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On Divisibility of Fibonomial Coefficients by 3

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Abstract

Let F_n be the nth Fibonacci number. For $1 \le k \le m-1$ let

$$\begin{bmatrix} m \\ k \end{bmatrix}_F = \frac{F_m F_{m-1} \cdots F_{m-k+1}}{F_1 \cdots F_k} \tag{1}$$

be the corresponding Fibonomial coefficient. In this paper, we present some divisibility properties of $\begin{bmatrix} sn \\ n \end{bmatrix}_F$ by 3, for some positive integers n and s. In particular, among other things, we prove that $3 \mid \begin{bmatrix} 3^{a+1} \\ 3^a \end{bmatrix}_F$, for all $a \ge 1$.

1 Introduction

Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence given by the recurrence relation $F_{n+2} = F_{n+1} + F_n$, with $F_0 = 0$ and $F_1 = 1$. These numbers are well-known for possessing amazing properties (consult [7] together with its very extensive annotated bibliography for additional references and history).

In 1915 Fontené published a one–page note [3] suggesting a generalization of binomial coefficients, replacing the natural numbers by the terms of an arbitrary sequence (A_n) of real or complex numbers.

Since 1964, there has been an accelerated interest in the *Fibonomial coefficients* $\left[{m \atop k}\right]_F$, which correspond to the choice $A_n = F_n$, thus are defined, for $1 \leq k \leq m$, in the following way

$$\begin{bmatrix} m \\ k \end{bmatrix}_F = \frac{F_m F_{m-1} \cdots F_{m-k+1}}{F_1 \cdots F_k}.$$

It is surprising that this quantity will always take integer values. This can be shown by an induction argument and the recursion formula

$$\begin{bmatrix} m \\ k \end{bmatrix}_F = F_{k+1} \begin{bmatrix} m-1 \\ k \end{bmatrix}_F + F_{m-k-1} \begin{bmatrix} m-1 \\ k-1 \end{bmatrix}_F,$$

which is a consequence of the formula $F_m = F_{k+1}F_{m-k} + F_k F_{m-k-1}$.

Several authors became interested in the divisibility properties of binomial coefficients. Among several interesting results on this subject, we mention the following facts:

- An integer $n \geq 2$ is prime if and only if all the binomial coefficients $\binom{n}{1}, \ldots, \binom{n}{n-1}$ are divisible by n.

- A surprising result, proved by D. Singmaster [13], is that any integer divides almost all binomial coefficients. More precisely, let d be an integer and let $f(N)$ be the number of binomial coefficients $\binom{n}{k}$ divisible by d, with $n < N$. Then

$$\lim_{N\to\infty} \frac{f(N)}{N(N+1)/2} = 1.$$

  Since there are $N(N+1)/2$ binomial coefficients $\binom{n}{k}$, with $n < N$, the density of the set of binomial coefficients divisible by d is 1.

- Recently Zhi-Wei Sun [16] proved, for example, that for any positive integers k, ℓ and n the following holds

$$\ell n + 1 \mid k\binom{kn + \ell n}{\ell n}.$$

Other interesting results concerning divisibility properties of binomial coefficients can be found in [2, 4]. For example the following holds: $3 \mid \binom{sn}{n}$, for all $n \geq 1$ if and only if $3 \mid s$.

In a very recent paper, the authors [10] proved, among other things, that $2 \mid \left[{2n \atop n}\right]_F$ for all integers $n > 1$. However, the same is not valid when we replace 2 by 3, as can be seen by the example $3 \nmid \left[{3\cdot 2 \atop 2}\right]_F = 40$.

In this paper, we shall study similar problems for the Fibonomial coefficients. Thus we shall deal with the divisibility of $\begin{bmatrix} sn \\ n \end{bmatrix}_F$ by 3 for some positive integers n and s.

Our first result gives a necessary and sufficient condition for that $3 \mid \begin{bmatrix} 3n \\ n \end{bmatrix}_F$.

