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Impulse Response Sequences and Construction of Number Sequence Identities

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Abstract

In this paper, we investigate impulse response sequences over the integers by presenting their generating functions and expressions. We also establish some of the corresponding identities. In addition, we give the relationship between an impulse response sequence and all linear recurring sequences satisfying the same linear recurrence relation, which can be used to transfer the identities among different sequences. Finally, we discuss some applications of impulse response sequences to the structure of Stirling numbers of the second kind, the Wythoff array, and the Boustrophedon transform.

1 Introduction

Many number and polynomial sequences can be defined, characterized, evaluated, and classified by linear recurrence relations with certain orders. A number sequence $\{a_n\}$ is called a sequence of order r if it satisfies the linear recurrence relation of order r

$$a_n = \sum_{j=1}^{r} p_j a_{n-j}, \quad n \geq r, \tag{1}$$

for some constants p_j, $j = 1, 2, \ldots, r$, $p_r \neq 0$, and initial conditions a_j $(j = 0, 1, \ldots, r-1)$. Linear recurrence relations with constant coefficients are important in subjects including combinatorics, pseudo-random number generation, circuit design, and cryptography, and they have been studied extensively. To construct an explicit formula of the general term of a number sequence of order r, one may use a generating function, a characteristic equation, or a matrix method (See Comtet [4], Niven, Zuckerman, and Montgomery [12], Strang [15], Wilf [16], etc.) Let A_r be the set of all linear recurring sequences defined by the homogeneous linear recurrence relation (1) with coefficient set $E_r = \{p_1, p_2, \ldots, p_r\}$. To study the structure of A_r with respect to E_r, we consider the *impulse response sequence* (or, *IRS* for abbreviation) in A_r, which is a particular sequence in A_r with initials $a_0 = a_1 = \cdots = a_{r-2} = 0$ and $a_{r-1} = 1$. The study of impulse response sequences in a finite field has been considerably active in the theory of finite fields. A summary work on this subject can be found in Lidl and Niederreiter [10]. In this paper, we investigate the same concept in $\mathbb{Z}$. If $r = 2$, the IRS in $\mathbb{Z}$ is the well known *Lucas sequence*. In this sense, an IRS of order $r > 2$ is an extension of a Lucas sequence in a high order setting.

In the next section, we will give the generating function and the expression of the IRS of A_r and find out the relationship between the IRS and other sequences in the set A_r. Several examples of IRS in sets A_2 and A_3 will be given in Section 3. The work shown in Sections 2 and 3 is partially motivated by Shiue and the author [8], which presented a method for the construction of general term expressions and identities for the linear recurring sequences of order $r = 2$ by using the reduction order method. However, the reduction method is too complicated for the linear recurring sequences of order $r > 2$, although our method can be applied readily to linear recurring sequences with arbitrary order by using their relationship with impulse response sequences. In Section 4, we shall give some applications of IRS in the discussion of the structure of *Stirling numbers of the second kind*, the *Wythoff array*, and the *Boustrophedon transform*.

2 Impulse response sequences

Among all the homogeneous linear recurring sequences satisfying rth order homogeneous linear recurrence relation (1) with a nonzero p_r and arbitrary initial conditions $\{a_j\}_{j=0}^{r-1}$, we define the impulse response sequence (IRS) with respect to $E_r = \{p_j\}_{j=1}^{r}$ as the sequence with initial conditions $a_0 = a_{r-2} = 0$ and $a_{r-1} = 1$. In particular, when $r = 2$, the homogeneous linear recurring sequences with respect to $E_2 = \{p_1, p_2\}$ satisfy

$$a_n = p_1 a_{n-1} + p_2 a_{n-2}, \quad n \geq 2, \tag{2}$$

with arbitrary initial conditions a_0 and a_1, or, initial vector (a_0, a_1). If the initial vector $(a_0, a_1) = (0, 1)$, the corresponding sequence generated by using (2) is the IRS with respect to E_2, is called the Lucas sequence. For instance, the *Fibonacci sequence* $\{F_n\}$ is the IRS with respect to $\{1, 1\}$, the *Pell number sequence* $\{P_n\}$ is the IRS with respect to $\{2, 1\}$, and the *Jacobsthal number sequence* $\{J_n\}$ is the IRS with respect to $\{1, 2\}$.

