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Translated Whitney and r-Whitney Numbers: A Combinatorial Approach

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Abstract

Using a combinatorial approach, we introduce the translated Whitney numbers. This seems to be more natural than to write a product of anarithmetical progression in terms of a power variable and conversely. We also extend our ideas to translated r-Whitney numbers of both kinds and to translated Whitney-Lah numbers.

1 Introduction

It is well known that the Whitney numbers of the first and second kind are defined, respectively, by the following recurrence relations

$$w_\alpha(n,k) = w_\alpha(n-1,k-1) + (\alpha(n-1)+1)\, w_\alpha(n-1,k), \tag{1}$$

$$W_\alpha(n,k) = W_\alpha(n-1,k-1) + (\alpha k+1)\, W_\alpha(n-1,k). \tag{2}$$

Benoumhani [2] established the most properties of these numbers. The principal ones are

$$(x+1|\alpha)_n = \sum_{k=0}^{n} (-1)^{n+k} w_\alpha(n,k) x^k, \tag{3}$$

$$x^n = \sum_{k=0}^{n} W_\alpha(n,k)\,(x-1|-\alpha)_k\,, \tag{4}$$

where $(x|\alpha)_n = x\,(x+\alpha)\cdots(x+\alpha\,(n-1))$ and $(x|\alpha)_0 = 1$.

Our aim is to give a combinatorial interpretation of the coefficient of a translation of these two expressions, and we call them the translated Whitney numbers.

To understand the combinatorial interpretations given below, we introduce the nature of the elements we will use. When a given element occurs in the same part as another element, the two elements create a conflict situation of domination. One of them dominates the other, and then the dominated one can mutate into α ways to another color, for example. Then we can state that in each part containing more than one element, all the elements have α possibilities to mutate to another color, except the globally dominant element. Without loss of generality, we can consider that the elements are in an ordered situation of domination: the first one dominates the second, and so on. The n^{th} element is the globally dominated one.

2 Translated Whitney numbers: a combinatorial approach

In this section, we introduce the translated Whitney numbers of both kinds.

Definition 1. The translated Whitney numbers of first kind, denoted $\left[{n \atop k}\right]^{(\alpha)}$, count the number of permutations of n elements with k cycles such that the elements of each cycle can mutate in α ways, except the dominant one.

Theorem 2. *The translated Whitney numbers of first kind satisfy the following recurrence relation:*

$$\begin{bmatrix} n \\ k \end{bmatrix}^{(\alpha)} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{(\alpha)} + \alpha(n-1)\begin{bmatrix} n-1 \\ k \end{bmatrix}^{(\alpha)}, \tag{5}$$

where $\left[{0 \atop k}\right]^{(\alpha)} = \delta_{k,0}$ *and* $\left[{n \atop 0}\right]^{(\alpha)} = \delta_{n,0}$, *where* δ *is the Kronecker symbol.*

Proof. Let us focus on the last element n and, as specified in the introduction, it is the globally dominated element. If it constitutes a cycle, we just have to form $(k-1)$ cycles from the remaining $(n-1)$ elements such that the elements of each cycle can mutate in α ways except the dominant one, which gives $\left[{n-1 \atop k-1}\right]^{(\alpha)}$ possibilities. Otherwise, if it belongs to one of the cycles, we consider all the permutations of $(n-1)$ elements with k cycles such that the elements of each cycle can mutate in α ways except the dominant one, and we have $\left[{n-1 \atop k}\right]^{(\alpha)}$ possibilities; then adding the n^{th} element to one of the cycles with α possibilities of mutation, and we have $\alpha(n-1)$ possibilities. This gives the proof. $\square$

Remark 3. For $\alpha = 1$, we recover the classical Stirling numbers of the first kind.

The translated Whitney of the first kind appears as a special case of the generalization of the Stirling numbers given by Hsu and Shiue [6]; in fact $\left[{n \atop k}\right]^{(\alpha)} = S(n, k; -\alpha, 0, 0)$.

