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Identities for the Classical Polynomials Through Sums of Liouville Type

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Abstract

We prove identities involving Bernoulli, Euler, Fibonacci, Lucas, Chebychev, and Dickson polynomials by using sums of Liouville type.

1 Introduction

Polynomials defined recursively over the integers such as Dickson polynomials, Chebychev polynomials, Fibonacci polynomials, Lucas polynomials, Bernoulli polynomials, Euler polynomials, and many others have been extensively studied in the past. Most of these polynomials have some type of relationship between them and share a large number of interesting properties. They have been also found to be topics of interest in many different areas of pure and applied sciences. Most recently, some of these families of polynomials have been found to be useful in cryptography and related topics, which keep making them a very interesting area of research for many people in this era of communication. In this paper, we use a sum of Liouiville type to prove new properties concerning many of these families of polynomials. The following notations and definitions will be adopted throughout this paper.

The sets of positive integers, nonnegative integers, integers, real numbers, and complex numbers are respectively denoted by $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$. The sets of even positive integers and the set of odd positive integers will be written $2\mathbb{N}$ and $2\mathbb{N}-1$ respectively and the sets of even integers and odd integers will be written $2\mathbb{Z}$ and $2\mathbb{Z}-1$ respectively. The Euler phi function will be denoted by $\phi(n)$ and the sum of the kth powers of positive divisors of n will denoted by $\sigma_k(n)$.

Definition 1. For $n \in \mathbb{N}$ such that $n > 1$, let the sets $B(n)$ and $B'(n)$ be defined as follows:

$$B(n) = \{(a,b,u,v) \in \mathbb{N}^4 : \ au + bv = n\},$$
$$B'(n) = \{(a,b,u,v) \in \mathbb{N}^4 : \ au + bv = n, \ \gcd(a,b) = \gcd(u,v) = 1\},$$

and for $n \in 2\mathbb{N}$ let the sets $O(n)$ and $O'(n)$ be defined as follows:

$$O(n) = \{(a,b,u,v) \in (2\mathbb{N}-1)^4 : \ au + bv = n\},$$
$$O'(n) = \{(a,b,u,v) \in (2\mathbb{N}-1)^4 : \ au + bv = n, \ \gcd(a,b) = \gcd(u,v) = 1\}.$$

Arithmetic sums of the types

$$\sum_{(a,b,u,v)\in B(n)} \big(f(a-b) - f(a+b)\big) \text{ and } \sum_{(a,b,u,v)\in O(n)} \big(f(a-b) - f(a+b)\big) \tag{1}$$

were first investigated by Liouville in his seminal work on elementary methods in number theory, see for instance Liouville [8, 9]. A comprehensive treatment of Liouville's methods has been given in a new book by Williams [12]. Adapting the proof of Williams [12] for the sums (1), El Bachraoui [3] found the following related identities for the sums over the sets $B'(n)$ and $O'(n)$.

Theorem 2. *Let $n > 1$ be a positive integer.*

(a) If $f : \mathbb{Z} \to \mathbb{C}$ is an even function, then

$$\sum_{(a,b,u,v)\in B'(n)} \big(f(a-b) - f(a+b)\big) = \big(f(0) + 2f(1) - f(n)\big)\phi(n) - 2 \sum_{\substack{1\le l<n \\ (l,n)=1}} f(l).$$

(b) If moreover $n \in 2\mathbb{N}$, then

$$\sum_{(a,b,u,v)\in O'(n)} \big(f(a-b) - f(a+b)\big) = \big(f(0) - f(n)\big)\phi(n).$$

In this note we shall apply Theorem 2 to deduce identities involving Bernoulli, Euler, Lucas, Fibonacci, Chebychev, and Dickson polynomials.