In the following, we will present the structure of the linear recurring sequences defined by (1) using their characteristic polynomial. This structure provides a way to find the relationship between those sequences and the IRS.

Let A_r be the set of all linear recurring sequences defined by the homogeneous linear recurrence relation (1) with coefficient set $E_r = \{p_1, p_2, \ldots, p_r\}$, and let $\tilde{F}_n^{(r)}$ be the IRS of A_r, namely, $\tilde{F}_n^{(r)}$ satisfies (1) with initial conditions $\tilde{F}_0^{(r)} = \cdots = \tilde{F}_{r-2}^{(r)} = 0$ and $\tilde{F}_{r-1}^{(r)} = 1$.

Suppose $\{a_n\} \in A_r$. Let $B = |p_1| + |p_2| + \cdots + |p_r|$, and for $n \in \mathbb{N}_0$ let $M_n = \max\{|a_0|, |a_1|, \ldots, |a_n|\}$. Thus, for $n \geq r$, $M_n \leq BM_{n-1}$. By induction, $|a_n| \leq CB^n$ for some constant A and $n = 0, 1, 2, \ldots$. A sequence satisfying a bound of this kind is called a sequence having at most *exponential growth* (see [12]). Therefore, the generating function $\sum_{j\geq 0} a_j t^j$ of such sequence has a positive radius of convergence, $0 < c < 1/B$. More precisely, if $|t| \leq c$, then by comparison with the convergent geometric series $\sum_{j\geq 0} CB^j c^j$, the series $\sum_{j\geq 0} a_j t^j$ converges absolutely. In the following, we always assume that $\{a_n\}$ has at most exponential growth.

Proposition 1. *Let $\{a_n\} \in A_r$, i.e., let $\{a_n\}$ be the linear recurring sequence defined by (1). Then its generating function $P_r(t)$ can be written as*

$$P_r(t) = \{a_0 + \sum_{n=1}^{r-1} \left(a_n - \sum_{j=1}^{n} p_j a_{n-j} \right) t^n\} / \{1 - \sum_{j=1}^{r} p_j t^j\}. \tag{3}$$

Hence, the generating function for the IRS with respect to $\{p_j\}$ is

$$\tilde{P}_r(t) = \frac{t^{r-1}}{1 - \sum_{j=1}^{r} p_j t^j}. \tag{4}$$

Proof. Multiplying t^n on the both sides of (1) and summing from $n = r$ to infinity yields

$$\begin{aligned}
\sum_{n\geq r} a_n t^n &= \sum_{n\geq r} \sum_{j=1}^{r} p_j a_{n-j} t^n = \sum_{j=1}^{r} p_j \sum_{n\geq r} a_{n-j} t^n \\
&= \sum_{j=1}^{r-1} p_j \left(\sum_{n\geq j} a_{n-j} t^n - \sum_{n=j}^{r-1} a_{n-j} t^n \right) + p_r \sum_{n\geq r} a_{n-r} t^n \\
&= \sum_{j=1}^{r-1} p_j \left(t^j \sum_{n\geq 0} a_n t^n - \sum_{n=j}^{r-1} a_{n-j} t^n \right) + p_r t^r \sum_{n\geq 0} a_n t^n
\end{aligned}$$

Denote $P_r(t) = \sum_{n\geq 0} a_n t^n$. Then the equation of the leftmost and the rightmost sides can be written as

$$P_r(t) - \sum_{n=0}^{r-1} a_n t^n = P_r(t) \sum_{j=1}^{r-1} p_j t^j - \sum_{j=1}^{r-1} \sum_{n=j}^{r-1} p_j a_{n-j} t^n + p_r t^r P_r(t),$$

which implies

$$P_r(t)\left(1-\sum_{j=1}^{r} p_j t^j\right)=\sum_{n=0}^{r-1} a_n t^n-\sum_{n=1}^{r-1}\sum_{j=1}^{n} p_j a_{n-j} t^n.$$

Therefore, (4) follows immediately. By substituting $a_0=\cdots=a_{r-2}=0$ and $a_{r-1}=1$ into (3), we obtain (4). ☐

We now give the explicit expression of $\tilde{F}_n^{(r)}$ in terms of the zeros of the characteristic polynomial of recurrence relation shown in (1).