According to relation (3), the translated Whitney numbers of the first kind satisfy the following result:

Theorem 4. *For $n \geq 1$, we have*

$$(x|\alpha)_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}^{(\alpha)} x^k. \tag{6}$$

Proof. The proof is given by induction for the general case in [6]. □

Remark 5. Comparing to (3), we understand why we call these numbers "the translated Whitney numbers".

According to [4, Corollary 3.4], the translated Whitney numbers of the first kind satisfy the following combinatorial identity involving the binomial coefficients:

Corollary 6.

$$\begin{bmatrix} n \\ k \end{bmatrix}^{(\alpha)} = \sum_{\substack{i_1+\cdots+i_{n-1}=n-k \\ i_j \in \{0,1\}}} \binom{\alpha}{i_1} \binom{2\alpha}{i_2} \cdots \binom{(n-1)\,\alpha}{i_{n-1}}.$$

Now, we introduce the translated Whitney of the second kind.

Definition 7. The translated Whitney numbers of the second kind, denoted $\left\{{n \atop k}\right\}^{(\alpha)}$, count the partitions of the set $\{1, 2, \ldots, n\}$ into k subsets such that the elements of each subset can mutate in α ways, except the dominant one.

Theorem 8. *The translated Whitney numbers of second kind satisfy the following recurrence relation:*

$$\begin{Bmatrix} n \\ k \end{Bmatrix}^{(\alpha)} = \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix}^{(\alpha)} + \alpha k \begin{Bmatrix} n-1 \\ k \end{Bmatrix}^{(\alpha)}, \tag{7}$$

with $\left\{{0 \atop k}\right\}^{(\alpha)} = \delta_{k,0}$ and $\left\{{n \atop 0}\right\}^{(\alpha)} = \delta_{n,0}$.

Proof. Let us focus on the last element n. If it constitutes a singleton, then it remains to form $(k-1)$ subsets from the $(n-1)$ remaining elements such that the elements of each subset can mutate in α ways except the dominant one, and we have $\left\{{n-1 \atop k-1}\right\}^{(\alpha)}$ ways to do that. Otherwise, we constitute k subsets from the $(n-1)$ elements such that the elements of each subset can mutate in α ways except the dominant one, and we have $\left\{{n-1 \atop k}\right\}^{(\alpha)}$ possibilities; then adding the n^{th} element to one of the k subsets with α possibilities of mutation, we get $\alpha k\left\{{n-1 \atop k}\right\}^{(\alpha)}$ possibilities. This ends the proof. □

Remark 9. For $\alpha = 1$, we recover the classical Stirling numbers of the second kind.

The translated Whitney numbers of the second kind appear as a special case of the generalization of the Stirling numbers given by Hsu and Shiue [6]. In fact $\left\{ {n \atop k} \right\}^{(\alpha)} = S(n, k; 0, \alpha, 0)$.

According to relation (4), the translated Whitney numbers of the second kind satisfy the following result:

Theorem 10. *For $n \geq 1$, we have*

$$x^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}^{(\alpha)} (x| - \alpha)_k. \tag{8}$$

Proof. See [6]. $\square$

According to [4, Corollary 3.4], we have

Corollary 11.

$$\left\{ {n \atop k} \right\}^{(\alpha)} = \sum_{\substack{i_1+\cdots+i_{n-1}=n-k \\ i_j \in \{0,1\}}} \binom{\alpha}{i_1} \binom{\alpha(2-i_1)}{i_2} \cdots \binom{\alpha((n-1)-i_1-\cdots-i_{n-2})}{i_{n-1}}.$$

3 The translated Whitney-Lah numbers

In this section, we introduce the translated Whitney-Lah numbers using the same operation of counting on ordered lists.

Definition 12. The translated Whitney-Lah numbers, denoted $\left\lfloor {n \atop k} \right\rfloor^{(\alpha)}$, count the number of ways to distribute the set $\{1, 2, ..., n\}$ into k ordered lists such that the elements of each list can mutate with α ways, except the dominant one.

