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Infinite Products Involving $\zeta(3)$ and Catalan's Constant

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Abstract

We present some infinite product formulas for $e^{\frac{7\zeta(3)}{\pi^2}}$, $e^{\frac{4G}{\pi}}$ and $e^{\frac{2G}{\pi}\pm\frac{1}{2}}$, where G is Catalan's constant. We relate these formulas to similar ones obtained by Guillera and Sondow in the context of their systematic study of Lerch's transcendent. Our proofs are entirely elementary.

1 Introduction

This paper studies some infinite product formulas involving two classical constants, namely $\zeta(3)$ and Catalan's constant, whose definition we now recall:

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$$

and

$$G \;=\; \sum_{m=0}^{\infty} \frac{\left(-1\right)^m}{\left(2m+1\right)^2} \;.$$

The following formulas are reminiscent of similar formulas obtained by Guillera and Sondow in [5]:

Proposition 1. *The following formulas hold:*

$$e^{\frac{7\zeta(3)}{4\pi^2}+\frac{1}{4}} = \lim_{m\to\infty} \prod_{n=1}^{2m+1} \frac{1}{\sqrt[4]{e}} \left(1-\frac{1}{n+1}\right)^{\frac{n(n+1)}{2}(-1)^n}. \tag{1}$$

$$e^{\frac{7\zeta(3)}{4\pi^2}-\frac{1}{4}} = \lim_{m\to\infty} \prod_{n=1}^{2m} \sqrt[4]{e} \left(1-\frac{1}{n+1}\right)^{\frac{n(n+1)}{2}(-1)^n}. \tag{2}$$

$$e^{\frac{7\zeta(3)}{\pi^2}} = \lim_{m\to\infty} \left(\frac{2^{2^2}\cdot 4^{4^2}\cdot 6^{6^2}\cdots (2m)^{(2m)^2}}{1^{1^2}\cdot 3^{3^2}\cdot 5^{5^2}\cdots (2m-1)^{(2m-1)^2}}\right)^4 \left(\frac{(2m+2)^{4m+5}}{(2m+1)^{12m+9}}\right)^m. \tag{3}$$

Proposition 2. *The following formulas hold:*

$$e^{\frac{2G}{\pi}-\frac{1}{2}} = \lim_{m\to\infty} \prod_{n=1}^{2m} \left(1-\frac{2}{2n+1}\right)^{n(-1)^n}. \tag{4}$$

$$e^{\frac{2G}{\pi}+\frac{1}{2}} = \lim_{m\to\infty} \prod_{n=1}^{2m+1} \left(1-\frac{2}{2n+1}\right)^{n(-1)^n}. \tag{5}$$

$$e^{\frac{4G}{\pi}} = \lim_{m\to\infty} \left(\frac{3^3\cdot 7^7\cdot 11^{11}\cdots (4m-1)^{4m-1}}{1^1\cdot 5^5\cdot 9^9\cdots (4m-3)^{4m-3}}\right)^2 \frac{(4m+3)^{2m+1}}{(4m+1)^{6m+1}}. \tag{6}$$

We claim no novelty for the formulas themselves; our only purpose here is to present completely elementary proofs of these formulas and to establish the not-so-obvious facts below:

Fact 3. Formula (3) is equivalent to the following formula given by Guillera and Sondow [5, Example 5.3]:

$$\begin{aligned}
e^{\frac{7\zeta(3)}{4\pi^2}} &= e^{\sum\limits_{n=1}^{\infty} \frac{n(n+1)}{2^{n+3}} \sum\limits_{k=0}^{n} (-1)^{k+1} \binom{n}{k} \log(k+1)} \\
&= \prod_{n=1}^{\infty} \left(\prod_{k=0}^{n} \left(k+1\right)^{(-1)^{k+1}\binom{n}{k}}\right)^{\frac{n(n+1)}{2^{n+3}}} \\
&= \left(\frac{2^1}{1^1}\right)^{\frac{1\cdot 2}{2^4}} \left(\frac{2^2}{1^1\cdot 3^1}\right)^{\frac{2\cdot 3}{2^5}} \left(\frac{2^3\cdot 4^1}{1^1\cdot 3^3}\right)^{\frac{3\cdot 4}{2^6}} \left(\frac{2^4\cdot 4^4}{1^1\cdot 3^6\cdot\ 5^1}\right)^{\frac{4\cdot 5}{2^7}} \cdots .
\end{aligned}$$

Fact 4. Formula (4) follows from rearranging the factors of the following formula given by Guillera and Sondow [5, Example 5.5]:

$$
\begin{aligned}
e^{\frac{G}{\pi}} &= e^{\sum\limits_{n=1}^{\infty} \frac{n}{2^{n+2}} \sum\limits_{k=0}^{n} (-1)^{k+1} \binom{n}{k} \log(2k+1)} \\
&= \prod_{n=1}^{\infty} \Big(\prod_{k=0}^{n} \Big(2k+1\Big)^{(-1)^{k+1} \binom{n}{k}} \Big)^{\frac{n}{2^{n+2}}} \\
&= \left(\frac{3^1}{1^1}\right)^{\frac{1}{2^3}} \left(\frac{3^2}{1^1 \cdot 5^1}\right)^{\frac{2}{2^4}} \left(\frac{3^3 \cdot 7^1}{1^1 \cdot 5^3}\right)^{\frac{3}{2^5}} \left(\frac{3^4 \cdot 7^4}{1^1 \cdot 5^6 \cdot 9^1}\right)^{\frac{4}{2^6}} \cdots,
\end{aligned}
$$

which in turn is equivalent to formula (6).