2 Preliminaries

Let $(B_n)_{n=0}^{\infty}$ be the sequence of Bernoulli numbers for which we have $B_1 = -\frac{1}{2}$, $B_{2n+1} = 0$ for any $n \in \mathbb{N}$, and the first few terms for even n are

$$B_0 = 1, \ B_2 = \frac{1}{6}, \ B_4 = -\frac{1}{30}, B_6 = \frac{1}{42}, \ B_8 = -\frac{1}{30}, \ B_{10} = \frac{5}{66}.$$

Further we have the recursive formula

$$B_n = \sum_{l=0}^{n} \binom{n}{l} B_l, \quad (n \ge 2). \tag{2}$$

Definition 3. If $n \in \mathbb{N}$, then the *Bernoulli polynomial* is given by

$$B_n(x) = \sum_{l=0}^{n} \binom{n}{l} B_l x^{n-l} = nB_1 x^{n-1} + \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2l} B_{2l} x^{n-2l}.$$

A well-known property of Bernoulli polynomial is

$$B_n(x+1) - B_n(x) = nx^{n-1}, \quad (n \geq 1)$$

which by iteration yields

$$B_n(x+m) - B_n(x) = n \sum_{l=0}^{m-1} (x+l)^{n-1}, \quad (m, n \geq 1). \tag{3}$$

Let $(E_n)_{n=0}^{\infty}$ be the sequence of Euler numbers. We have $E_{2n-1} = 0$ for are all $n \in \mathbb{N}_0$ and the first few terms for even n are

$$E_0 = 1,\ E_2 = -1,\ E_4 = 5,\ E_6 = -61,\ E_8 = 1385,\ E_{10} = -50521,\ E_{12} = 2702765.$$

Definition 4. For $n \in \mathbb{N}_0$ the *Euler polynomial* is given by

$$E_n(x) = \sum_{l=0}^{n} \binom{n}{l} \frac{E_l}{2^l} \cdot \left(x - \frac{1}{2}\right)^{n-l} = \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2l} \frac{E_{2l}}{4^l} \cdot \left(x - \frac{1}{2}\right)^{n-2l}.$$

A well-known property of $E_n(x)$ is

$$E_n(x+1) - E_n(x) = 2(x^n - E_n(x)),$$

which by iteration yields

$$E_n(x+m) - E_n(x) = 2 \sum_{l=0}^{m-1} \big((x+l)^n - E_n(x+l)\big). \tag{4}$$

Definition 5. For $n \in \mathbb{N}$, the *Dickson polynomials* of the first kind and of the second kind of degree n with parameter a are respectively defined as follows:

$$\begin{aligned} D_n(x,a) &= \sum_{j=1}^{\lfloor n/2 \rfloor} \frac{k}{k-j} \binom{n-j}{j} (-a)^j x^{n-2j}, \text{ and } D_0(x,a) = 2. \\ \mathrm{D}_n^{(2)}(x,a) &= \sum_{j=1}^{\lfloor n/2 \rfloor} \binom{n-j}{j} (-a)^j x^{n-2j}, \text{ and } \mathrm{D}_0^{(2)}(x,a) = 1. \end{aligned}$$

The first few Dickson polynomials of the first and of the second kind of degree n with parameter a are respectively:

$$D_0(x) = 2,\ D_1(x) = x,\ D_2(x) = x^2 - 2a,\ D_3(x) = x^3 - 3xa,$$

$$D_4(x) = x^4 - 4x^2a + 2a^2,\ D_5(x) = x^5 - 5x^3a + 5xa^2,$$
$$\mathrm{D}_0^{(2)}(x) = 1,\ \mathrm{D}_1^{(2)}(x) = x,\ \mathrm{D}_2^{(2)}(x) = x^2 - a,\ \mathrm{D}_3^{(2)}(x) = x^3 - 2xa,$$
$$\mathrm{D}_4^{(2)}(x) = x^4 - 3x^2a + a^2,\ \mathrm{D}_5^{(2)}(x) = x^5 - 4x^3a + 3xa^2.$$

Dickson polynomials can also be defined recursively and the definition extended to all $\mathbb{Z}$ as follow:

$$\begin{aligned} D_0(x,a) &= 2,\ D_1(x,a) = x,\ D_{n+1}(x,a) = xD_n(x,a) - aD_{n-1}(x,a) \\ \mathrm{D}_0^{(2)}(x,a) &= 1,\ \mathrm{D}_1^{(2)}(x,a) = x,\ \mathrm{D}_{n+1}^{(2)}(x,a) = x\mathrm{D}_n^{(2)}(x,a) - a\mathrm{D}_{n-1}^{(2)}(x,a) \end{aligned}$$

