{2} \tag{7}$$

since $b - n\lambda < a + (n+1)\lambda$ if and only if $(2n+1)\lambda > b - a$.

As in subcase (i), we may show that

$$\min\left\{|ay|_m, |by|_m, |n(a+b)y|_m\right\} \leq \tfrac{m-\{a+(n+1)\lambda\}}{2}$$

for each y, $1 \leq y \leq \frac{m}{2}$. The argument is similar and we omit the proof. This completes subcase (ii).

To compute $\kappa(M)$ in Case I, we need to compare the expressions in (4) and (7). If we let $m_1 = a + n(a+b)$ and $m_2 = b + n(a+b)$, then a lengthy but easy computation shows that

$$\tfrac{m_2-\{(a+(n+1)\lambda\}}{2m_2} = \tfrac{1}{2} - \tfrac{1}{2}\tfrac{a+(n+1)\lambda}{b+n(a+b)} < \tfrac{1}{2} - \tfrac{1}{2}\tfrac{a+n\lambda}{a+n(a+b)} = \tfrac{m_1-(a+n\lambda)}{2m_1}$$

if and only if $(2n+1)\lambda > b - a$. This completes the proof of Case I.

Case II: $\left(\lambda \not\equiv a + b \ \textbf{(mod 2)}\right)$

Subcase (i): $(m = a + n(a+b))$ Choose x such that

$$(a+b)x \equiv \tfrac{a+b-\lambda+1}{2} \pmod{m}.$$

Then

$$-ax \equiv n(a+b)x \equiv \tfrac{m-\{a+n(\lambda-1)\}}{2} \pmod{m},$$

and

$$bx = (a+b)x - ax \equiv \tfrac{m+\{b-(n+1)(\lambda-1)\}}{2} \pmod{m}.$$

Therefore

$$\min\left\{|ax|_m, |bx|_m, |n(a+b)x|_m\right\} = \tfrac{m-\{b-(n+1)(\lambda-1)\}}{2} \tag{8}$$

since $a + n(\lambda-1) \leq b - (n+1)(\lambda-1)$ if and only if $(2n+1)(\lambda-1) \leq b - a$.

As in subcase (i) of Case I, we may show that

$$\min\left\{|ay|_m, |by|_m, |n(a+b)y|_m\right\} \leq \tfrac{m-\{b-(n+1)(\lambda-1)\}}{2}$$

for each y, $1 \leq y \leq \frac{m}{2}$. This completes subcase (i).

Subcase (ii): $(m = b + n(a+b))$ Choose x such that

$$(a+b)x \equiv \tfrac{a+b+\lambda-1}{2} \pmod{m}.$$

Then

$$-bx \equiv n(a+b)x \equiv \tfrac{m-\{b-n(\lambda-1)\}}{2} \pmod{m},$$

and

$$ax = (a+b)x - bx \equiv \tfrac{m+\{a+(n+1)(\lambda-1)\}}{2} \pmod{m}.$$

Therefore

$$\min\left\{|ax|_m, |bx|_m, |n(a+b)x|_m\right\} = \tfrac{m-\{b-n(\lambda-1)\}}{2} \tag{9}$$

since $a+(n+1)(\lambda-1) \leq b-n(\lambda-1)$ if and only if $(2n+1)(\lambda-1) \leq b-a$.

As in subcase (i) of Case I, we may show that

$$\min\left\{|ay|_m, |by|_m, |n(a+b)y|_m\right\} \leq \tfrac{m-\{b-n(\lambda-1)\}}{2}$$

for each y, $1 \leq y \leq \frac{m}{2}$. This completes subcase (ii).

To compute $\kappa(M)$ in Case II, we need to compare the expressions in (8) and (9). If we let $m_1 = a+n(a+b)$ and $m_2 = b+n(a+b)$, then a lengthy but easy computation shows that

$$\tfrac{m_1-\{b-(n+1)(\lambda-1)\}}{2m_1} = \tfrac{1}{2} - \tfrac{1}{2}\tfrac{b-(n+1)(\lambda-1)}{a+n(a+b)} \leq \tfrac{1}{2} - \tfrac{1}{2}\tfrac{b-n(\lambda-1)}{b+n(a+b)} = \tfrac{m_2-\{b-n(\lambda-1)\}}{2m_2}$$

if and only if $(2n+1)(\lambda-1) \leq b-a$. This completes the proof of Case II, and of the theorem. □

Corollary 2. (Liu & Zhu [13])

Let $M = \{a, b, a+b\}$, *where* $a < b$ *and* $\gcd(a,b) = 1$. *Then*

$$\kappa(M) = \begin{cases} \frac{1}{3}, & \text{if } b \equiv a \pmod{3}; \\ \frac{2a+b-1}{3(2a+b)}, & \text{if } b \equiv a+1 \pmod{3}; \\ \frac{a+2b-1}{3(a+2b)}, & \text{if } b \equiv a+2 \pmod{3}. \end{cases}$$

Proof. This is a direct consequence of Theorem 1. Set $b-a = 3k+r$, where $0 \leq r \leq 2$. Then $\lambda = k+1$, so that we are in Case I when $b \equiv a+1 \pmod 3$ and Case II otherwise. The calculation is routine, and the details are omitted. □

Remark 3. Haralambis [11] conjectured that $\mu(M) = \kappa(M)$ when $|M| = 3$. If this is true, Theorem 1 actually determines $\mu(\{a, b, n(a+b)\})$. Observe that, if a, b are of opposite parity and if $n \geq (b-a)/2$, Theorem 1 reduces to

$$\kappa(\{a, b, n(a+b)\}) = \frac{a+b-1}{2(a+b)+(2a/n)},$$

which is asymptotic to $\mu(\{a,b\})$. This may be an indication that the conjecture of Haralambis may hold, at least for the special case $M = \{a, b, n(a+b)\}$ when n is large enough, and perhaps even for $M = \{a, b, c\}$ when c is sufficiently large even if not of the form $n(a+b)$.

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