Theorem 1. *We have $3 \nmid \begin{bmatrix} 3n \\ n \end{bmatrix}_F$ if and only if $n = 1$ or $n = 2 \cdot 3^k$ for $k \geq 0$.*

As we said before, we have $3 \mid \binom{sn}{n}$ for all $n \geq 1$ if and only if $3 \mid s$. Our next theorem gives a related result in the Fibonomial context.

Theorem 2. *Let $s > 0$ be an integer. The number $\begin{bmatrix} sn \\ n \end{bmatrix}_F$ is a multiple of 3 for all $n \geq 1$ if and only if $s \equiv 0 \pmod{12}$.*

We organize this paper as follows. In Section 2, we will recall some useful properties of the Fibonacci numbers such as a result concerning the 3-adic order of F_n. Sections 3 and 4 are devoted to the proof of Theorems 1 and 2, respectively.

2 Auxiliary results

Before proceeding further, we recall some facts about the Fibonacci numbers for the convenience of the reader.

Lemma 3. *We have*

(a) *$F_n \mid F_m$ if and only if $n \mid m$.*

(b) *If $m > k > 1$, then*

$$\begin{bmatrix} m \\ k \end{bmatrix}_F = \frac{F_m}{F_k} \begin{bmatrix} m-1 \\ k-1 \end{bmatrix}_F.$$

Item (a) can be proved by using the well-known Binet formula

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \text{ for } n \geq 0,$$

where $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$. The proof of item (b) follows directly from definition (1). We refer the reader to [1, 6, 15, 11] for more properties and additional bibliography.

The p-adic order (or valuation) of r, $\nu_p(r)$, is the exponent of the highest power of a prime p which divides r. The p-adic order of Fibonacci numbers was completely characterized, see [5, 9, 12, 14]. For instance, from the main results of Lengyel [9], we extract the following result.

Lemma 4. *For $n \geq 1$, we have*

$$\nu_3(F_n) = \begin{cases} \nu_3(n) + 1, & \text{if } n \equiv 0 \pmod{4}; \\ 0, & \text{if } n \not\equiv 0 \pmod{4}. \end{cases}$$

A proof of a more general result can be found in [9, pp. 236–237 and Section 5].

Lemma 5. *For any integer $k \geq 1$ and p prime, we have*

$$\frac{k}{p-1} - \left\lfloor \frac{\log k}{\log p} \right\rfloor - 1 \leq \nu_p(k!) \leq \frac{k-1}{p-1}, \tag{2}$$

where, as usual, $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.

Proof. Recall the well-known Legendre formula [8]:

$$\nu_p(k!) = \frac{k - s_p(k)}{p-1}, \tag{3}$$

where $s_p(k)$ is the sum of digits of k in base p. Since k has $\lfloor \log k / \log p \rfloor + 1$ digits in base p, and each digit is at most $p-1$, we get

$$1 \leq s_p(k) \leq (p-1)\left(\left\lfloor \frac{\log k}{\log p} \right\rfloor + 1\right). \tag{4}$$

Therefore, the inequality in (2) follows from (3) and (4). $\square$

Now we are ready to deal with the proofs of our theorems.

3 Proof of Theorem 1

In order to make our proof clearer, we shall split the statement of Theorem 1 in four propositions.

Proposition 6. (*The "if" part*) *For all integers $k \geq 0$, we have that $3 \nmid \left[\begin{smallmatrix} 2\cdot 3^{k+1} \\ 2\cdot 3^k \end{smallmatrix}\right]_F$.*

Proof. Using the definition of the Fibonomial coefficients, we have

$$\begin{bmatrix} 2\cdot 3^{k+1} \\ 2\cdot 3^k \end{bmatrix}_F = \prod_{i=1}^{2\cdot 3^k} \frac{F_{2\cdot 3^{k+1} - 2\cdot 3^k + i}}{F_i} = \prod_{i=1}^{2\cdot 3^k} \frac{F_{4\cdot 3^k + i}}{F_i}$$

and hence

$$\nu_3\left(\begin{bmatrix} 2\cdot 3^{k+1} \\ 2\cdot 3^k \end{bmatrix}_F\right) = \nu_3\left(\prod_{i=1}^{2\cdot 3^k} \frac{F_{4\cdot 3^k + i}}{F_i}\right) = \sum_{i=1}^{2\cdot 3^k} \left(\nu_3\left(F_{4\cdot 3^k + i}\right) - \nu_3\left(F_i\right)\right).$$

