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On Lower Order Extremal Integral Sets Avoiding Prime Pairwise Sums

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Abstract

Let A be a subset of $\{1, 2, \ldots, n\}$ such that the sum of no two distinct elements of A is a prime number. Such a subset is called a prime-sumset-free subset of $\{1, 2, \ldots, n\}$. A prime-sumset-free subset is called an extremal prime-sumset-free subset of $\{1, 2, \ldots, n\}$ if $A \cup \{a\}$ is not a prime-sumset-free subset for any $a \in \{1, 2, \ldots, n\} \setminus A$. We prove that if $n \geq 10$ then there is no any extremal prime-sumset-free subset of $\{1, 2, \ldots, n\}$ of order 2 and if $n \geq 13$ then there is no any extremal prime-sumset-free subset of $\{1, 2, \ldots, n\}$ of order 3. Moreover, we prove that for each integer $k \geq 2$, there exists a n_k such that if $n \geq n_k$ then there does not exist any extremal prime-sumset-free subset of $\{1, 2, \ldots, n\}$ of length k. Furthermore, for some small values of n, we give the orders of all extremal prime-sumset-free subset of $\{1, 2, \ldots, n\}$ along with an example of each order, and we give all extremal prime-sumset-free subsets of $\{1, 2, \ldots, n\}$ of orders 2 and 3 for $n \leq 13$.

1 Introduction

Let A be a subset of $\{1, 2, \ldots, n\}$. Then a combinatorial problem posed by Chen [1] is to study the subsets A such that

$$(A \hat{+} A) \cap P = \emptyset, \tag{1}$$

where $A \hat{+} A = \{a + b : a, b \in A, a \neq b\}$ and P is the set of all prime numbers. Such a set is called a *prime-sumset-free subset* of the set $\{1, 2, \ldots, n\}$. If we replace the set P above by a given set T of positive integers then A is called a T*-sumset-free set*. Chen [1] determined all prime-sumset-free subsets of $\{1, 2, \ldots, n\}$ with the largest cardinality. Let the largest cardinality be U_n. Chen [1] proved the following theorem.

Theorem 1. *For all $n \geq 1$ we have $U_n = \lfloor \frac{1}{2}(n+1) \rfloor$. Furthermore, if $A \subset \{1, 2, \ldots, n\}$ is a prime-sumset-free set with $|A| = U_n$, then all elements of A have the same parity.*

With this, the natural and more challenging question coming in mind is that what is the largest cardinality of $A \subset \{1, 2, \ldots, n\}$ satisfying (1) when A contains elements of both parities?

A prime-sumset-free subset A of $\{1, 2, \ldots, n\}$ is called an extremal prime-sumset-free subset of $\{1, 2, \ldots, n\}$ if $A \cup \{a\}$ is not a prime-sumset-free subset for any $a \in \{1, 2, \ldots, n\} \setminus A$. Let $(PF_k(n))_{k \geq 1}$ be the sequence of cardinalities of all the extremal prime-sumset-free subsets of $\{1, 2, \ldots, n\}$ with $PF_1(n) > PF_2(n) > \cdots$. Then by the theorem of Chen we have $PF_1(n) = U_n = \lfloor \frac{1}{2}(n+1) \rfloor$.

In this paper we provide the sequence $(PF_k(n))$ for a few small values of n and we study the finite monotonic strictly decreasing sequence $PF_k(n)$ with the largest term $\lfloor \frac{1}{2}(n+1) \rfloor$ and the smallest term 2 (if it exists) from the lower end of the sequence for all sufficiently large values of n. In particular, we show that if $n \geq 10$ then there is no extremal prime-sumset-free subset of $\{1, 2, \ldots, n\}$ of order 2 and if $n \geq 13$ then there is no extremal prime-sumset-free subset of $\{1, 2, \ldots, n\}$ of order 3. That is in the case $n \geq 10$ the sequence $PF_k(n)$ will not take the value 2 and hence will terminate at 3 (if it exists) and in the case $n \geq 13$ the sequence $PF_k(n)$ will not take the value 3 and hence will terminate at 4 (if it exists) as $PF_k(n) \neq 1$ for any n. We also prove that for each integer $k \geq 2$, there exists a n_k such that if $n \geq n_k$ then there does not exist any extremal prime-sumset-free subset of $\{1, 2, \ldots, n\}$ of length k.

