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An Asymptotic Formula for Short Sums of Gcd-Sum Functions

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Abstract

In this note, we prove an asymptotic formula for sums, over short segments, of the composition of the gcd and arithmetic functions belonging to certain classes. Two examples are also given.

1 Introduction and main results

There has been renewed interest in gcd-sum functions over the last few years. We refer to [4, 6] for a survey. In [1], it is shown that

$$\sum_{n \le x} \left(\sum_{i=1}^{n} f(\gcd(i,n)) \right) = \frac{x^2 F(2)}{2\zeta(2)} + O\left\{ x \prod_{p \le x} \left(1 + \sum_{l=1}^{\infty} \frac{\left| f\left(p^l\right) - f\left(p^{l-1}\right) \right|}{p^l} \right) + x(\log x)^{\alpha} \right\}$$

under the sole hypothesis

$$\sum_{n \le x} |(f \star \mu)(n)| \ll x(\log x)^{\alpha} \tag{1}$$

where $\alpha \ge 0$, $F(s)$ is the Dirichlet series of the arithmetic function f and the Dirichlet convolution product $f \star g$ of f and g is defined by $(f \star g)(n) = \sum_{d|n} f(d) g(n/d)$.

It can easily be deduced that, if f satisfies (1), then for large real numbers x, y such that $x^{1/2} \le y \le x$, we have

$$\begin{aligned} \sum_{x < n \le x+y} \left(\sum_{i=1}^{n} f(\gcd(i,n)) \right) &= y\,(2x+y)\,\frac{F(2)}{2\zeta(2)} \\ &\quad + O\left\{ x \prod_{p \le x} \left(1 + \sum_{l=1}^{\infty} \frac{\left| f\left(p^l\right) - f\left(p^{l-1}\right) \right|}{p^l} \right) + x(\log x)^{\alpha} \right\}. \end{aligned}$$

The aim of the present work is to study similar short sums when f lies into a wider class of arithmetic functions. More precisely, we consider the class of real-valued multiplicative functions satisfying

$$\sum_{n\le x} \frac{|(f\star\mu)(n)|}{n} \ll x^a \quad \left(0\le a\le \tfrac{1}{2}\right) \tag{2}$$

$$\sum_{n\le x} \frac{(f\star\mu)(n)^2}{n^2} \ll x^b \quad (0\le b\le 1) \tag{3}$$

for all $x\ge 1$. We will prove the following asymptotic formula.

Theorem 1. *Let f be a real-valued multiplicative function satisfying (2) and (3) with Dirichlet series $F(s)$. Then for $x^{1/3}\le y\le x$, we have*

$$\sum_{x<n\le x+y}\left(\sum_{i=1}^{n} f(\gcd(i,n))\right) = y\,(2x+y)\,\frac{F(2)}{2\zeta(2)} + O\left(x^{1+a/2+b/4}y^{1/4}(\log x)^{1/4}\right).$$

Note that this result is non-trivial whenever $x^{(4+2a+b)/7}(\log x)^{1/7}\le y\le x$.

Let $k\ge 2$ be a fixed integer and s_k be the characteristic function of the set of k-full numbers. Denote $\mathrm{Id}: n\longmapsto n$ and consider the multiplicative function $\mathrm{Id}\cdot s_k$. Similarly, let $M_k(n)$ be the maximal k-full divisor of n (see [5] for instance). It can easily be seen that both functions $\mathrm{Id}\cdot s_k$ and M_k satisfy the assumptions (2) and (3) with $a=b=\frac{1}{k}$, and hence do not satisfy (1). Theorem 1 implies the following result.