Dickson polynomials are closely related to some well-known polynomials like the first and the second *Chebychev polynomials* $T_n(x)$ and $U_n(x)$ defined as follows:

$$T_n(x) = \begin{cases} 1, & \text{if } n = 0; \\ x, & \text{if } n = 1; \\ 2xT_{n-1}(x) + T_{n-2}(x), & \text{if } n \geq 2; \end{cases} \quad \text{and } T_{-n}(x) = (-1)^n T_n(x),$$
$$U_n(x) = \begin{cases} 1, & \text{if } n = 0; \\ 2x, & \text{if } n = 1; \\ 2xU_{n-1}(x) + U_{n-2}(x), & \text{if } n \geq 2; \end{cases} \quad \text{and } U_{-n}(x) = (-1)^n U_n(x).$$

Dickson polynomials are also related to *Lucas polynomials* $L_n(x)$ and the *Fibonacci polynomials* $F_n(x)$, see for instance Piero [10]. These are defined as follows:

Definition 6. For $n \in \mathbb{N}$,

$$\begin{aligned} L_n(x) &= \sum_{j=0}^{\lfloor k/2 \rfloor} \frac{k}{k-j} \binom{k-j}{j} x^j \\ F_{n+1}(x) &= \sum_{j=1}^{\lfloor k/2 \rfloor} \binom{k-j}{j} x^j \end{aligned}$$

Like the Dickson polynomials, these two classes of polynomials can also be defined recursively in $\mathbb{Z}$ as follow:

$$L_n(x) = \begin{cases} 2, & \text{if } n = 0; \\ x, & \text{if } n = 1; \\ xL_{n-1}(x) + L_{n-2}(x), & \text{if } n \geq 2; \end{cases} \quad \text{and } L_{-n}(x) = (-1)^n L_n(x),$$
$$F_n(x) = \begin{cases} 0, & \text{if } n = 0; \\ 1, & \text{if } n = 1; \\ xF_{n-1}(x) + F_{n-2}(x), & \text{if } n \geq 2; \end{cases} \quad \text{and } F_{-n}(x) = (-1)^{n-1} F_n(x).$$

The first few Lucas and Fibonacci polynomials are

$$L_0(x) = 2,\ L_1(x) = x,\ L_2(x) = x^2 + 2,\ L_3(x) = x^3 + 3x,$$

$$L_4(x) = x^4 + 4x^2 + 2,\ L_5(x) = x^5 + 5x^3 + 5x.$$

$$F_0(x) = 0,\ F_1(x) = 1,\ F_2(x) = x,\ F_3(x) = x^2 + 1,$$

$$F_4(x) = x^3 + 2x,\ F_5(x) = x^4 + 3x^2 + 1.$$

Note that $\big(L_n(1)\big)_{n=0}^{\infty}$ is the Lucas sequence $(L_n)_{n=0}^{\infty}$ and that $\big(F_n(1)\big)_{n=0}^{\infty}$ is the Fibonacci sequence $(F_n)_{n=0}^{\infty}$. It is important for the current purposes to note that Dickson polynomials $D_n(x,a)$ and Lucas polynomials $L_n(x)$ are even functions for even n and odd functions for odd n, whereas Fibonacci polynomials $F_n(x)$ are even functions for odd n and odd functions for even n. For the Lucas and Fibonacci polynomials, it is well-known that for $m,n \in \mathbb{Z}$,

$$\begin{aligned} L_{m+n}(x) &= L_m(x)L_n(x) - (-1)^n L_{m-n}(x), && (5)\\ F_{m+n}(x) &= F_m(x)F_n(x) + F_{m-1}(x)F_{n-1}(x). && (6)\end{aligned}$$