2 Proof of Proposition 2

We begin with the following formula which is a classically known Fourier expansion (see, for example, Exercise 11.15(c) in [1, p. 338]):

Formula 5. *Let* $\sigma \in \big[-\frac{1}{2}, \frac{1}{2}\big] \setminus \{0\}$*. Then*

$$\sum_{m=0}^{\infty} \frac{\cos\Big(\pi\big(2m+1\big)\ \sigma\Big)}{2m+1} \ = \ \frac{1}{2} \ \log \Big| \ \cot \ \Big(\frac{\pi}{2} \ \sigma\Big) \ \Big|.$$

The following formula, which follows directly from Formula 5 by integrating both sides over the interval $\Big[0, \ \frac{1}{2}\Big]$, is also well-known (see, for example, [2, p. 239]):

Formula 6.

$$G \ = \ \int_{\theta=0}^{\pi/4} \log \Big(\cot\theta\Big) \ d\theta.$$

By applying integration by parts to the latter integral, we obtain

Corollary 7.

$$G \ = \ \frac{1}{2} \int_{\alpha=0}^{1/2} \frac{\pi^2 \ \alpha}{\sin\big(\pi \ \alpha\big)} \ d\alpha.$$

The following formula is also well-known (see, for example, [8, p. 155]):

Formula 8. *Let* $\alpha \in \mathbb{R} \setminus \mathbb{Z}$ *and* $s \in \Big(-\frac{1}{2}, \frac{1}{2}\Big)$*. Then*

$$\cos\big(2\pi\alpha s\big) \ = \ \frac{\sin\big(\pi\alpha\big)}{\pi} \ \Big(\frac{1}{\alpha} \ + \ 2\alpha \sum_{m=1}^{\infty} \frac{\big(-1\big)^m}{\alpha^2 - m^2} \ \cos\big(2\pi m s\big)\Big).$$

Setting $s = 0$ in Formula 8 gives:

Corollary 9. *Let* $\alpha \in \mathbb{R} \setminus \mathbb{Z}$. *Then*

$$1 \;=\; \frac{\sin\big(\pi\alpha\big)}{\pi}\,\Big(\frac{1}{\alpha} \;+\; 2\alpha \sum_{m=1}^{\infty} \frac{\big(-1\big)^m}{\alpha^2 - m^2}\Big).$$

Lemma 10. *Let* $m \in \mathbb{Z}, \;\; m \geq 1$. *Then*

$$\int_{\alpha=0}^{1/2} \frac{\alpha^2}{\alpha^2 - m^2}\, d\alpha \;=\; \frac{1}{2} \;+\; \frac{m}{2}\;\log\,\frac{2m-1}{2m+1}.$$

Proof. This is straightforward:

$$\begin{aligned}
\int_{\alpha=0}^{1/2} \frac{\alpha^2}{\alpha^2 - m^2}\, d\alpha &= \frac{1}{2}\int_{\alpha=0}^{1/2} \Big(2 \;+\; \frac{m}{\alpha - m} \;-\; \frac{m}{\alpha + m}\Big)\, d\alpha \\
&= \frac{1}{2}\Bigg[2\alpha \;+\; m\;\log\Big(-\alpha + m\Big) \;-\; m\;\log\Big(\alpha + m\Big)\Bigg]_{\alpha=0}^{1/2} \\
&= \frac{1}{2} \;+\; \frac{m}{2}\;\log\,\frac{2m-1}{2m+1}.
\end{aligned}$$

∎

We now proceed with the proof of formula (4). By Corollary 9, we have

$$\frac{\pi^2\alpha}{\sin\big(\pi\alpha\big)} \;=\; \pi \;+\; 2\pi\alpha^2 \sum_{m=1}^{\infty} \frac{\big(-1\big)^m}{\alpha^2 - m^2}.$$

Integrating both sides with respect to α over the interval $\Big[0, \frac{1}{2}\Big]$ gives

$$\int_{\alpha=0}^{1/2} \frac{\pi^2\alpha}{\sin\big(\pi\alpha\big)}\, d\alpha \;=\; \frac{\pi}{2} \;+\; 2\pi \int_{\alpha=0}^{1/2} \Big(\sum_{m=1}^{\infty} \frac{(-1)^m\,\alpha^2}{\alpha^2 - m^2}\Big)\, d\alpha. \tag{7}$$

Consider the sequence of functions

$$f_m\big(\alpha\big) \;=\; \frac{\big(-1\big)^m\,\alpha^2}{\alpha^2 - m^2}$$

on the interval $I = [0, \frac{1}{2}]$, where $m = 1, 2, \ldots$. Since $\alpha \in I$, we clearly have

$$\big|f_m\big(\alpha\big)\big| \;=\; \frac{\alpha^2}{\big|\alpha^2 - m^2\big|} \;\leq\; \frac{\frac{1}{4}}{m^2 - \frac{1}{4}} \;=\; \frac{1}{4m^2 - 1} \;\leq\; \frac{1}{2m^2},$$

for all m. Since

$$\sum_{m=1}^{\infty} \frac{1}{2m^2}$$

converges, it follows from the Weierstrass M-test that the series

$$\sum_{m=1}^{\infty} f_m(\alpha)$$

converges uniformly on I, and, by well-known principles, (see, for example, [1, Thm. 9.9, p. 226]), can therefore be integrated term by term. In other words, if we set

$$a_m = (-1)^m \left(1 + m \log \frac{2m-1}{2m+1}\right),$$

then (7) and Lemma 10 imply that

$$\int_{\alpha=0}^{1/2} \frac{\pi^2 \alpha}{\sin(\pi\alpha)}\, d\alpha = \frac{\pi}{2} + \pi \sum_{m=1}^{\infty} a_m. \tag{8}$$