2 Main Results

Sometimes we use $[n]$ for $\{1, 2, \ldots, n\}$ and PSFS for prime-sumset-free subset in this section. Observe that any proper subset of an extremal prime-sumset-free subset of $\{1, 2, \ldots, n\}$ of order l can not be an extremal prime-sumset-free subset of $\{1, 2, \ldots, n\}$. Thus if we have extremal prime-sumset-free subsets of $\{1, 2, \ldots, n\}$ of both orders k and l where $k < l$ then the extremal prime-sumset-free subset of $\{1, 2, \ldots, n\}$ of order k will not be a subset of the extremal prime-sumset-free subset of $\{1, 2, \ldots, n\}$ of order l. In the table below we provide the sequence $(PF_k(n))$ and an example of extremal prime-sumset-free subset of $[n]$ for each term of the sequence for all $1 \leq n \leq 14$.

n	Sequence $PF_k(n)$ with an extremal prime-sumset-free subset of $[n]$ for each term
1	not defined
2	not defined
3	$(PF_k(3)) \equiv (2)$; ({1,3})
4	$(PF_k(4)) \equiv (2)$; ({1,3})
5	$(PF_k(5)) \equiv (3,2)$; ({1,3,5}, {2,4})
6	$(PF_k(6)) \equiv (3,2)$; ({1,3,5}, {4,5})
7	$(PF_k(7)) \equiv (4,3,2)$; ({1,3,5,7}, {2,4,6}, {4,5})
8	$(PF_k(8)) \equiv (4,3,2)$; ({1,3,5,7}, {2,7,8}, {4,5})
9	$(PF_k(9)) \equiv (5,4,3,2)$; ({1,3,5,7,9}, {2,4,6,8},{2,7,8}, {4,5})
10	$(PF_k(10)) \equiv (5,3)$; ({1,3,5,7,9},{2,7,8})
11	$(PF_k(11)) \equiv (6,5,4,3)$; ({1,3,5,7,9,11}, {2,4,6,8,10}, {4,5,10,11}, {1,7,8})
12	$(PF_k(12)) \equiv (6,4,3)$; ({1,3,5,7,9,11}, {4,5,10,11}, {1,7,8})
13	$(PF_k(13)) \equiv (7,6,4)$; ({1,3,5,7,9,11,13}, {2,4,6,8,10,12}, {1,7,8,13})
14	$(PF_k(14)) \equiv (7,5,4)$; ({1,3,5,7,9,11,13}, {1,7,11,13,14}, {3,9,12,13})

Table 1: $PF_k(n)$ and extremal-prime-sumset-free subset of $[n]$ for $1 \leq n \leq 14$

After having a look at the above table a natural question coming in mind is that for a given positive integer n does there exist an extremal prime-sumset-free subset of $[n]$ of each order l, where $2 \leq l < \lfloor \frac{1}{2}(n+1) \rfloor$? We shall see that if $n \geq 10$ then there is no extremal prime-sumset-free subset of $[n]$ of order 2 and if $n \geq 13$ then there is no extremal prime-sumset-free subset of $[n]$ of order 3. Before we prove these results we give all extremal prime-sumset-free subset of $[n]$ of order 2 and 3 for each n where $1 \leq n \leq 13$ in the following table.