In particular, if $m,n \in 2\mathbb{N}-1$, then the relation (6) implies

$$F_{m+n+1}(x) - F_{m-n+1}(x) = F_{m+1}(x)\left(F_{n+1}(x) + F_{n-1}(x)\right). \tag{7}$$

Similar properties can be easily proved by induction for the Dickson polynomials. More precisely, for $n \in \mathbb{Z}$, we have

$$\begin{aligned} D_{m+n}(x,a) &= D_m(x,a)D_n(x,a) - aD_{m-n}(x,a), && (8)\\ \mathrm{D}^{(2)}_{m+n}(x,a) &= \mathrm{D}^{(2)}_{m}(x,a)\mathrm{D}^{(2)}_{n}(x,a) - a\mathrm{D}^{(2)^2}_{m-1}(x,a)\mathrm{D}^{(2)^2}_{n-1}(x,a), && (9)\end{aligned}$$

which imply

$$\begin{aligned} D_{2n}(x,a) &= D_n^2(x,a) - 2. && (10)\\ \mathrm{D}^{(2)}_{2n}(x,a) &= \mathrm{D}^{(2)}_{n}(x,a) - a\mathrm{D}^{(2)^2}_{n-1}(x,a). && (11)\end{aligned}$$

Further for any $n \in \mathbb{Z}$, we have:

$$D_n(2xa, a^2) = 2a^nT_n(x) \text{ and } \mathrm{D}^{(2)}_n(2xa; a^2) = a^nU_n(x), \tag{12}$$

in particular

$$D_n(x;1) = 2T_n(x/2) \text{ and } \mathrm{D}^{(2)}_n(x;1) = U_n(x/2). \tag{13}$$

Many other interesting properties about Dickson polynomials can be found in Lidl, Mullen, and Turnwald [6].

3 An identity involving Bernoulli polynomials

Theorem 7. *If $n \in 2\mathbb{N}$, then*

$$\sum_{(a,b,u,v)\in O'(n)} \left(\sum_{l=0}^{2b-1}(a-b+l)^{n-1} + \frac{(a+b)^{n-1}-(a-b)^{n-1}}{2}\right)$$

$$= \left(\sum_{l=0}^{n-1}\binom{n}{l}B_l n^{n-l-1} + \frac{1}{2}n^n\right)\frac{\phi(n)}{n}.$$

Proof. Let $n \in 2\mathbb{N}$ and let $f : \mathbb{Z} \to \mathbb{C}$ be defined by:

$$f(k) = B_n(k) - nB_1 k^{n-1} = B_n(k) + \frac{n}{2}k^{n-1} = \sum_{l=0}^{\frac{n}{2}} \binom{n}{2l} B_{2l} k^{n-2l}.$$

Then clearly f is an even function and thus by Theorem 2 we find

$$\sum_{(a,b,u,v)\in O'(n)} \Big(\big(B_n(a+b) - B_n(a-b)\big) + \frac{n}{2}(a+b)^{n-1} - \frac{n}{2}(a-b)^{n-1} \Big) \tag{14}$$

$$= \big(B_n(n) + \frac{n}{2}n^{n-1} - B_n(0)\big)\phi(n) = \big(B_n(n) + \frac{n^n}{2} - B_n\big)\phi(n).$$

By the relations (2) we have

$$B_n(n) - B_n = \sum_{l=0}^{n} \binom{n}{l} B_l n^{n-l} - B_n = n \sum_{l=0}^{n-1} \binom{n}{l} B_l n^{n-l-1}$$

and by the relation (3) we have

$$B_n(a+b) - B_n(a-b) = B_n(a-b+2b) - B_n(a-b) = n \sum_{l=0}^{2b-1} (a-b+l)^{n-1}.$$

Now put in (14) and divide by n to obtain the desired identity. $\square$

Remark 8. If $n \in 2\mathbb{N}$, then by Theorem 2 applied to the function $f(k) = k^n$ we have

$$\sum_{(a,b,u,v)\in O'(n)} \big((a+b)^n - (a-b)^n\big) = n^n \phi(n) \quad (n \in 2\mathbb{N}).$$

On the other hand, if $n \in 2\mathbb{N}$, we do not know an evaluation for the sum

$$\sum_{(a,b,u,v)\in O'(n)} \big((a+b)^{n-1} - (a-b)^{n-1}\big)$$

which appears in the left hand side of the formula in Theorem 7.