The left-hand side of (8) is a definite integral of the continuous function $\frac{\pi^2\alpha}{\sin(\pi\alpha)}$ over the interval $[0, \frac{1}{2}]$. Hence the left-hand side of (8) is a real number which implies that

$$\lim_{m\to\infty} a_m = 0.$$

Keeping this in mind, define

$$A_n = \sum_{m=1}^{n} a_m.$$

Then, by (8), we have

$$\begin{aligned}
\int_{\alpha=0}^{1/2} \frac{\pi^2\alpha}{\sin(\pi\alpha)}\, d\alpha &= \frac{\pi}{2} + \pi \lim_{n\to\infty} A_n = \frac{\pi}{2} + \pi \lim_{N\to\infty} A_{2N} = \frac{\pi}{2} + \pi \lim_{N\to\infty} \sum_{m=1}^{N} (a_{2m-1} + a_{2m}) \\
&= \pi \left(\frac{1}{2} + \lim_{N\to\infty} \sum_{m=1}^{N} \left(-(2m-1)\log\frac{4m-3}{4m-1} + 2m \log \frac{4m-1}{4m+1} \right)\right) \\
&= \pi \left(\frac{1}{2} + \lim_{N\to\infty} \log \prod_{m=1}^{N} \frac{(4m-1)^{4m-1}}{(4m-3)^{2m-1}(4m+1)^{2m}} \right) \\
&= \pi \left(\frac{1}{2} + \log \prod_{m=1}^{\infty} \frac{(4m-1)^{4m-1}}{(4m-3)^{2m-1}(4m+1)^{2m}} \right).
\end{aligned}$$

By Corollary 7, the left-hand side equals $2G$, therefore

$$\frac{2G}{\pi} - \frac{1}{2} = \log \prod_{m=1}^{\infty} \frac{(4m-1)^{4m-1}}{(4m-3)^{2m-1}(4m+1)^{2m}}.$$

Therefore,

$$
\begin{aligned}
e^{\frac{2G}{\pi}-\frac{1}{2}} &= \lim_{m\to\infty} \frac{3^3}{1^1\cdot 5^2}\cdot\frac{7^7}{5^3\cdot 9^4}\cdot\frac{11^{11}}{9^5\cdot 13^6}\cdots\frac{(4m-1)^{4m-1}}{(4m-3)^{2m-1}(4m+1)^{2m}} \\
&= \lim_{m\to\infty} \frac{3^3\cdot 7^7\cdot 11^{11}\cdots(4m-1)^{4m-1}}{5^5\cdot 9^9\cdot 13^{13}\cdots(4m-3)^{4m-3}\cdot(4m+1)^{2m}} \\
&= \lim_{m\to\infty}\left(\frac{1}{3}\right)^{-1}\left(\frac{3}{5}\right)^{2}\left(\frac{5}{7}\right)^{-3}\cdots\left(\frac{4m-1}{4m+1}\right)^{2m} = \lim_{m\to\infty}\prod_{n=1}^{2m}\left(1-\frac{2}{2n+1}\right)^{n(-1)^n},
\end{aligned}
$$

and this completes the proof of formula (4). Multiplying both sides of the latter formula by e and using the fact that

$$
e = \lim_{m\to\infty}\left(1-\frac{2}{4m+3}\right)^{-(2m+1)}
$$

gives formula (5). Finally, multiplying formulas (4) and (5) together and expanding gives formula (6).

3 Proof of Proposition 1

We will first prove formula (1).

Lemma 11. *Let $m \in \mathbb{N}$ and $\delta \in \left(0, \frac{1}{2}\right)$. Then*

$$
\begin{aligned}
&\pi^2 \int_{\sigma=\delta}^{1/2} \left(\frac{1}{2} - \sigma\right) \frac{\cos\Big(\pi\big(2m+1\big)\sigma\Big)}{2m+1}\, d\sigma \\
&= \frac{\cos\Big(\pi\big(2m+1\big)\delta\Big)}{\big(2m+1\big)^3} + \frac{\pi\left(\delta-\frac{1}{2}\right)\sin\Big(\pi\big(2m+1\big)\delta\Big)}{\big(2m+1\big)^2}.
\end{aligned}
$$

Proof. This is straightforward integration by parts:

$$
\begin{aligned}
&\int_{\sigma=\delta}^{1/2} \left(\frac{1}{2} - \sigma\right) \frac{\cos\Big(\pi\big(2m+1\big)\sigma\Big)}{2m+1}\, d\sigma \\
&= \left[\left(\frac{1}{2}-\sigma\right)\frac{\sin\Big(\pi\big(2m+1\big)\sigma\Big)}{\pi\big(2m+1\big)^2}\right]_{\sigma=\delta}^{1/2} + \int_{\sigma=\delta}^{1/2}\frac{\sin\Big(\pi\big(2m+1\big)\sigma\Big)}{\pi\big(2m+1\big)^2}\, d\sigma \\
&= \left(\delta-\frac{1}{2}\right)\frac{\sin\Big(\pi\big(2m+1\big)\delta\Big)}{\pi\big(2m+1\big)^2} - \left[\frac{\cos\Big(\pi\big(2m+1\big)\sigma\Big)}{\pi^2\big(2m+1\big)^3}\right]_{\sigma=\delta}^{1/2},
\end{aligned}
$$

and the claim follows. □

Corollary 12. *Let $m \in \mathbb{N}$. Then*

$$
\pi^2 \int_{\sigma=0}^{1/2} \left(\frac{1}{2}-\sigma\right)\frac{\cos\Big(\pi\big(2m+1\big)\sigma\Big)}{2m+1}\, d\sigma = \frac{1}{\big(2m+1\big)^3}.
$$