Theorem 2. *Sequence* $\{\tilde{F}_n^{(r)}\}_n$ *is the IRS with respect to* $\{p_j\}$ *defined by (1) with* $\tilde{F}_0^{(r)}=\tilde{F}_1^{(r)}=\cdots=\tilde{F}_{r-2}^{(r)}=0$ *ane* $\tilde{F}_{r-1}^{(r)}=1$ *if and only if*

$$\tilde{F}_n^{(r)}=\sum_{j=1}^{\ell}\frac{\binom{n}{m_j-1}\alpha_j^{n-m_j+1}}{\prod_{k=1,k\neq j}^{\ell}(\alpha_j-\alpha_k)} \tag{5}$$

for $n\geq r$, *where* $\alpha_1,\alpha_2,\ldots,\alpha_\ell$ *are the distinct real or complex zeros of characteristic polynomial,* $p(t)=t^r-p_1t^{r-1}-\cdots-p_r$, *of the recurrence relation (1) with the multiplicities* $m_1,m_2,\ldots,m_\ell$, *respectively, and* $m_1+m_2+\cdots+m_\ell=r$. *In particular, if all* $\alpha_1,\alpha_2,\ldots,\alpha_r$ *are of multiplicity one, then (5) becomes to*

$$\tilde{F}_n^{(r)}=\sum_{j=1}^{r}\frac{\alpha_j^n}{\prod_{k=1,k\neq j}^{r}(\alpha_j-\alpha_k)},\quad n\geq r, \tag{6}$$

Proof. First, we prove (6). Then we use it to prove the necessity of (5). Let α_j, $j=1,2,\ldots,\ell$, be ℓ zeros of the characteristic polynomial $t^n-\sum_{i=1}^{r}p_it^{n-i}$. Then from (4)

$$\begin{aligned}\tilde{P}_r(t)&=\frac{t^{r-1}}{1-\sum_{j=1}^{r}p_jt^j}=\frac{1/t}{(1/t)^r-\sum_{j=1}^{r}p_j(1/t)^{n-j}}\\ &=\frac{1/t}{\prod_{j=1}^{r}(1/t-\alpha_j)}=\frac{t^{r-1}}{\prod_{j=1}^{r}(1-\alpha_jt)}.\end{aligned} \tag{7}$$

We now prove (6) from (7) by using mathematical induction for r under the assumption that all solutions α_j are distinct. Because $[t^n]\tilde{P}_r(t)=\tilde{F}_n^{(r)}$, we need to show

$$[t^n]\tilde{P}_r(t)=[t^n]\frac{t^{r-1}}{\prod_{j=1}^{r}(1-\alpha_jt)}=\sum_{j=1}^{r}\frac{\alpha_j^n}{\prod_{k=1,k\neq j}^{r}(\alpha_j-\alpha_k)},\quad n\geq r. \tag{8}$$

(8) is obviously true for $r=1$. Assume (8) holds for $r=m$. We find

$$[t^n]\tilde{P}_{m+1}(t)=[t^n]\frac{t^m}{\prod_{j=1}^{m+1}(1-\alpha_jt)},$$

which implies

$$[t^n](1-\alpha_{m+1}t)\tilde{P}_{m+1}(t)=[t^{n-1}]\frac{t^{m-1}}{\prod_{j=1}^{m}(1-\alpha_j t)}=\sum_{j=1}^{m}\frac{\alpha_j^{n-1}}{\prod_{k=1,k\neq j}^{m}(\alpha_j-\alpha_k)}$$

because of the induction assumption. Noting $[t^n]\tilde{P}_{m+1}(t)=\tilde{F}_n^{(m+1)}$, from the last equations one may write