4 An identity for Euler polynomials

Theorem 9. *If* $n \in 2\mathbb{N}$, *then*

$$\sum_{(a,b,u,v)\in O'(n)} \left(\sum_{l=0}^{2b-1} (a-b+\frac{1}{2}+l)^n - E_n(a-b+\frac{1}{2}+l) \right)$$

$$= \phi(n) \sum_{l=0}^{n-1} \big((\frac{1}{2}+l)^n - E_n(\frac{1}{2}+l)\big).$$

Proof. Let $n \in 2\mathbb{N}$ and let $f : \mathbb{Z} \to \mathbb{C}$ be defined as follows:

$$f(k) = E_n(k + \frac{1}{2}) = \sum_{l=0}^{\frac{n}{2}} \binom{n}{2l} \frac{E_{2l}}{4^l} k^{n-2l}.$$

Then clearly f is an even function and so by Theorem 2 we get

$$\begin{aligned} \sum_{(a,b,u,v)\in O'(n)} \left(E_n(a + b + \frac{1}{2}) - E_n(a - b + \frac{1}{2}) \right) \\ = \left(E_n(n + \frac{1}{2}) - E_n(\frac{1}{2}) \right) \phi(n). \end{aligned} \tag{15}$$

By the property (4) we have

$$\begin{aligned} E_n(a + b + 1/2) - E_n(a - b + 1/2) &= E_n(a - b + 1/2 + 2b) - E_n(a - b + 1/2) \\ &= 2 \left(\sum_{l=0}^{2b-1} (a - b + 1/2 + l)^n - E_n(a - b + 1/2 + l) \right) \end{aligned}$$

and

$$E_n(n + 1/2) - E_n(1/2) = 2 \sum_{l=0}^{n-1} \big((1/2 + l)^n - E_n(1/2 + l) \big),$$

which put in (15) yield the desired formula. □

5 Identities for Lucas polynomials

Theorem 10. *If $n \in 2\mathbb{N}$ and $x \in \mathbb{R}$, then*

$$\sum_{(a,b,u,v)\in O'(n)} L_a(x)L_b(x) = \phi(n)(L_n(x) - 2).$$

Proof. Let $x \in \mathbb{R}$ and let the function $f(k)$ be defined as follows:

$$f(k) = \begin{cases} 0, & \text{if } k \in 2\mathbb{Z} - 1; \\ L_k(x), & \text{if } k \in 2\mathbb{Z}. \end{cases}$$

Clearly f is an even function. Then application of Theorem 2 to the function f yields

$$\begin{aligned} \sum_{(a,b,u,v)\in O'(n)} (L_{a+b}(x) - L_{a-b}(x)) &= (L_n(x) - L_0(x))\, \phi(n) \\ &= \big(L_n(x) - 2\big)\phi(n), \end{aligned} \tag{16}$$

which by virtue of the relation (5) and the fact that $L_0(x) = 2$ gives

$$\sum_{(a,b,u,v)\in O'(n)} \big(L_a(x)L_b(x) - (-1)^b L_{a-b}(x) - L_{a-b}(x)\big) = (L_n(x) - 2)\, \phi(n),$$

and the desired formula follows since $(-1)^b = -1$. □

Taking $x = 1$ in Theorem 10 we have the following result on Lucas numbers.