Proof. In Lemma 11, let $\delta \to 0+$. $\square$

We now recall the following basic formula:

$$\sum_{m=0}^{\infty} \frac{1}{\left(2m+1\right)^3} \;=\; \frac{7}{8}\,\zeta\left(3\right). \tag{9}$$

We will establish the following:

Formula 13.

$$\zeta\left(3\right) \;=\; \frac{4}{7}\,\pi\;G \;-\; \frac{2}{7}\,\pi^2 \int_{\sigma=0}^{1/2} \frac{\pi\sigma^2}{\sin\left(\pi\sigma\right)}\;d\sigma.$$

Proof. First, we may rewrite Formula 6 as

$$G \;=\; \int_{\sigma=0}^{1/2} \frac{\pi}{2}\;\log\left(\;\cot\left(\frac{\pi}{2}\sigma\right)\right)\;d\sigma. \tag{10}$$

Second, by (9) and Corollary 12, we have

$$\frac{7}{8}\,\zeta\left(3\right) \;=\; \sum_{m=0}^{\infty} \pi^2 \int_{\sigma=0}^{1/2} \left(\frac{1}{2} \;-\; \sigma\right) \frac{\cos\left(\pi\left(2m+1\right)\sigma\right)}{2m+1}\;d\sigma. \tag{11}$$

Fix $\delta \in \left(0, \frac{1}{2}\right)$. For each $n \in \mathbb{N}$, define the function

$$F_n\left(\sigma\right) \;=\; \sum_{m=0}^{n} \left(\frac{1}{2} \;-\; \sigma\right) \frac{\cos\left(\pi(2m+1)\sigma\right)}{2m+1}$$

on the interval $I \;=\; \left[\delta, \dfrac{1}{2}\right]$. The sequence

$$\left\{\sum_{m=0}^{n} \cos\left(\pi\left(2m+1\right)\sigma\right)\right\}_{n\in\mathbb{N}}$$

of functions is uniformly bounded on I by $(2\sin(\pi\delta))^{-1}$ (see [1, Formula (15), p. 198] or [6, Item 185.5, p. 316]), whereas the sequence

$$\left\{\left(\frac{1}{2} \;-\; \sigma\right) \frac{1}{2m+1}\right\}_{m\in\mathbb{N}}$$

clearly tends monotonically to 0 uniformly on I. Hence by applying Dirichlet's test for uniform convergence (see [1, Thm. 9.15, p. 230] or or [6, p. 347]), it follows that the sequence of functions $F_n\left(\sigma\right)$ converges uniformly on I. Therefore, the series

$$\sum_{m=0}^{\infty} \left(\frac{1}{2} \;-\; \sigma\right) \frac{\cos\left(\pi(2m+1)\sigma\right)}{2m+1}$$

can be integrated term by term on I. Hence, Lemma 11 establishes the following

Formula 14.

$$\pi^2 \int_{\sigma=\delta}^{1/2} \sum_{m=0}^{\infty} \left(\frac{1}{2} - \sigma\right) \frac{\cos\Big(\pi(2m+1)\sigma\Big)}{2m+1}\, d\sigma$$
$$= \sum_{m=0}^{\infty} \frac{\cos\Big(\pi(2m+1)\delta\Big)}{(2m+1)^3} + \pi\left(\delta - \frac{1}{2}\right) \sum_{m=0}^{\infty} \frac{\sin\Big(\pi(2m+1)\delta\Big)}{(2\,m\,+\,1)^2}.$$

Now take the limits of both sides of the latter formula as $\delta \to 0+$. By the Weierstrass M-test, both series on the right-hand side of Formula 14 are uniformly convergent series of functions of δ on the interval $I = [\delta, \frac{1}{2}]$. Therefore, we can interchange limits and infinite sums on the right-hand side of Formula 14 (see [1, Thm. 9.7, p. 220]). By (11), it follows that

$$\pi^2 \int_{\sigma=0}^{1/2} \sum_{m=0}^{\infty} \left(\frac{1}{2} - \sigma\right) \frac{\cos\Big(\pi(2m+1)\sigma\Big)}{2m+1} = \frac{7}{8}\,\zeta(3). \tag{12}$$

Combining (10), (12) and Formula 5 gives

$$\frac{7}{8}\,\zeta(3) = \pi^2 \int_{\sigma=0}^{1/2} \left(\frac{1}{2} - \sigma\right) \frac{1}{2} \log\left(\cot\left(\frac{\pi}{2}\sigma\right)\right) d\sigma$$
$$= \frac{\pi^2}{4} \left(\int_{\sigma=0}^{1/2} \log\left(\cot\left(\frac{\pi}{2}\sigma\right)\right) d\sigma - \int_{\sigma=0}^{1/2} 2\sigma \log\left(\cot\left(\frac{\pi}{2}\sigma\right)\right) d\sigma \right)$$
$$= \frac{\pi}{2} G - \frac{\pi^2}{2} \int_{\sigma=0}^{1/2} \sigma\, \log\left(\cot\left(\frac{\pi}{2}\sigma\right)\right) d\sigma.$$

In short,

$$\zeta(3) = \frac{4}{7}\,\pi\, G - \frac{4}{7}\,\pi^2 \int_{\sigma=0}^{1/2} \sigma\, \log\left(\cot\left(\frac{\pi}{2}\sigma\right)\right) d\sigma. \tag{13}$$