$$\tilde{F}_n^{(m+1)}-\alpha_{m+1}\tilde{F}_{n-1}^{(m+1)}=\sum_{j=1}^{m}\frac{\alpha_j^{n-1}}{\prod_{k=1,k\neq j}^{m}(\alpha_j-\alpha_k)}, \tag{9}$$

which has the solution

$$\tilde{F}_n^{(m+1)}=\sum_{j=1}^{m+1}\frac{\alpha_j^{n}}{\prod_{k=1,k\neq j}^{m+1}(\alpha_j-\alpha_k)}. \tag{10}$$

Indeed, we have

$$\begin{aligned}
&\alpha_{m+1}\tilde{F}_{n-1}^{(m+1)}+\sum_{j=1}^{m}\frac{\alpha_j^{n-1}}{\prod_{k=1,k\neq j}^{m}(\alpha_j-\alpha_k)}\\
=\ &\alpha_{m+1}\sum_{j=1}^{m+1}\frac{\alpha_j^{n-1}}{\prod_{k=1,k\neq j}^{m+1}(\alpha_j-\alpha_k)}+\sum_{j=1}^{m}\frac{\alpha_j^{n-1}}{\prod_{k=1,k\neq j}^{m}(\alpha_j-\alpha_k)}\\
=\ &\sum_{j=1}^{m}\frac{\alpha_j^{n-1}}{\prod_{k=1,k\neq j}^{m+1}(\alpha_j-\alpha_k)}\left(\alpha_{m+1}+\alpha_j-\alpha_{m+1}\right)+\frac{\alpha_{m+1}^{n}}{\prod_{k=1}^{m}(\alpha_{m+1}-\alpha_k)}\\
=\ &\sum_{j=1}^{m+1}\frac{\alpha_j^{n}}{\prod_{k=1,k\neq j}^{m+1}(\alpha_j-\alpha_k)}=\tilde{F}_n^{(m+1)},
\end{aligned}$$

and (10) is proved.

To prove (5), it is sufficient to show it holds for the case of one multiple zero, say α_i with the multiplicity m_i. The formula of $\tilde{F}_n^{(r)}$ in this case can be derived from (6) by using the limit process. More precisely, let m_i zeros of the characteristic polynomial, denoted by α_{i_j}, $1\leq j\leq m_i$, approach α_i. Then (6) will be reduced to the corresponding (5) with respect to α_i. Indeed, taking the limit as α_{i_j}, $1\leq j\leq m_i$, approaches α_i on both sides of (6), yields

$$\begin{aligned}
&\lim_{(\alpha_{i_1},\ldots,\alpha_{i_{m_i}})\to(\alpha_i,\ldots,\alpha_i)}\tilde{F}_n^{(r)}\\
=\ &\frac{1}{\prod_{k=1,k\neq i}^{r-m_i+1}(\alpha_i-\alpha_k)}\lim_{(\alpha_{i_1},\ldots,\alpha_{i_{m_i}})\to(\alpha_i,\ldots,\alpha_i)}\left(\frac{\alpha_{i_1}^n}{\prod_{k=2}^{m_i}(\alpha_{i_1}-\alpha_{i_k})}+\frac{\alpha_{i_2}^n}{\prod_{k=1,k\neq 2}^{m_i}(\alpha_{i_2}-\alpha_{i_k})}\right.\\
&\left.+\cdots+\frac{\alpha_{i_{m_i}}^n}{\prod_{k=1,k\neq m_i}^{m_i}(\alpha_{i_{m_i}}-\alpha_{i_k})}\right)+\sum_{j=1,j\neq i}^{r}\frac{\alpha_j^n}{\prod_{k=1,k\neq j}^{r}(\alpha_j-\alpha_k)}.
\end{aligned} \tag{11}$$