Corollary 11. *If $n \in 2\mathbb{N}$, then*

$$\sum_{(a,b,u,v)\in O'(n)} L_a L_b = \phi(n)(L_n - 2).$$

Theorem 12. *If $n \in 2\mathbb{N}$, then*

$$\begin{aligned}\sum_{(a,b,u,v)\in O'(n)} \big(L_{a+b}(a+b) - L_{a-b}(a-b)\big) &= \sum_{(a,b,u,v)\in O'(n)} \big(L_n(a+b) - L_n(a-b)\big)\\ &= \sum_{(a,b,u,v)\in O'(n)} \big(L_{a+b}(n) - L_{a-b}(n)\big).\end{aligned}$$

Proof. Note first that by formula (16) the rightmost sum equals

$$\big(L_n(n) - 2\big)\phi(n).$$

Next let $f(k) = L_k(-k)$ and $g(k) = L_n(k)$. Then it is easily verified that both f and g are even functions. Thus by Theorem 2 applied to f and g we find respectively

$$\sum_{(a,b,u,v)\in O'(n)} \big(L_{a+b}(-a-b) - L_{a-b}(b-a)\big) = \big(L_n(n) - L_0(0)\big)\phi(n)$$

and

$$\sum_{(a,b,u,v)\in O'(n)} \big(L_n(a+b) - L_n(a-b)\big) = \big(L_n(n) - L_n(0)\big)\phi(n).$$

Now combine the previous two relations with the facts that $L_n(0) = L_0(0) = 2$, $L_{a+b}(-a-b) = L_{a+b}(a+b)$, and $L_{a-b}(b-a) = L_{a-b}(a-b)$ to deduce the equality of the three sums. □

6 Identities for Fibonacci polynomials

Theorem 13. *Let $n \in 2\mathbb{N}$ and $x \in \mathbb{R}$. Then*

$$\sum_{(a,b,u,v)\in O'(n)} \big(F_{a+1}(x)\left(F_{b+1}(x) + F_{b-1}(x)\right)\big) = \phi(n)(F_{n+1}(x) - 1).$$

Proof. Note first that if f is even, then

$$\begin{aligned}&\sum_{(a,b,u,v)\in O'(n)} (f(a+b) - f(a-b))\\ &\qquad = (f(2) - f(0))\,\phi(n) + 2\sum_{\substack{(a,b,u,v)\in O'(n)\\ a>b}} (f(a+b) - f(a-b)). \quad (17)\end{aligned}$$

Let f be the function defined by:

$$f(k) = \begin{cases} 0, & \text{if } k \in 2\mathbb{Z} - 1; \\ F_{k+1}(x), & \text{if } 0 \leq k \in 2\mathbb{Z}; \\ F_{k-1}(x), & \text{otherwise.} \end{cases}$$

It is easily verified that f is an even function. Then by Theorem 2 we obtain

$$\sum_{(a,b,u,v)\in O'(n)} (F_{a+b+1}(x) - F_{a-b+1}(x)) \tag{18}$$

$$= \phi(n)\big(F_{n+1}(x) - F_1(x)\big) = \phi(n)\big(F_{n+1}(x) - 1\big).$$

By virtue of the relation (17) the previous formula is equivalent to

$$(F_3(x) - F_1(x))\,\phi(n) + 2 \sum_{\substack{(a,b,u,v)\in O'(n)\\ a>b}} (F_{a+b+1}(x) - F_{a-b+1}(x))$$

$$= (F_{n+1}(x) - F_1(x))\,\phi(n).$$

That is,

$$\sum_{\substack{(a,b,u,v)\in O'(n)\\ a>b}} (F_{a+b+1}(x) - F_{a-b+1}(x)) = \frac{F_{n+1}(x) - F_3(x)}{2}\phi(n),$$

which by identity (6) and the fact that $F_3(x) = x^2 + 1$ gives

$$\sum_{\substack{(a,b,u,v)\in O'(n)\\ a>b}} \big(F_{a+1}(x)(F_{b+1}(x) + F_{b-1}(x))\big) = \frac{F_{n+1}(x) - x^2 - 1}{2}\phi(n).$$

Now use the previous formula together with the facts that $F_0(x) = 0$, $F_2(x) = x$, and

$$\sum_{(a,b,u,v)\in O'(n)} (F_{a+1}(x)(F_{b+1}(x) + F_{b-1}(x))) = F_2(x)\,(F_2(x) + F_0(x))\,\phi(n)$$

$$+ 2 \sum_{\substack{(a,b,u,v)\in O'(n)\\ a>b}} (F_{a+1}(x)(F_{b+1}(x) + F_{b-1}(x)))$$

to derive the result. □

Taking $x = 1$ in Theorem 13 we have the following identity involving Fibonacci numbers.