Theorem 13. *We have the following recurence relation:*

$$\left\lfloor {n \atop k} \right\rfloor^{(\alpha)} = \left\lfloor {n-1 \atop k-1} \right\rfloor^{(\alpha)} + \alpha(n+k-1) \left\lfloor {n-1 \atop k} \right\rfloor^{(\alpha)}. \tag{9}$$

Proof. Let us consider the last element n. If it constitutes a list, we have to order the remaining $(n-1)$ elements into k lists such that the elements of each list mutate with α ways except the dominant one, and we have $\left\lfloor {n-1 \atop k-1} \right\rfloor^{(\alpha)}$ ways to do that. Otherwise, if n belongs to one of k constituted lists, we have $\left\lfloor {n-1 \atop k} \right\rfloor^{(\alpha)}$ possibilities. So, we insert the n^{th} element in one of the lists with $(n+k-1)$ choices ($(n-1)$ ways of putting the element after each ordered element and k ways as a head list). The α possibilities of mutation gives us the total number of $\alpha(n+k-1)\left\lfloor {n-1 \atop k} \right\rfloor^{(\alpha)}$ possibilities. $\square$

Corollary 14. *We have*

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor^{(\alpha)} = \sum_{j=k}^{n} \left[\begin{matrix} n \\ j \end{matrix} \right]^{(\alpha)} \left\{ \begin{matrix} j \\ k \end{matrix} \right\}^{(\alpha)}. \tag{10}$$

Proof. From (6) and (8), we can deduce our result. □

The translated Whitney-Lah numbers appear as a special case of the generalization of the Stirling numbers given by Hsu and Shiue [6]. In fact $\left\lfloor \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rfloor^{(\alpha)} = S(n, k; -\alpha, \alpha, 0)$.

According to relation (10), the translated Whitney-Lah numbers satisfy the following result:

Corollary 15.

$$(x|\alpha)_n = \sum_{k=0}^{n} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor^{(\alpha)} (x| - \alpha)_k. \tag{11}$$

Remark 16. For $\alpha = 1$, we get the Lah numbers. See, for instance, [1].

According to [4, Corollary 3.4], we have

Corollary 17.

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor^{(\alpha)} = \sum_{\substack{i_1+\cdots+i_{n-1}=n-k \\ i_j \in \{0,1\}}} \binom{2\alpha}{i_1} \binom{\alpha(4-i_1)}{i_2} \cdots \binom{\alpha(2(n-1) - i_1 - \cdots - i_{n-2}}{i_{n-1}}.$$

4 The translated r-Whitney numbers

Broder [3] gives a generalization of the Stirling numbers of both kinds by adding a certain restriction the treatment of elements. Using this generalization, Mezo [7] treats the corresponding Bell numbers: the r-Bell numbers. We give a corresponding extension of the translated Whitney numbers of both kinds and the translated Whitney-Lah numbers in the sense of Broder's approach.

Definition 18. The generalized translated Whitney numbers of the first kind (respectively the second kind, Lah numbers), denoted $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]_r^{(\alpha)}$ (respectively $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_r^{(\alpha)}$, $\left\lfloor \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rfloor_r^{(\alpha)}$), count the number of permutations (respectively partitions, arrangements) of n elements with k cycles (respectively parts, ordered lists) such that the r first elements are in distinct cycles (respectively parts, lists) and the elements of each cycle (respectively parts, lists) can mutate in α ways, except the dominant one.