n	All extremal PSFSs of $[n]$ of order 2	All extremal PSFSs of $[n]$ of order 3
3	{1,3}	not defined
4	{1,3,}, {2,4}	not defined
5	{2,4}, {4,5}	{1,3,5}
6	{6,3}, {4,5}	{1,3,5}, {2,4,6}
7	{2,7}, {6,3}, {4,5}	{2,4,6}
8	{6,3}, {4,5}	{1,7,8}, {2,7,8}
9	{4,5}	{1,7,8}, {2,7,8}, {3,6,9}
10	not exists	{1,7,8}, {2,7,8}, {3,6,9}, {4,5,10}
11	not exists	{1,7,8}, {2,7,8}, {3,6,9}
12	not exists	{1,7,8}, {2,7,8}
13	not exists	not exists

Table 2: Extremal prime-sumset-free subset of $[n]$ of orders 2 and 3 for $1 \leq n \leq 13$

Theorem 2. *If $n \geq 10$ then there does not exist any extremal prime-sumset-free subset of*

$[n]$ *of order* 2.

Proof. The proof is by induction on n. If $n = 10$ then there does not exist any extremal prime-sumset-free subset of $[10]$ of order 2. Indeed, we know that if $n \geq 6$ then an extremal prime-sumset-free subset of order 2 (if exists) will contain integers of opposite parity. Therefore, the only possibilities of two elements subsets of $[10]$ to be an extremal prime-sumset-free subset are the following sets:

$$\{1, 8\}, \{3, 6\}, \{5, 4\}, \{5, 10\}, \{7, 2\}, \{7, 8\}, \{9, 6\}$$

But none of the above sets is an extremal prime-sumset-free subset as each one is a subset of some prime-sumset-free subsets of order 3 given below.

$$\{1, 8, 7\}, \{3, 6, 9\}, \{5, 4, 10\}, \{7, 2, 8\}.$$

Hence the theorem is true for $n = 10$. Now let the theorem is true for $k \geq 10$. Without loss of generality we can assume that k is odd. Take all subsets $\{k + 1, a\}$ where a is odd and $1 \leq a \leq k$ such that $k + 1 + a$ is a composite integer. We shall show that there exists an integer t such that $t \in [k + 1] \setminus \{k + 1, a\}$ and both $k + 1 + t$ and $a + t$ are composite.

CASE I: $1 \leq a \leq k - 1$. By induction the subset $\{k, a + 1\}$ is not an extremal prime-sumset-free subset of $[k]$. Hence there exists an integer $l \in [k] \setminus \{k, a + 1\}$ such that all $l + k, k + a + 1$ and $a + 1 + l$ are composite integers. Now if l is odd then taking $t = l + 1$ and if l is even then taking $t = l - 1$, we see that $\{k + 1, a, t\}$ is a prime-sumset-free subset of $[k + 1]$. Hence the theorem is true in this case.

CASE II: $a = k$. The subset $\{k + 1, a\}$ becomes $\{k + 1, k\}$. In this case the integer t is given in the following table depending on k.

Units digit of k	t
1	4
3	2
5	10
7	8
9	6

Table 3: t as a function of k

This completes the proof of the theorem. □

Theorem 3. *If $n \geq 13$ then there does not exist any extremal prime-sumset-free subset of $[n]$ of order* 3.

Proof. Let $A = \{a_1, a_2, a_3\}$ be a prime-sumset-free subset of $[n]$ of order 3. We are required to show that A is not an extremal prime-sumset-free subset of $[n]$. If all elements of A are of same parity then clearly, A is not an extremal prime-sumset-free subset of $[n]$ as $n \geq 13$. So, let us assume that A contains elements of both parities.

CASE I: A contains two odd integers and an even integer. Without loss of generality let a_1 be even.