Corollary 14. *If $n \in 2\mathbb{N}$, then*

$$\sum_{(a,b,u,v)\in O'(n)} \big(F_{a+1}\big(F_{b+1} + F_{b-1}\big)\big) = \phi(n)(F_{n+1} - 1).$$

Theorem 15. *Let $n \in 2\mathbb{N}$. Then we have*

$$(a) \qquad \sum_{(a,b,u,v)\in O'(n)} \big(F_{a+b}(a+b) - F_{a-b}(a-b)\big) = F_n(n)\phi(n).$$

$$(b) \qquad \sum_{(a,b,u,v)\in O'(n)} \big(F_{n+1}(a+b) - F_{n+1}(a-b)\big) = \sum_{(a,b,u,v)\in O'(n)} \big(F_{a+b+1}(n) - F_{a-b+1}(n)\big).$$

Proof. To prove part (a) apply Theorem 2 to the even functions $f(k) = F_k(k)$. As to part (b) note first that by the relation (18) the sum on the right equals

$$\big(F_{n+1}(n) - 1\big)\phi(n).$$

Next by Theorem 2 applied to the even function $g(k) = F_{n+1}(k)$ we have

$$\sum_{(a,b,u,v)\in O'(n)} \big(F_{n+1}(a+b) - F_{n+1}(a-b)\big) = \big(F_{n+1}(n) - F_{n+1}(0)\big)\phi(n) = \big(F_{n+1}(n) - 1\big)\phi(n).$$

This completes the proof. □

7 Identities involving Dickson polynomials

To simplify the notation in this section, we write $D_n(x)$ for $D_n(x,1)$.

Theorem 16. *Let $n \in \mathbb{N}$ and $x \in \mathbb{R}$. Then*

$$\sum_{(a,b,u,v)\in O'(n)} \big(D_a(x)D_b(x)\big)\big(D_a(x)D_b(x) - 2D_{a-b}(x)\big) = \phi(n)\big(D_{2n}(x) - 2\big).$$

Proof. Clearly,

$$D_{2n}(x) = D_n^2(x) - 2$$

is an even function. Therefore if we define f on $\mathbb{Z}$ by

$$f(k) = \begin{cases} 0, & \text{if } k \le 0; \\ D_{2k}(x), & \text{if } k > 0; \end{cases}$$

then f is also even. By Theorem 2 we have

$$\sum_{(a,b,u,v)\in O'(n)} f(a+b) - f(a-b) = (f(n) - f(0))\phi(n),$$

or equivalently,

$$\sum_{(a,b,u,v)\in O'(n)} \big(D_{2(a+b)}(x) - D_{2(a-b)}(x)\big) = (D_{2n}(x) - D_0(x))\phi(n),$$

which by identity (10) and the fact that $D_0(x) = 2$ gives

$$\begin{aligned}\sum_{(a,b,u,v)\in O'(n)} \left(D^2_{a+b}(x) - D^2_{a-b}(x)\right) &= (D_{2n}(x) - 2)\phi(n).\\ \sum_{(a,b,u,v)\in O'(n)} \left(D_{a+b}(x) + D_{a-b}(x)\right)\left(D_{a+1}(x) - D_{a-b}(x)\right) &= (D_{2n}(x) - 2)\phi(n).\end{aligned}$$