Theorem 19. *The translated r-Whitney numbers satisfy the following recurrence relation:*

$$\begin{bmatrix} n \\ k \end{bmatrix}_r^{(\alpha)} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_r^{(\alpha)} + \alpha(n-1)\begin{bmatrix} n-1 \\ k \end{bmatrix}_r^{(\alpha)}, \tag{12}$$

$$\begin{Bmatrix} n \\ k \end{Bmatrix}_r^{(\alpha)} = \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix}_r^{(\alpha)} + \alpha k\begin{Bmatrix} n-1 \\ k \end{Bmatrix}_r^{(\alpha)}, \tag{13}$$

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r^{(\alpha)} = \left\lfloor \begin{matrix} n-1 \\ k-1 \end{matrix} \right\rfloor_r^{(\alpha)} + \alpha(n-1+k)\left\lfloor \begin{matrix} n-1 \\ k \end{matrix} \right\rfloor_r^{(\alpha)}, \tag{14}$$

with $\left[\begin{smallmatrix} 0 \\ k \end{smallmatrix}\right]_r^{(\alpha)} = \left\{\begin{smallmatrix} 0 \\ k \end{smallmatrix}\right\}_r^{(\alpha)} = \left\lfloor\begin{smallmatrix} 0 \\ k \end{smallmatrix}\right\rfloor_r^{(\alpha)} = \delta_{k,0}$ *and* $\left[\begin{smallmatrix} n \\ 0 \end{smallmatrix}\right]_r^{(\alpha)} = \left\{\begin{smallmatrix} n \\ 0 \end{smallmatrix}\right\}_r^{(\alpha)} = \left\lfloor\begin{smallmatrix} n \\ 0 \end{smallmatrix}\right\rfloor_r^{(\alpha)} = \delta_{n,0}$.

Remark 20. For $\alpha = 1$, we obtain, respectively, the r-Stirling numbers of the first kind, the second kind and the r-Lah numbers.

Corollary 21. *We have*

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r^{(\alpha)} = \sum_{j=k}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_r^{(\alpha)} \begin{Bmatrix} j \\ k \end{Bmatrix}_r^{(\alpha)}. \tag{15}$$

The translated r-Whitney of the first kind, the second kind and the r-Whitney-Lah numbers appear as special cases of the generalization of the Stirling numbers given by Hsu and Shiue [6]. Indeed, $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]^{(\alpha)} = S(n,k;-\alpha,0,r)$, $\left\{\begin{smallmatrix} n \\ k \end{smallmatrix}\right\}_r^{(\alpha)} = S(n,k;0,\alpha,r)$ and $\left\lfloor\begin{smallmatrix} n \\ k \end{smallmatrix}\right\rfloor_r^{(\alpha)} = S(n,k;-\alpha,\alpha,r)$ and they satisfy the following results:

Corollary 22. *For any nonnegative integers $0 \leq r \leq k \leq n$, we have*

$$(x+r|\alpha)_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_r^{(\alpha)} (x+r)^k, \tag{16}$$

$$(x+r)^n = \sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix}_r^{(\alpha)} (x-r|-\alpha)_k, \tag{17}$$

$$(x+r|\alpha)_n = \sum_{k=0}^{n} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r^{(\alpha)} (x-r|-\alpha)_k. \tag{18}$$

According to [4, Corollary 3.4], we have

Corollary 23.

$$\begin{bmatrix} n \\ k \end{bmatrix}_r^{(\alpha)} = \sum_{\substack{i_0+\cdots+i_{n-1}=n-k \\ i_j\in\{0,1\}}} \binom{r}{i_0}\binom{r+\alpha}{i_1}\cdots\binom{r+(n-1)\,\alpha}{i_{n-1}},$$

$$\begin{Bmatrix} n \\ k \end{Bmatrix}_r^{(\alpha)} = \sum_{\substack{i_0+\cdots+i_{n-1}=n-k \\ i_j\in\{0,1\}}} \binom{r}{i_0}\binom{r+\alpha(1-i_0)}{i_1}\cdots\binom{r+\alpha((n-1)-i_0-i_1-\cdots-i_{n-2})}{i_{n-1}},$$

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r^{(\alpha)} = \sum_{\substack{i_0+\cdots+i_{n-1}=n-k \\ i_j\in\{0,1\}}} \binom{r}{i_0}\binom{r+(2-i_0)\alpha}{i_1}\cdots\binom{r+\alpha(2\,(n-1)-i_1-\cdots-i_{n-2}}{i_{n-1}}.$$

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