Subcase (i): $2 \leq a_1 \leq 10$. First let $a_2, a_3 \in \{1, 3, 5, \dots, 13\}$. In this case the only possibilities of the set A for each a_1 is $\{2, 7, 13\}$, $\{4, 5, 11\}$, $\{6, 3, 9\}$, $\{8, 1, 7\}$, $\{8, 1, 13\}$, $\{8, 7, 13\}$, $\{10, 5, 11\}$. None of the above subsets is an extremal prime-sumset-free subset because of the following prime-sumset-free subsets.

$$\{2, 7, 13, 8\}, \{4, 5, 11, 10\}, \{6, 3, 9, 12\}, \{8, 1, 7, 13\}.$$

Secondly, let a_2, a_3 be any odd numbers in A not necessarily from the set $\{1, 3, 5, \dots, 13\}$ and the set A be distinct from the sets given in the above list: $\{2, 7, 13\}$, $\{4, 5, 11\}$, $\{6, 3, 9\}$, $\{8, 1, 7\}$, $\{8, 1, 13\}$, $\{8, 7, 13\}$, $\{10, 5, 11\}$. We have the following:
$2 \in A \Rightarrow A \cup \{a\}$ is a prime-sumset-free subset for some $a \in \{7, 13\}$,
$4 \in A \Rightarrow A \cup \{a\}$ is a prime-sumset-free subset for some $a \in \{5, 11\}$,
$6 \in A \Rightarrow A \cup \{a\}$ is a prime-sumset-free subset for some $a \in \{3, 9\}$,
$8 \in A \Rightarrow A \cup \{a\}$ is a prime-sumset-free subset for some $a \in \{1, 7, 13\}$,
$10 \in A \Rightarrow A \cup \{a\}$ is a prime-sumset-free subset for some $a \in \{5, 11\}$.
Consequently, A is not an extremal prime-sumset-free subset. Thus we have seen that if $2 \leq a_1 \leq 10$ then A is not an extremal prime-sumset-free subset.

Subcase (ii): $12 \leq a_1 \leq 28$. In the list below, for each a_1 we provide at least three distinct odd integers from $[n]$ such that their sum with a_1 give composite integers. Hence if $12 \leq a_1 \leq 28$, we do not have extremal prime-sumset-free subsets of $[n]$ of length 3.

List: $\{12, 3, 9, 13\}$, $\{14, 1, 7, 11\}$, $\{16, 5, 9, 11\}$, $\{18, 3, 7, 9\}$, $\{20, 1, 5, 15\}$, $\{22, 3, 5, 13\}$, $\{24, 1, 3, 11\}$, $\{26, 1, 9, 19\}$, $\{28, 5, 7, 17\}$.

Subcase (iii): $a_1 \geq 30$. Depending on a_1 we have at least three distinct odd integers from $[n]$ such that their sum with a_1 give composite integers. Hence if $a_1 \geq 30$, we do not have extremal prime-sumset-free subsets of $[n]$ of length 3.

Units digit of a_1	Three distinct odd integers
0	5,15,25
2	3,13,23
4	1,11,21
6	9,19,29
8	7,17,27

Table 4: Examples for Subcase (iii)

CASE II: A contains two even integers and an odd integer. Without loss of generality let a_1 be odd.

Subcase (i): $1 \leq a_1 \leq 11$. First let $a_2, a_3 \in \{2, 4, 6, \dots, 12\}$. In this case the only possibility of the set A for each a_1 is $\{3, 6, 12\}, \{5, 4, 10\}, \{7, 2, 8\}, \{9, 6, 12\}, \{11, 4, 10\}$. None of the above subsets is an extremal prime-sumset-free subset because of the following prime-sumset-free subsets.

$$\{3,6,12,9\},\{5,4,10,11\},\{7,2,8,13\}.$$

Secondly, let a_2, a_3 be any even numbers of A not necessarily from the set $\{2,4,\ldots,12\}$ and the set A be distinct from the sets given in the above list: $\{3,6,12\}$, $\{5,4,10\}$, $\{7,2,8\}$, $\{9,6,12\}$, $\{11,4,10\}$. We have the following:
$1 \in A$ and $A = \{1,8,14\}$ then A is not an extremal prime-sumset-free subset because $\{1,8,14,7\}$ is a prime-sumset-free subset.
$1 \in A$ and $A \neq \{1,8,14\} \Rightarrow A \cup \{a\}$ is a prime-sumset-free subset for some $a \in \{8,14\}$,
$3 \in A \Rightarrow A \cup \{a\}$ is a prime-sumset-free subset for some $a \in \{6,12\}$,
$5 \in A \Rightarrow A \cup \{a\}$ is a prime-sumset-free subset for some $a \in \{4,10\}$,
$7 \in A \Rightarrow A \cup \{a\}$ is a prime-sumset-free subset for some $a \in \{2,8\}$,
$9 \in A \Rightarrow A \cup \{a\}$ is a prime-sumset-free subset for some $a \in \{6,12\}$,
$11 \in A \Rightarrow A \cup \{a\}$ is a prime-sumset-free subset for some $a \in \{4,10\}$.
Consequently, A is not an extremal prime-sumset-free subset.