Now application of the formula (8) yields

$$\left(D_{a+b}(x) + D_{a-b}(x)\right) = D_a(x)D_b(x),$$

and

$$\left(D_{a+b}(x) - D_{a-b}(x)\right) = D_a(x)D_b(x) - 2D_{a-b}(x).$$

That is,

$$\begin{aligned}\sum_{(a,b,u,v)\in O'(n)} \left(D_{a+b}(x) + D_{a-b}(x)\right)\left(D_{a+1}(x) - D_{a-b}(x)\right) &= (D_{2n}(x) - 2)\phi(n) \quad \text{or}\\ \sum_{(a,b,u,v)\in O'(n)} \left(D_a(x)D_b(x)\right)\left(D_a(x)D_b(x) - 2D_{a-b}(x)\right) &= \phi(n)\left(D_{2n}(x) - 2\right),\end{aligned}$$

which completes the proof. □

Theorem 17. *Let $n \in 2\mathbb{N}$ and $x \in \mathbb{R}$. Then*

$$\sum_{(a,b,u,v)\in O'(n)} \left(D_a(x)D_b(x) - 2D_{a-b}(x)\right)\left(D_a(x)D_b(x) - 1\right) = \phi(n)(D_{2n}(x) - D_n(x)).$$

Proof. Using the same argument as in the previous theorem with

$$g(k) = \begin{cases} 0, & \text{if } k \in 2\mathbb{Z} - 1;\\ D_k(x), & \text{if } k \in 2\mathbb{Z}; \end{cases}$$

we obtain:

$$\sum_{(a,b,u,v)\in O'(n)} \left(D_{(a+b)}(x) - D_{(a-b)}(x)\right) = (D_n(x) - D_0(x))\phi(n),$$

or equivalently

$$\sum_{(a,b,u,v)\in O'(n)} \left(D_a(x)D_b(x) - 2D_{a-b}(x)\right) = \phi(n)\left(D_n(x) - 2\right).$$

Combining with the result of Theorem 16 we obtain:

$$\sum_{(a,b,u,v)\in O'(n)} \left(D_a(x)D_b(x)\right)\left(D_a(x)D_b(x) - 2D_{a-b}(x)\right) \\ + \sum_{(a,b,u,v)\in O'(n)} \left(D_a(x)D_b(x) - 2D_{a-b}(x)\right)$$

$$= \phi(n)\big(D_{2n}(x) - 2\big) + \phi(n)\big(D_n(x) - 2\big).$$

or equivalently

$$\sum_{(a,b,u,v)\in O'(n)} \big(D_a(x)D_b(x) - 2D_{a-b}(x)\big)\big(D_a(x)D_b(x) - 1\big)\big) = \phi(n)(D_{2n}(x) - D_0(x)).$$

□

As a corollary, we have

Corollary 18. *Let $n \in 2\mathbb{N}$ and $x \in \mathbb{R}$. Then*

$$\sum_{(a,b,u,v)\in O'(n)} \big(T_a(x)T_b(x) - 2T_{a-b}(x)\big)\big(T_a(x)T_b(x) - 1\big)\big) = \frac{1}{2}\phi(n)(T_{2n}(x) - T_n(x)).$$

Proof. The proof is immediate from the previous theorem and relation (13). □

Remark 19. Similar identities can be proven for the Dickson polynomials of the second kind.

8 Concluding remarks

It should be noticed that in deducing our formulas involving the classical polynomials, we restricted ourselves to sums over the set $O'(n)$ as the formulas we obtain when considering sums over the sets $B(n)$, $O(n)$, and $B'(n)$ are not easy to simplify. However, under some restrictions on the positive integer n one can get to nice identities. For our example in this matter we need the following particular case of a sum of Liouville [8] (Problem 13 of Chapter 10 in Williams [12]). If N is twice an odd positive integer, then

$$\sum_{(a,b,x,y)\in O(N)} (f(a+b) - f(a-b)) = -f(0)\sigma(N/2) + \sum_{d|N/2} df(2d). \tag{19}$$

Let N be twice an odd positive integer. Using relation (19) one can show that the analogue of Theorem 7 is

$$\sum_{(a,b,u,v)\in O(N)} \left(\sum_{l=0}^{2b-1} (a-b+l)^{N-1} + \frac{(a+b)^{N-1} - (a-b)^{N-1}}{2} \right)$$
$$= 2^{N-2}\sigma_N(N/2) - \frac{B_N}{N}\sigma(N/2) + \frac{1}{N}\sum_{d|N/2} dB_N(2d).$$

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