Thus, for proving that the assertion holds, it suffices to show that $\nu_3(F_i) = \nu_3(F_{4\cdot 3^k + i})$ for all $i = 1, 2, \ldots, 2\cdot 3^k$. Since $4\cdot 3^k + i \equiv i \pmod 4$ and $3 \mid F_n$ if and only if $4 \mid n$ (Lemma 3 (a)), we need only to consider the case when $4 \mid i$, that is, when $i = 4t_i$, for some positive integer t_i. From this fact together with Lemma 4, we obtain

$$\nu_3(F_i) = \nu_3(F_{4t_i}) = \nu_3(t_i) + 1$$

while

$$\nu_3(F_{4\cdot 3^k+i}) = \nu_3(F_{4(3^k+t_i)}) = \nu_3(3^k+t_i)+1 = \nu_3(t_i)+1,$$

where in the last equality above, we used that $t_i < 3^k$ (since $4t_i = i \le 2\cdot 3^k$) and the clear identity $\nu_p(a+b) = \min\{\nu_p(a), \nu_p(b)\}$, when $\nu_p(a) \neq \nu_p(b)$, where p is any prime. This completes the proof. $\square$

For the "only if" part, we have

Proposition 7. *For all integers $a \ge 2$ and $k \ge 1$, we have that $3 \mid \left[{2^a\cdot 3^{k+1} \atop 2^a\cdot 3^k}\right]_F$.*

Proof. By Lemma 3 (b), we can write

$$\begin{bmatrix} 2^a\cdot 3^{k+1} \\ 2^a\cdot 3^k \end{bmatrix}_F = \frac{F_{2^a\cdot 3^{k+1}}}{F_{2^a\cdot 3^k}} \begin{bmatrix} 2^a\cdot 3^{k+1}-1 \\ 2^a\cdot 3^k-1 \end{bmatrix}_F$$

and so it suffices to prove that $3 \mid F_{2^a\cdot 3^{k+1}}/F_{2^a\cdot 3^k}$. Indeed, using Lemma 4 and the fact that $a \ge 2$ to get

$$\nu_3\left(\frac{F_{2^a\cdot 3^{k+1}}}{F_{2^a\cdot 3^k}}\right) = \nu_3(F_{2^a\cdot 3^{k+1}}) - \nu_3(F_{2^a\cdot 3^k}) = \nu_3(2^a\cdot 3^{k+1}) - \nu_3(2^a\cdot 3^k) = 1.$$

$\square$

Proposition 8. *For all integers $a \ge 1$, we have that $3 \mid \left[{3^{a+1} \atop 3^a}\right]_F$.*

Proof. Let us suppose, without loss of generality, that a is even (the case of a odd can be handled in much the same way). Since $3 \mid \left[{27 \atop 9}\right]_F$, we may assume that $a > 2$. By definition of the Fibonomial coefficient, we have

$$\begin{bmatrix} 3^{a+1} \\ 3^a \end{bmatrix}_F = \frac{F_{3^{a+1}}\cdots F_{2\cdot 3^a+1}}{F_1\cdots F_{3^a}}.$$

So we must to compare the 3-adic order of the numerator and denominator of the previous fraction. Since $3 \mid F_n$ if and only if $4 \mid n$, we need only to consider the 3-adic order of the $\lfloor 3^a/4\rfloor$ numbers $F_4, \ldots, F_{3^a-1}$, for the denominator, and $F_{2\cdot 3^a+2}, \ldots, F_{3^{a+1}-3}$, for the numerator. So, in the first case, we use Lemma 4 to obtain

$$\begin{aligned}
\mathcal{S}_1 &:= \nu_3(F_1\cdots F_{3^a}) \\
&= \nu_3(F_4)+\nu_3(F_8)+\cdots+\nu_3(F_{3^a-1}) \\
&= (\nu_3(4)+1)+(\nu_3(8)+1)+\cdots+(\nu_3(3^a-1)+1) \\
&= \nu_3(4)+\nu_3(8)+\cdots+\nu_3(3^a-1)+\left\lfloor\frac{3^a}{4}\right\rfloor. \qquad (5)
\end{aligned}$$