It is obvious that the summation in the parentheses of equation (11) is the expansion of the *divided difference* of the function $f(t) = t^n$, denoted by $f[\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_{m_i}}]$, at nodes $\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_{m_i}}$. Using the mean value theorem (see, for example, [13]) for divided difference (see, for example, [2, Thm. 3.6]), one may obtain

$$\lim_{(\alpha_{i_1},\ldots,\alpha_{i_{m_i}})\to(\alpha_i,\ldots,\alpha_i)} f[\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_{m_i}}] = \frac{f^{(m_i-1)}(\alpha_i)}{(m_i-1)!} = \binom{n}{m_i-1}\alpha_i^{n-m_i+1}.$$

Therefore, by taking the limit as $\alpha_{i_j} \to \alpha_i$ for $1 \le j \le m_i$ on both sides of (11), we obtain

$$\lim_{(\alpha_{i_1},\ldots,\alpha_{i_{m_i}})\to(\alpha_i,\ldots,\alpha_i)} \tilde{F}_n^{(r)} = \frac{\binom{n}{m_i-1}\alpha_i^{n-m_1+1}}{\prod_{k=1,k\neq i}^{r-m_i+1}(\alpha_i-\alpha_k)} + \sum_{j=1,j\neq i}^{r} \frac{\alpha_j^n}{\prod_{k=1,k\neq j}^{r}(\alpha_j-\alpha_k)},$$

which implies the correctness of (5) for the multiple zero α_i of multiplicity m_i. Therefore, (5) follows by taking the same argument for each multiple zero of the characteristic polynomial $P_r(t)$ shown in Proposition 1.

Finally, we prove sufficiency. For $n \ge r$,

$$\begin{aligned}\sum_{i=1}^{r} p_i \tilde{F}_{n-i}^{(r)} &= \sum_{j=1}^{\ell} \frac{\sum_{i=1}^{r} p_i \binom{n}{m_j-1}\alpha_j^{n-i-m_j+1}}{\prod_{k=1,k\neq j}^{\ell}(\alpha_j-\alpha_k)} \\ &= \sum_{j=1}^{\ell} \frac{\binom{n}{m_j-1}\alpha_j^{-m_j+1}\sum_{i=1}^{r} p_i\alpha_j^{n-i}}{\prod_{k=1,k\neq j}^{\ell}(\alpha_j-\alpha_k)} \\ &= \sum_{j=1}^{\ell} \frac{\binom{n}{m_j-1}\alpha_j^{n-m_j+1}}{\prod_{k=1,k\neq j}^{\ell}(\alpha_j-\alpha_k)} = \tilde{F}_n^{(r)},\end{aligned}$$

where we have used $\alpha_j^n = \sum_{i=1}^{r} p_i\alpha_j^{n-i}$ because of $p(\alpha_j) = 0$. Therefore, the sequence $\{\tilde{F}_n^{(r)}\}$ satisfies the linear recurrence relation (1) and is the IRS with respect to $\{p_j\}$. $\square$

The IRS of a set of linear recurring sequences is a kind of basis so that every sequence in the set can be represented in terms of the IRS. For this purpose, we need to extend the indices of IRS $\tilde{F}_n^{(r)}$ of A_r to the negative indices down to till $n = -r+1$, where A_r is the set of all linear recurring sequences defined by the homogeneous linear recurrence relation (1) with coefficient set $E_r = \{p_1, p_2, \ldots, p_r\}$, $p_r \neq 0$. For instance, $\tilde{F}_{-1}^{(r)}$ is defined by

$$\tilde{F}_{r-1}^{(r)} = p_1\tilde{F}_{r-2}^{(r)} + p_2\tilde{F}_{r-3}^{(r)} + \cdots + p_r\tilde{F}_{-1}^{(r)},$$

which implies $\tilde{F}_{-1}^{(r)} = 1/p_r$ because $\tilde{F}_0^{(r)} = \cdots = \tilde{F}_{r-2}^{(r)} = 0$ and $\tilde{F}_{r-1}^{(r)} = 1$. Similarly, we may use

$$\tilde{F}_{r+j}^{(r)} = p_1\tilde{F}_{r+j-1}^{(r)} + p_2\tilde{F}_{r+j-2}^{(r)} + \cdots + p_r\tilde{F}_j^{(r)}$$

to define $\tilde{F}_j^{(r)}$, $j = -2, \ldots, -r+1$ successively. The following theorem shows how to represent linear recurring sequences in terms of the extended IRS $\{\tilde{F}_n^{(r)}\}_{n\ge -r+1}$.