Corollary 2. *Let $k\ge 2$ be an integer. For $x^{1/3}\le y\le x$, we have*

$$\sum_{x<n\le x+y}\left(\sum_{\substack{i=1\\ \gcd(i,n)\ k\text{-full}}}^{n} \gcd(i,n)\right) = \frac{y\,(2x+y)}{2\zeta(2)}\prod_p\left(1+\frac{1}{p^{k-1}(p-1)}\right) + O\left(x^{1+\frac{3}{4k}}y^{1/4}(\log x)^{1/4}\right)$$

and

$$\sum_{x<n\le x+y}\left(\sum_{i=1}^{n} M_k(\gcd(i,n))\right) = \frac{C_k\,y\,(2x+y)}{2\zeta(2)} + O\left(x^{1+\frac{3}{4k}}y^{1/4}(\log x)^{1/4}\right)$$

where

$$C_k := \prod_p\left(1+\frac{p^{2k}+p^{k+1}(p+1)-p^2}{p^{2k}(p^2-1)}\right).$$

2 Notation

In what follows, $1\le y\le x$ are large real numbers and $N\ge 1$ is a large integer. Let f be a real-valued multiplicative function satisfying (2) and (3), and let $g := f\star\mu$ be the Eratosthenes transform of f. According to the usual practice, for all $x\in\mathbb{R}$, $\lfloor x\rfloor$ is the integer part of x and $\|x\|$ is the distance from x to its nearest integer. φ is the Euler totient function and μ is the Möbius function.

3 Proof of Theorem 1

First, note that by (2) and Abel summation, we have for all $z \geq 1$

$$\sum_{n>z} \frac{|g(n)|}{n^2} \ll z^{-1+a}. \tag{4}$$

Next, using Cesáro's identity [3]

$$\sum_{i=1}^{n} f(\gcd(i,n)) = (f \star \varphi)(n)$$

and using $\varphi = \mu \star \mathrm{Id}$, we get

$$\begin{aligned}
\sum_{x<n\leq x+y} \left(\sum_{i=1}^{n} f(\gcd(i,n)) \right) &= \sum_{x<n\leq x+y} (g \star \mathrm{Id})(n) \\
&= \sum_{d\leq x+y} g(d) \sum_{x/d<k\leq (x+y)/d} k \\
&= \frac{1}{2} \sum_{d\leq x+y} g(d) \left(\left\lfloor \frac{x+y}{d} \right\rfloor - \left\lfloor \frac{x}{d} \right\rfloor \right) \left(\left\lfloor \frac{x+y}{d} \right\rfloor + \left\lfloor \frac{x}{d} \right\rfloor + 1 \right) \\
&= \frac{1}{2} \sum_{d\leq 2y} g(d) \left(\left\lfloor \frac{x+y}{d} \right\rfloor - \left\lfloor \frac{x}{d} \right\rfloor \right) \left(\left\lfloor \frac{x+y}{d} \right\rfloor + \left\lfloor \frac{x}{d} \right\rfloor + 1 \right) \\
&\quad + \sum_{2y<d\leq x+y} g(d) \left\lfloor \frac{x+y}{d} \right\rfloor \left(\left\lfloor \frac{x+y}{d} \right\rfloor - \left\lfloor \frac{x}{d} \right\rfloor \right) \\
&:= \Sigma_1 + \Sigma_2.
\end{aligned}$$

3.1 The sum Σ_1

Since

$$\frac{1}{2} \left(\left\lfloor \frac{x+y}{d} \right\rfloor - \left\lfloor \frac{x}{d} \right\rfloor \right) \left(\left\lfloor \frac{x+y}{d} \right\rfloor + \left\lfloor \frac{x}{d} \right\rfloor + 1 \right) = \frac{y}{2d^2} (2x+y) + O\left(\frac{x}{d} \right)$$

we get using (2) and (4)

$$\begin{aligned}
\Sigma_1 &= \frac{y}{2} (2x+y) \sum_{d\leq 2y} \frac{g(d)}{d^2} + O\left(x \sum_{d\leq 2y} \frac{|g(d)|}{d} \right) \\
&= \frac{y}{2} (2x+y) \sum_{d=1}^{\infty} \frac{g(d)}{d^2} - \frac{y}{2} (2x+y) \sum_{d>2y} \frac{g(d)}{d^2} + O(xy^a) \\
&= y(2x+y) \frac{F(2)}{2\zeta(2)} + O(xy^a).
\end{aligned}$$

3.2 The sum Σ_2

Lemma 3. *If $x^{1/3} \le y \le x$, then*

$$\sum_{2y<n\le x+y} \left(\left\lfloor \frac{x+y}{n} \right\rfloor - \left\lfloor \frac{x}{n} \right\rfloor \right) \ll y \log x.$$