We note that (5) could be rewritten as

$$\begin{aligned}
\nu_3(F_1\cdots F_{3^a}) &= \nu_3(12)+\nu_3(24)+\cdots+\nu_3\left(12\left\lfloor\frac{3^a-1}{12}\right\rfloor\right)+\left\lfloor\frac{3^a}{4}\right\rfloor \\
&= \nu_3\left(\left\lfloor\frac{3^a-1}{12}\right\rfloor!\right)+\left\lfloor\frac{3^a-1}{12}\right\rfloor+\left\lfloor\frac{3^a}{4}\right\rfloor.
\end{aligned}$$

For the 3-adic order of numerator, we proceed as before to get

$$\begin{aligned}
\mathcal{S}_2 &:= \nu_3(F_{3^{a+1}} \cdots F_{2\cdot 3^a+1}) = \nu_3(F_{3^{a+1}-3}) + \cdots + \nu_3(F_{2\cdot 3^a+2}) \\
&= \nu_3(3^{a+1}-3) + \cdots + \nu_3(2\cdot 3^a + 2) + \left\lfloor \frac{3^a}{4} \right\rfloor \\
&= \nu_3(3(3^a-1)) + \cdots + \nu_3(3(3^a - (3^{a-1}-2))) + \left\lfloor \frac{3^a}{4} \right\rfloor \\
&= \nu_3(3^a-1) + \cdots + \nu_3(3^a - (3^{a-1}-2)) + \\
&+ \frac{3^{a-1}+1}{4} + \left\lfloor \frac{3^a}{4} \right\rfloor . \qquad (6)
\end{aligned}$$

Observe that there exist several common terms in sums (5) and (6), so combining them gives

$$\begin{aligned}
\mathcal{S}_2 - \mathcal{S}_1 &= \frac{3^{a-1}+1}{4} - (\nu_3(4) + \cdots + \nu_3(3^a - (3^{a-1}+3))) \\
&= \frac{3^{a-1}+1}{4} - \left(\nu_3(12) + \cdots + \nu_3\left(12\left\lfloor \frac{2\cdot 3^{a-1}-3}{12} \right\rfloor\right)\right) \\
&= \frac{3^{a-1}+1}{4} - \left\lfloor \frac{2\cdot 3^{a-1}-3}{12} \right\rfloor - \nu_3\left(\left\lfloor \frac{2\cdot 3^{a-1}-3}{12} \right\rfloor !\right). \qquad (7)
\end{aligned}$$

Hence, when a is even, we have

$$\nu_3\left(\begin{bmatrix} 3^{a+1} \\ 3^a \end{bmatrix}_F\right) = \frac{3^{a-1}+1}{4} - \left\lfloor \frac{2\cdot 3^{a-1}-3}{12} \right\rfloor - \nu_3\left(\left\lfloor \frac{2\cdot 3^{a-1}-3}{12} \right\rfloor !\right). \qquad (8)$$

The fact that $\lfloor x \rfloor \leq x$ yields the following estimate

$$\nu_3\left(\begin{bmatrix} 3^{a+1} \\ 3^a \end{bmatrix}_F\right) \geq \frac{3^{a-1}+6}{12} - \nu_3\left(\left\lfloor \frac{2\cdot 3^{a-1}-3}{12} \right\rfloor !\right). \qquad (9)$$

By applying Lemma 5 to the 3-adic order in the right-hand side of (9), we obtain

$$\nu_3\left(\left\lfloor \frac{2\cdot 3^{a-1}-3}{12} \right\rfloor !\right) \leq \frac{2\cdot 3^{a-1}-15}{24}. \qquad (10)$$