Formula 13 now follows because

$$\int_{\sigma=0}^{1/2} \sigma\, \log\left(\cot\left(\frac{\pi}{2}\sigma\right)\right) d\sigma$$
$$= \left[\frac{\sigma^2}{2} \log\left(\cot\left(\frac{\pi}{2}\sigma\right)\right)\right]_{\sigma=0}^{1/2} - \int_{\sigma=0}^{1/2} \frac{\sigma^2}{2} \frac{1}{\cot\left(\frac{\pi}{2}\,\sigma\right)} \frac{-1}{\left(\sin\left(\frac{\pi}{2}\sigma\right)\right)^2} \frac{\pi}{2}\, d\,\sigma$$
$$= 0 + \int_{\sigma=0}^{1/2} \frac{\sigma^2}{2} \frac{1}{\cos\left(\frac{\pi}{2}\sigma\right)\, \sin\left(\frac{\pi}{2}\sigma\right)} \frac{\pi}{2}\, d\sigma = \frac{1}{2} \int_{\sigma=0}^{1/2} \frac{\pi\sigma^2}{\sin\left(\pi\sigma\right)}\, \sigma.$$

□

The following statement is similar to Lemma 10.

Lemma 15. *Let $m \in \mathbb{Z}$, $m \geq 1$. Then*

$$\int_{\sigma=0}^{1/2} \frac{\sigma^3}{\sigma^2 \; - \; m^2} \, d\sigma \;=\; \frac{1}{8} \;+\; \frac{m^2}{2} \; \log \frac{4m^2 - 1}{4m^2}.$$

Proof. This is straightforward:

$$\begin{aligned}\int_{\sigma=0}^{1/2} \frac{\sigma^3}{\sigma^2 \; - \; m^2} \, d\sigma &= \int_{\sigma=0}^{1/2} \left(\sigma \;+\; m^2 \; \frac{\sigma}{\sigma^2 \; - m^2} \right) d\sigma \\ &= \left[\frac{1}{2}\sigma^2 \;+\; m^2 \; \frac{1}{2} \; \log\Big(-\sigma^2 + m^2 \Big) \right]_{\sigma=0}^{1/2} \\ &= \frac{1}{8} \;+\; \frac{m^2}{2} \; \log \frac{m^2 \; - \; \frac{1}{4}}{m^2}.\end{aligned}$$

$\square$

Lemma 16.

$$\lim_{n \to \infty} \frac{e^n}{\left(1 + \frac{1}{n}\right)^{n^2}} \;=\; \lim_{n \to \infty} \frac{e^{-n}}{\left(1 - \frac{1}{n}\right)^{n^2}} \;=\; \sqrt{e}.$$

Proof. By taking logarithms, it suffices to show that

$$\lim_{n \to \infty} \left(n \;-\; n^2 \; \log\left(1 \;+\; \frac{1}{n}\right)\right) \;=\; \frac{1}{2} \;=\; \lim_{n \to \infty} \left(-\, n \;-\; n^2 \; \log\left(1 \;-\; \frac{1}{n}\right)\right).$$

This follows by substituting $x \;=\; \pm\frac{1}{n}$ in the Maclaurin series of the function $\log\big(1 \;+\; x\big)$ and using continuity. $\square$

We now proceed with the proof of formula (1). By Corollary 9, we have

$$\frac{\pi \, \sigma^2}{\sin\big(\pi\sigma\big)} \;=\; \sigma \;+\; 2\sigma^3 \sum_{m=1}^{\infty} \frac{\big(-1\big)^m}{\sigma^2 - m^2}.$$

Integrating both sides with respect to σ over the interval $\left[0, \frac{1}{2}\right]$ and using formula (4) (and its proof) and Lemma 15 gives

$$\int_{\sigma=0}^{1/2} \frac{\pi\sigma^2}{\sin\left(\pi\sigma\right)}\ d\sigma\ =\ \int_{\sigma=0}^{1/2} \left(\sigma\ +\ 2\sigma^3 \sum_{m=1}^{\infty} \frac{\left(-1\right)^m}{\sigma^2 - m^2}\right)\ d\sigma$$

$$= \int_{\sigma=0}^{1/2} \sigma\ d\sigma\ +\ 2\int_{\sigma=0}^{1/2} \left(\sum_{m=1}^{\infty}\left(-1\right)^m \frac{\sigma^3}{\sigma^2 - m^2}\right)\ d\sigma$$

$$= \frac{1}{8}\ +\ 2\sum_{m=1}^{\infty}\left(-1\right)^m \int_{\sigma=0}^{1/2} \frac{\sigma^3}{\sigma^2 - m^2}\ d\sigma$$

$$= \frac{1}{8}\ +\ 2\sum_{m=1}^{\infty}\left(-1\right)^m \left(\frac{1}{8}\ +\ \frac{m^2}{2}\ \log\frac{4m^2-1}{4m^2}\right)$$

$$= \frac{1}{8}\ +\ \sum_{m=1}^{\infty}\left(-1\right)^m \left(\frac{1}{4}\ +\ m^2\ \log\ \frac{4m^2-1}{4m^2}\right),$$

which equals

$$\frac{1}{8}\ +\ \sum_{\ell=1}^{\infty}\ \left(-\left(\frac{1}{4}\ +\ \left(2\ell-1\right)^2\ \log\ \frac{4\left(2\ell-1\right)^2-1}{4\left(2\ell-1\right)^2}\right)\right.$$

$$\left.+\left(\frac{1}{4}\ +\ \left(2\ell\right)^2\ \log\ \frac{4\left(2\ell\right)^2-1}{4\left(2\ell\right)^2}\right)\right)$$

$$= \frac{1}{8}\ +\ \sum_{\ell=1}^{\infty}\ \left(-\left(2\ell-1\right)^2\ \log\ \frac{4\left(2\ell-1\right)^2-1}{4\left(2\ell-1\right)^2}\right.$$