Proof. If $N \ge 1$ is a large integer, $\delta \in \left(0, \frac{1}{2}\right)$ and if $h : [N, 2N] \longrightarrow \mathbb{R}$ is any function, then let $\mathcal{R}(h, N, \delta)$ be the number of integers $n \in [N, 2N]$ such that $\|h(n)\| < \delta$. The second derivative test for $\mathcal{R}(h, N, \delta)$ (see [2, Corollaire]) states that, if $h \in C^2[N, 2N]$ such that there exists $T \ge 1$ such that $|h'(x)| \asymp TN^{-1}$ and $|h''(x)| \asymp TN^{-2}$, then

$$\mathcal{R}(h, N, \delta) \ll (NT)^{1/3} + N\delta + (T\delta)^{1/2}.$$

Now using this estimate with $T = xN^{-1}$ and $\delta = yN^{-1}$, we get

$$\begin{aligned}
\sum_{2y<n\le x+y} \left(\left\lfloor \frac{x+y}{n} \right\rfloor - \left\lfloor \frac{x}{n} \right\rfloor \right) &= \left(\sum_{2y<n\le x} + \sum_{x<n\le x+y} \right) \left(\left\lfloor \frac{x+y}{n} \right\rfloor - \left\lfloor \frac{x}{n} \right\rfloor \right) \\
&\ll \max_{2y<N\le x} \sum_{N<n\le 2N} \left(\left\lfloor \frac{x+y}{n} \right\rfloor - \left\lfloor \frac{x}{n} \right\rfloor \right) \log x + y \\
&\ll \max_{2y<N\le x} \mathcal{R}\left(\frac{x}{n}, N, \frac{y}{N} \right) \log x + y \\
&\ll \max_{2y<N\le x} \left(x^{1/3} + y + (xy)^{1/2} N^{-1} \right) \log x + y \\
&\ll \left(x^{1/3} + y + \left(xy^{-1} \right)^{1/2} \right) \log x + y \ll y \log x
\end{aligned}$$

since $y \ge x^{1/3}$. □

Now using the Cauchy-Schwarz inequality and Lemma 3, we get

$$\begin{aligned}
|\Sigma_2| &\le 2x \sum_{2y<d\le x+y} \frac{|g(d)|}{d} \left(\left\lfloor \frac{x+y}{d} \right\rfloor - \left\lfloor \frac{x}{d} \right\rfloor \right) \\
&\ll x \left(\sum_{2y<d\le x+y} \frac{g(d)^2}{d^2} \right)^{1/2} \left(\sum_{2y<d\le x+y} \left(\left\lfloor \frac{x+y}{d} \right\rfloor - \left\lfloor \frac{x}{d} \right\rfloor \right) \right)^{1/2} \\
&\ll xy^{1/2} \left(\sum_{2y<d\le x+y} \frac{g(d)^2}{d^2} \right)^{1/2} (\log x)^{1/2}
\end{aligned}$$

whenever $x^{1/3} \le y \le x$, and the hypothesis (3) provides

$$|\Sigma_2| \ll x^{1+b/2} y^{1/2} (\log x)^{1/2}.$$

We also have trivially using (2)

$$|\Sigma_2| \ll x \sum_{2y<d\le x+y} \frac{|g(d)|}{d} \ll x^{1+a}$$

and hence

$$|\Sigma_2| \ll \min\left(x^{1+a}, x^{1+b/2}y^{1/2}(\log x)^{1/2}\right) \ll x^{1+a/2+b/4}y^{1/4}(\log x)^{1/4}.$$

We conclude the proof with the following remarks: if $0 \le a \le \frac{1}{4}$, then

$$xy^a \le xy^{1/4} \le x^{1+a/2+b/4}y^{1/4}.$$

If $\frac{1}{4} < a \le \frac{1}{2}$, then $x^{1+a/2+b/4}y^{1/4} \ge xy^a$ as soon as $y \le x^{\frac{2a+b}{4a-1}}$ which is trivially true since in that case we have

$$\frac{2a+b}{4a-1} \ge b+1 \ge 1$$

since $b \ge 0$. □

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