Now, we combine (9) and (10) to derive

$$\nu_3\left(\begin{bmatrix} 3^{a+1} \\ 3^a \end{bmatrix}_F\right) \geq \frac{3^{a-1}+6}{12} - \frac{2\cdot 3^{a-1}-15}{24} = \frac{27}{24} > 0$$

as desired. Since $27/24 = 1.125$, we actually proved that $\nu_3\left(\begin{bmatrix} 3^{a+1} \\ 3^a \end{bmatrix}_F\right) \geq 2$, when $a > 2$ is even. $\square$

For the sake of completeness, we remark that the related formula to (8), for a odd is

$$\nu_3\left(\begin{bmatrix}3^{a+1}\\ 3^a\end{bmatrix}_F\right) = \frac{3^{a-1}-1}{4} - \left\lfloor \frac{2\cdot 3^{a-1}-2}{12} \right\rfloor - \nu_3\left(\left\lfloor \frac{2\cdot 3^{a-1}-2}{12} \right\rfloor !\right). \tag{11}$$

To finish the "only if" case, all that remains is to prove the following.

Proposition 9. *For all integers $k \geq 1$ and every prime $p > 3$, we have that $3 \mid \begin{bmatrix}3pk\\ pk\end{bmatrix}_F$.*

Proof. To prove this assertion, we take the same approach as in the proof of Proposition 8. Instead of demonstrating the general case, which is notationally complicated, we restrict ourselves to a particular case that captures the exact essence of our idea. For that, we shall consider $p \equiv k \equiv 1 \pmod{12}$. Although there are several cases to consider (48 cases depending on the residue of p and k modulo 12), the proofs are very similar.

First, we write

$$\begin{bmatrix}3pk\\ pk\end{bmatrix}_F = \frac{F_{3pk}\cdots F_{2pk-1}}{F_1\cdots F_{pk}}.$$

We note that again, by Lemma 3 (a) (for $n = 4$), we need to take care only of the following sequences of indexes: $4, 8, \ldots, pk-1$ and $2pk+2, \ldots, 3pk-3$ which correspond to indexes of the denominator and numerator respectively, having non-zero 3-adic valuation. Thus

$$\begin{aligned}\mathcal{M}_1 := \nu_3(F_1F_2\cdots F_{pk}) &= \nu_3(F_4) + \nu_3(F_8) + \cdots + \nu_3(F_{pk-1})\\ &= (\nu_3(4)+1) + \cdots + (\nu_3(pk-1)+1)\\ &= \nu_3(4) + \cdots + \nu_3(pk-1) + \left\lfloor \frac{pk}{4} \right\rfloor \end{aligned} \tag{12}$$

and

$$\begin{aligned}\mathcal{M}_2 &:= \nu_3(F_{3pk}F_{3pk-1}\cdots F_{2pk+1})\\ &= \nu_3(F_{3pk-3}) + \nu_3(F_{3pk-7}) + \cdots + \nu_3(F_{2pk+2})\\ &= \nu_3(3pk-3) + \cdots + \nu_3(3pk-(pk-2)) + \left\lfloor \frac{pk}{4} \right\rfloor\\ &= \nu_3(3(pk-1)) + \cdots + \nu_3\left(3\left(pk - \frac{pk-10}{3}\right)\right) + \left\lfloor \frac{pk}{4} \right\rfloor\\ &= \nu_3(pk-1) + \cdots + \nu_3\left(pk - \frac{pk-10}{3}\right) + \frac{pk-1}{12} +\\ &+ \left\lfloor \frac{pk}{4} \right\rfloor. \end{aligned} \tag{13}$$