$$\left.+\left(2\ell\right)^2\ \log\ \frac{4\left(2\ell\right)^2-1}{4\left(2\ell\right)^2}\right)$$

$$= \frac{1}{8}\ +\ \sum_{\ell=1}^{\infty}\ \log\ \left(\frac{\left(4\ell-1\right)^{4\ell-1}\ \left(4\ell+1\right)^{(2\ell)^2}\ \left(4\ell-2\right)^{2(2\ell-1)^2}}{\left(4\ell\right)^{2(2\ell)^2}\ \left(4\ell-3\right)^{(2\ell-1)^2}}\right)$$

$$= \frac{1}{8}\ +\ \sum_{\ell=1}^{\infty}\ \log\ \left(\frac{\left(4\ell-1\right)^{4\ell-1}}{\left(4\ell-3\right)^{2\ell-1}\ \left(4\ell+1\right)^{2\ell}}\right)$$

$$+\sum_{\ell=1}^{\infty}\ \log\ \left(\frac{\left(4\ell+1\right)^{4\ell^2+2\ell}\ \left(4\ell-2\right)^{2(2\ell-1)^2}}{\left(4\ell\right)^{2(2\ell)^2}\ \left(4\ell-3\right)^{4\ell^2-6\ell+2}}\right)$$

$$= \frac{2G}{\pi}\ -\ \frac{3}{8}\ +\ 2\ \sum_{\ell=1}^{\infty}\ \log\ \left(\frac{\left(4\ell+1\right)^{2\ell^2+\ell}\ \left(4\ell-2\right)^{(2\ell-1)^2}}{\left(4\ell\right)^{(2\ell)^2}\ \left(4\ell-3\right)^{2\ell^2-3\ell+1}}\right).$$

Therefore, by Formula 13, it follows that

$$\frac{7}{4\pi^2}\,\zeta\Big(3\Big) \;=\; \frac{3}{16} \;+\; \log \prod_{\ell=1}^{\infty} \frac{\Big(4\ell\Big)^{(2\ell)^2}\,\Big(4\ell-3\Big)^{2\ell^2-3\ell+1}}{\Big(4\ell+1\Big)^{2\ell^2+\ell}\,\Big(4\ell-2\Big)^{(2\ell-1)^2}}\;. \tag{14}$$

Now the latter infinite product can be written as

$$\lim_{N\to\infty}\prod_{\ell=1}^{N} 2^{4\ell-1}\;\frac{\Big(2\ell\Big)^{(2\ell)^2}}{\Big(2\ell-1\Big)^{(2\ell-1)^2}}\;\frac{\Big(4\Big(\ell-1\Big)+1\Big)^{2(\ell-1)^2+(\ell-1)}}{\Big(4\ell+1\Big)^{2\ell^2+\ell}},$$

which equals

$$\begin{aligned}
&\lim_{N\to\infty}\left(\frac{2}{4N+1}\right)^{2N^2+N}\;\prod_{\ell=1}^{N}\frac{\Big(2\ell\Big)^{(2\ell)^2}}{\Big(2\ell-1\Big)^{(2\ell-1)^2}}\\
&=\lim_{N\to\infty}\frac{2^{2N^2+N}\,\Big(2N+1\Big)^{(2N+1)^2}}{\Big(4N+1\Big)^{2N^2+N}}\;\prod_{\ell=1}^{N}\frac{\Big(2\ell\Big)^{(2\ell)^2}}{\Big(2\ell+1\Big)^{(2\ell+1)^2}}\\
&=\lim_{N\to\infty}\Bigg(\frac{2^{2N^2+N}\,\Big(2N+1\Big)^{(2N+1)^2}}{\Big(4N+1\Big)^{2N^2+N}}\;\frac{2}{\Big(2N+2\Big)^{(2N+1)(N+1)}}\\
&\times\prod_{\ell=1}^{N}\frac{\Big(2\ell\Big)^{(2\ell+1)\ell}\,\Big(2\ell+2\Big)^{(2\ell+1)(\ell+1)}}{\Big(2\ell+1\Big)^{(2\ell+1)^2}}\Bigg)\\
&=\lim_{N\to\infty}\Bigg(e^{\frac{N+1}{2}}\,\left(\frac{4N+2}{4N+1}\right)^{2N^2+N}\,\left(\frac{2N+1}{2N+2}\right)^{(2N+1)(N+1)}\,\frac{2}{\sqrt{e}}\\
&\times\prod_{\ell=1}^{N}\frac{\Big(2\ell\Big)^{(2\ell+1)\ell}\,\Big(2\ell+2\Big)^{(2\ell+1)(\ell+1)}}{\sqrt{e}\,\Big(2\ell+1\Big)^{(2\ell+1)^2}}\Bigg).
\end{aligned}$$

We claim that

$$\lim_{N\to\infty}\frac{e^{\frac{N+1}{2}}}{\Big(1-\frac{1}{4N+2}\Big)^{(2N+1)N}\,\Big(1+\frac{1}{2N+1}\Big)^{(2N+1)(N+1)}} \;=\; e^{-\frac{3}{16}}. \tag{15}$$