Subcase (ii): $13 \leq a_1 \leq 29$. In the list below, for each a_1 we provide at least three distinct even integers from $[n]$ such that their sum with a_1 give composite integers. Hence if $13 \leq a_1 \leq 29$, we do not have extremal prime-sumset-free subsets of $[n]$ of length 3.

List: $\{13,2,8,12\}$, $\{15,6,10,12\}$, $\{17,4,8,10\}$, $\{19,6,8,16\}$, $\{21,4,6,14\}$, $\{23,2,4,12\}$, $\{25,2,10,20\}$, $\{27,8,18,6\}$, $\{29,4,6,16\}$.

Subcase (iii): $a_1 \geq 31$. Depending on a_1 we have at least three distinct even integers from $[n]$ such that their sum with a_1 give composite integers. Hence if $a_1 \geq 31$, we do not have extremal prime-sumset-free subsets of $[n]$ of length 3.

Units digit of a_1	Three distinct even integers
1	4,14,24
3	2,12,22
5	10,20,30
7	8,18,28
9	6,16,26

Table 5: Examples for Subcase (iii)

This completes the proof of the theorem. □

Inspired by the above two theorems, we have the following:

Theorem 4. *For each integer $k \geq 2$, there exists a n_k such that if $n \geq n_k$ then there does not exist any extremal prime-sumset-free subsets of $[n]$ of length k.*

One can consult [2] for definitions, notation and results used in the proof below.

Proof. Suppose that $A = \{a_1, a_2, \ldots, a_k\}$ be a prime-sumset-free subset of $[n]$. Clearly, by the definition of psfs we have that $a_i + a_j = m_{ij}$ is a composite integer for each $1 \leq i, j \leq k$. Let $1 < m^i_j < m_{ij}$ and $m^i_j | m_{ij}$ for each fixed $1 \leq i \leq k$ and for each $1 \leq j \leq k$. Let us consider the following k simultaneous congruences

$$\begin{aligned} x &\equiv -a_1 \pmod{m^i_1} \\ x &\equiv -a_2 \pmod{m^i_2} \\ &\vdots \\ x &\equiv -a_k \pmod{m^i_k} \end{aligned}$$

for each $1 \leq i \leq k$. Each one of the k simultaneous congruences is consistent being a_i is the common solution for the ith simultaneous congruence for each $1 \leq i \leq k$. Set

$$a^i_{k+1} = a_i + t[m^i_1, m^i_2, \ldots, m^i_k],$$

for each $1 \leq i \leq k$ and $t \geq 0$. We see that a^i_{k+1} is a solution of the ith simultaneous congruence and hence all $a_1 + a^i_{k+1}, a_2 + a^i_{k+1}, \ldots, a_k + a^i_{k+1}$ are composite integers for each $1 \leq i \leq k$. Choosing t minimum such that $a^i_{k+1} \notin \{a_1, a_2, \ldots, a_k\}$, we see that for $n \geq n_k = \max_i a^i_{k+1} := a_{k+1}$ we have all $a_1 + a_{k+1}, a_2 + a_{k+1}, \ldots, a_k + a_{k+1}$ are composite integers. Consequently, the set $\{a_1, a_2, \ldots, a_{k+1}\}$ is a prime-sumset-free subset of $[n]$ and hence A is not an extremal prime-sumset-free subset of $[n]$ of length k. This proves the theorem. □

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