Observe that there exist several common terms in sums (12) and (13), thus combining them

$$\begin{aligned}\mathcal{M}_2 - \mathcal{M}_1 &= \frac{pk-1}{12} - \left(\nu_3(4) + \cdots + \nu_3\left(\frac{2pk+2}{3}\right)\right)\\ &= \frac{pk-1}{12} - \left(\nu_3(12) + \cdots + \nu_3\left(12\left\lfloor\frac{2pk+2}{36}\right\rfloor\right)\right)\\ &= \frac{pk-1}{12} - \left\lfloor\frac{2pk+2}{36}\right\rfloor - \nu_3\left(\left\lfloor\frac{2pk+2}{36}\right\rfloor!\right). \qquad (14)\end{aligned}$$

Hence

$$\begin{aligned}\nu_3\left(\begin{bmatrix}3pk\\ pk\end{bmatrix}_F\right) &= \frac{pk-1}{12} - \left\lfloor\frac{2pk+2}{36}\right\rfloor - \nu_3\left(\left\lfloor\frac{2pk+2}{36}\right\rfloor!\right)\\ &\geq \frac{pk-5}{36} - \nu_3\left(\left\lfloor\frac{2pk+2}{36}\right\rfloor!\right) \qquad (15)\\ &\geq \frac{pk-5}{36} - \frac{pk-17}{36} \qquad (16)\\ &= \frac{1}{3} > 0,\end{aligned}$$

where we used that $\lfloor x\rfloor \leq x$ (in (15)) and that $\nu_3(\lfloor(2pk+2)/36\rfloor!) \leq (pk-17)/36$, by Lemma 5 (in (16)). The proof is then complete. $\square$

4 Proof of Theorem 2

Proof. For the "if" part, we write $s = 12k$, then

$$\begin{bmatrix}sn\\ n\end{bmatrix}_F = \begin{bmatrix}12kn\\ n\end{bmatrix}_F = \frac{F_{12kn}}{F_n}\begin{bmatrix}12kn-1\\ n-1\end{bmatrix}_F.$$

Now, it suffices to prove that $3 \mid F_{12kn}/F_n$. For that we use Lemma 4 to obtain

$$\nu_3\left(\frac{F_{12kn}}{F_n}\right) = \nu_3(F_{12kn}) - \nu_3(F_n) = \nu_3(kn) + 2 - \nu_3(F_n)$$

and so

$$\nu_3\left(\frac{F_{12kn}}{F_n}\right) = \begin{cases}2 + \nu_3(kn), & \text{if } 4 \nmid n;\\ 1 + \nu_3(k), & \text{if } 4 \mid n.\end{cases}$$

Summarizing, we conclude that $\nu_3\left(F_{12kn}/F_n\right) \geq 1$ and this completes the proof of this case.

Let k be an integer belonging to $\{1, \ldots, 11\}$. Suppose that $s \equiv k \pmod{12}$, in order to prove the "only if" part, it suffices to exhibit a positive integer N_k such that $3 \nmid \begin{bmatrix}sN_k\\ N_k\end{bmatrix}_F$. Of course, $N_k = 1$ is an example of such number for $k = 1, 2, 3, 5, 6, 7, 9, 10, 11$, because $\begin{bmatrix}s\\ 1\end{bmatrix}_F = F_s$ is not a multiple of 3, if $4 \nmid s$. We claim that $N_4 = N_8 = 4$ are also examples. In fact, we have

$$\nu_3\left(\begin{bmatrix}4s\\4\end{bmatrix}_F\right)=\nu_3\left(\frac{F_{4s}F_{4s-1}F_{4s-2}F_{4s-3}}{F_1F_2F_3F_4}\right)=\nu_3\left(\frac{F_{4s}}{3}\right)=(\nu_3(4s)+1)-1=0,$$

where we used that $3\nmid s$ when $s\equiv 4,8 \pmod{12}$. □

5 Conclusion

In this paper, we study divisibility properties of the Fibonomial coefficients $\left[{m \atop k}\right]_F$ by 3. Among other things, we give necessary and sufficient conditions for $\left[{sn \atop n}\right]_F$ being divisible by 3, for some integers s and n. Our method is effective and possibly can be used to work on divisibility by larger primes. However, it is important to get noticed that for each prime, this study brings a lot of particular technicalities.

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