Indeed, by Lemma 16, we have

$$\lim_{N\to\infty} \frac{e^{N+\frac{1}{4}}}{\left(1+\frac{1}{2N+1}\right)^{\frac{(2N+1)^2}{2}}} = 1 = \lim_{N\to\infty} \frac{e^{\frac{1}{2}}}{\left(1+\frac{1}{2N+1}\right)^{\frac{2N+1}{2}}}$$

and

$$\lim_{N\to\infty} \frac{e^{-\frac{N}{2}-\frac{5}{16}}}{\left(1-\frac{1}{4N+2}\right)^{\frac{(4N+2)^2}{8}}} = 1 = \lim_{N\to\infty} \frac{e^{\frac{1}{4}}}{\left(1-\frac{1}{4N+2}\right)^{-\frac{4N+2}{4}}},$$

hence (15) follows. Combining (14) with (15) gives

$$e^{\frac{7\zeta(3)}{4\pi^2}} = \frac{2}{\sqrt{e}} \prod_{l=1}^{\infty} \frac{(2l)^{(2l+1)l}\ (2l+2)^{(2l+2)(l+1)}}{\sqrt{e}\ (2l+1)^{2l+1)^2}}.$$

Therefore,

$$\begin{aligned} e^{\frac{7\zeta(3)}{4\pi^2}} &= \frac{2}{\sqrt{e}} \lim_{m\to\infty} \prod_{n=1}^{m} \frac{(2n)^{(2n+1)n}\ (2n+2)^{(2n+2)(n+1)}}{\sqrt{e}\ (2n+1)^{(2n+1)^2}} \\ &= \frac{(2m+2)^{(2m+1)(m+1)}}{e^{\frac{m+1}{2}}} \frac{2^{2^2}\cdot 4^{4^2}\cdot 6^{6^2}\cdots (2m)^{(2m)^2}}{3^{3^2}\cdot 5^{5^2}\cdot 7^{7^2}\cdots (2m+1)^{(2m+1)^2}} \\ &= e^{-\frac{1}{4}} \lim_{m\to\infty} \prod_{n=1}^{2m+1} \frac{1}{\sqrt[4]{e}}\left(1-\frac{1}{n+1}\right)^{\frac{n(n+1)}{2}(-1)^n}. \end{aligned}$$

and this completes the proof of formula (1).

It remains to prove formulas (2) and (3). Note that

$$\prod_{n=1}^{2m} \sqrt[4]{e}\left(1-\frac{1}{n+1}\right)^{\frac{n(n+1)}{2}(-1)^n} = \frac{\left(1-\frac{1}{2m+2}\right)^{(2m+1)(m+1)}}{e^{-(m+\frac{1}{4})}} \prod_{n=1}^{2m+1} \frac{1}{\sqrt[4]{e}}\left(1-\frac{1}{n+1}\right)^{\frac{n(n+1)}{2}(-1)^n}.$$

Therefore, formula (2) will follow from formula (1) once we show that

$$\lim_{m\to\infty} \frac{e^{-(2m+\frac{1}{2})}}{\left(1-\frac{1}{2m+2}\right)^{(2m+2)(2m+1)}} = e.$$

This follows by writing $(2m+2)(2m+1)$ as $(2m+2)^2-(2m+2)$ and using Lemma 16.

Now multiplying formulas (1) and (2) together and squaring gives

$$\begin{aligned} e^{\frac{7\zeta(3)}{\pi^2}} &= \lim_{m\to\infty} \frac{1}{\sqrt{e}}\left(\frac{2m+1}{2m+2}\right)^{-(2m+1)(2m+2)} \prod_{n=1}^{2m}\left(1-\frac{1}{n+1}\right)^{2n(n+1)(-1)^n} \\ &= \lim_{m\to\infty} \frac{1}{\sqrt{e}}\left(\frac{(2m+2)^{2m+2}}{(2m+1)^{6m+2}}\right)^{2m+1} \left(\frac{2^{2^2}\cdot 4^{4^2}\cdot 6^{6^2}\cdots (2m)^{(2m)^2}}{1^{1^2}\cdot 3^{3^2}\cdot 5^{5^2}\cdots (2m-1)^{(2m-1)^2}}\right)^4 \end{aligned}$$

Formula (3) is now a consequence of the equality

$$\frac{1}{\sqrt{e}} = \lim_{m\to\infty} \Big(\frac{2m+1}{2m+2}\Big)^{m+2}.$$

4 Proof of Facts 3 and 4

By formula (3) and its proof, it suffices to show that the total exponent of $k+1$ in the infinite product expansion given by Guillera and Sondow [5, Example 5.3] equals $\big(-1\big)^{k+1}\big(k+1\big)^{2}$, for all $k \in \mathbb{N}$. The exponent in question equals

$$\big(-1\big)^{k+1} \sum_{n=k}^{\infty} \binom{n}{k} \frac{n^2+n}{2^{n+3}} = \frac{\big(-1\big)^{k+1}}{8\ \cdot \big(k!\big)} \sum_{n=k}^{\infty} \frac{\big(n+1\big)\ n^2\ \big(n-1\big)\ \cdots\ \big(n-k+1\big)}{2^n}$$
$$= \frac{\big(-1\big)^{k+1}}{8\ \cdot \big(k!\big)} \Big(\sum_{n=k}^{\infty} \frac{\big(n+2\big)\ \big(n+1\big)\ \cdots\ \big(n-k+1\big)}{2^n} - 2\sum_{n=k}^{\infty} \frac{\big(n+1\big)\ n\ \cdots\ \big(n-k+1\big)}{2^n}\Big)$$
$$= \frac{\big(-1\big)^{k+1}}{8\ \cdot \big(k!\big)} \Big(\frac{1}{2^k} \sum_{m=0}^{\infty} \frac{\big(m+k+2\big)\ \big(m+k+1\big)\ \cdots\ \big(m+1\big)}{2^m}$$
$$- \frac{1}{2^{k-1}} \sum_{m=0}^{\infty} \frac{\big(m+k+1\big)\ \big(m+k\big)\ \cdots\ \big(m+1\big)}{2^m}\Big).$$

We have the following lemma:

Lemma 17. *For all $k \in \mathbb{N}$, we have*

$$\sum_{m=0}^{\infty} \frac{\big(m+k+1\big)\ \big(m+k\big)\ \cdots\ \big(m+1\big)}{2^m} = 2^{(k+2)}\ \cdot\ \big(\big(k+1\big)!\big).$$

Proof. This follows by term-by-term $\big(k+1\big)$-fold differentiation of the geometric series

$$\sum_{n=0}^{\infty} x^n = (1-x)^{-1}$$

and subsequent evaluation at $x = \frac{1}{2}$. ◻

Therefore, by Lemma 17, the exponent in question equals

$$\frac{(-1)^{k+1}}{8\ \cdot\ (k!)} \big(8\ \cdot\ \big(\big(k+2\big)!\big) - 8\ \cdot\ \big(\big(k+1\big)!\big)\big) = (-1)^{k+1}\ (k+1)^2,$$

which completes the proof of Fact 3.

We will now show that, apart from the factor $e^{-\frac{1}{2}}$ on the left-hand side of formula (4), the product expansion given by the latter formula and the product expansion given by Guillera and Sondow [5, Example 5.5] are equivalent. In other words, we will show that the total exponent of $2k+1$ in the infinite product expansion of $e^{\frac{G}{\pi}}$ given by Guillera and Sondow [5, Example 5.5] equals $(-1)^{k+1}\,(k+\frac{1}{2})$, for all $k \in \mathbb{N}$. Since the infinite series involved is only conditionally convergent, the discrepancy involving $e^{-\frac{1}{2}}$ can be explained by means of Riemann's theorem on rearrangements of conditionally convergent series. The exponent in question equals

$$
\begin{aligned}
&\big(-1\big)^{k+1} \sum_{n=k}^{\infty} \binom{n}{k} \frac{n}{2^{n+2}} = \frac{\big(-1\big)^{k+1}}{4 \cdot \big(k!\big)} \sum_{n=k}^{\infty} \frac{n^2\,\big(n-1\big)\,\cdots\,\big(n-k+1\big)}{2^n} \\
&= \frac{\big(-1\big)^{k+1}}{4 \cdot \big(k!\big)} \Big(\sum_{n=k}^{\infty} \frac{\big(n+1\big)\,n\,\cdots\,\big(n-k+1\big)}{2^n} - \sum_{n=k}^{\infty} \frac{n\,\big(n-1\big)\,\cdots\,\big(n-k+1\big)}{2^n} \Big) \\
&= \frac{\big(-1\big)^{k+1}}{4 \cdot \big(k!\big)} \Big(\frac{1}{2^k} \sum_{m=0}^{\infty} \frac{\big(m+k+1\big)\,\big(m+k\big)\,\cdots\,\big(m+1\big)}{2^m} \\
&- \frac{1}{2^k} \sum_{m=0}^{\infty} \frac{\big(m+k\big)\,\big(m+k-1\big)\,\cdots\,\big(m+1\big)}{2^m} \Big).
\end{aligned}
$$

By Lemma 17, this equals

$$\frac{(-1)^{k+1}}{4 \cdot (k!)} \big(4 \cdot \big((k+1)!\big) - 2 \cdot (k!)\big) = \big(-1\big)^{k+1}\,\big(k+\frac{1}{2}\big)\,,$$

as required, and this completes the proof of Fact 4.

5 Concluding remarks

Remark 18. The identities

$$\sum_{n=k}^{\infty} \binom{n}{k} \frac{n^2+n}{2^{n+3}} = (k+1)^2\,, \qquad \sum_{n=k}^{\infty} \binom{n}{k} \frac{n}{2^{n+2}} = k+\frac{1}{2}$$

which were used in the proofs of Facts 3 and 4 can also be very easily established by the Wilf-Zeilberger method via the use of Zeilberger's Maple package EKHAD (see [9]).

Remark 19. One way to account for the fact that the products discussed in this paper are so closely tied to the ones studied by Guillera and Sondow in [5] is by noticing that they are related via Euler transformations. For instance, using the latter formula in the previous remark, one has

$$\lim_{m\to\infty} \sum_{k=1}^{2m} (-1)^k k \log \frac{2k-1}{2k+1} = \lim_{m\to\infty} \sum_{k=1}^{2m} (-1)^k \log \frac{2k-1}{2k+1} \sum_{n=k}^{\infty} \binom{n}{k} \frac{n-1}{2^{n+2}}.$$

If we interchange the summation on the right-hand side (an Euler transformation) the relation between formula (4) and the formula given in Fact 4 becomes evident.

Remark 20. The formulas in Propositions 2 and 1 are reminiscent of some powerful statements that deserve to be more widely known. We refer the reader to Finch's book [4] for a wealth of information regarding such statements involving classical constants. For instance, the following function (first introduced by Borwein and Dykshoorn in [3]):

$$D(x) = \lim_{m\to\infty} \prod_{n=1}^{2m+1} \left(1+\frac{x}{n}\right)^{n(-1)^{n+1}} = e^x \lim_{m\to\infty} \prod_{n=1}^{2m} \left(1+\frac{x}{n}\right)^{n(-1)^{n+1}}.$$

Certain values of this function are related to some classical constants. Melzak proved in [7] that $D(2) = \frac{\pi e}{2}$. In [3], Borwein and Dykshoorn generalized Melzak's result and explicitly determined the values of $D(x)$ at all rational x having denominator 1, 2 or 3. Interestingly enough, some of the resulting evaluations involve Catalan's constant, the Glaisher-Kinkelin constant and $\Gamma(\frac{1}{4})$. We have not been able to show that any of the formulas in Propositions 2 or 1 is a direct consequence of the latter evaluations.

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