Theorem 3. *Let A_r be the set of all linear recurring sequences defined by the homogeneous linear recurrence relation (1) with coefficient set $E_r = \{p_1, p_2, \ldots, p_r\}$, $p_r \neq 0$, and let $\{\tilde{F}_n^{(r)}\}_{n \geq -r+1}$ be the (extended) IRS of A_r. Then, for any $a_n \in A_r$, one may write a_n as*

$$a_n = a_{r-1}\tilde{F}_n^{(r)} + \sum_{j=0}^{r-2}\sum_{k=j}^{r-2} a_k p_{r+j-k}\tilde{F}_{n-1-j}^{(r)}. \tag{12}$$

Proof. Considering the sequences on the both sides of (12), we find that both sequences satisfy the same recurrence relation (1). Hence, to prove the equivalence of two sequences, we only need to show that they have the same initial vector $(a_0, a_1, \ldots, a_{r-1})$ because of the uniqueness of the linear recurring sequence defined by (1) with fixed initial conditions. First, for $n = r-1$, the conditions $\tilde{F}_0^{(r)} = \cdots = \tilde{F}_{r-2}^{(r)} = 0$ and $\tilde{F}_{r-1}^{(r)} = 1$ are applied on the right-hand side of (12) to obtain its value as a_{r-1}, which shows (12) holds for $n = r-1$. Secondly, for all $0 \leq n \leq r-2$, one may write the right-hand side RHS of (12) as

$$RHS = 0 + \sum_{k=0}^{r-2} a_k \left(\sum_{j=0}^{k} p_{r+j-k}\tilde{F}_{n-1-j}^{(r)}\right) = \sum_{k=0}^{r-2} a_k \delta_{k,n} = a_n,$$

where the Kronecker delta symbol $\delta_{k,n}$ is 1 when $k = n$ and zero otherwise, and the following formula is used:

$$\sum_{j=0}^{k} p_{r+j-k}\tilde{F}_{n-1-j}^{(r)} = \delta_{k,n}. \tag{13}$$

(13) can be proved by splitting it into three cases: $n =, >$, and $< k$, respectively. If $n = k$, we have

$$\begin{aligned} \sum_{j=0}^{k} p_{r+j-k}\tilde{F}_{n-1-j}^{(r)} &= \sum_{j=0}^{k} p_{r+j-k}\tilde{F}_{k-1-j}^{(r)} \\ &= p_r\tilde{F}_{-1}^{(r)} + p_{r-1}\tilde{F}_0^{(r)} + p_{r-2}\tilde{F}_1^{(r)} + \cdots + p_{r-k}\tilde{F}_{k-1}^{(r)} \\ &= p_r\tilde{F}_{-1}^{(r)} = 1, \end{aligned}$$

where we use $\tilde{F}_{-1}^{(r)} = 1/p_r$ and the fact $\tilde{F}_j^{(r)} = 0$ for all $j = 0, 1, \ldots, k-1$ due to $k-1 \leq r-3$ and the initial values of $\tilde{F}_n^{(r)}$ being zero for all $0 \leq n \leq r-2$.

If $n > k$, we have

$$\sum_{j=0}^{k} p_{r+j-k}\tilde{F}_{n-1-j}^{(r)} = p_r\tilde{F}_{n-k-1}^{(r)} + p_{r-1}\tilde{F}_{n-k}^{(r)} + \cdots + p_{r-k}\tilde{F}_{n-1}^{(r)} = 0$$

because $\tilde{F}_j^{(r)} = 0$ for all $j = n-k-1, n-k, \ldots, n-1$ due to the definition of $\tilde{F}_n^{(r)}$ and the condition of $-1 < n-k-1 \leq j \leq n-1 \leq r-3$.

Finally, for $0 \le n < k$, we may find

$$\begin{aligned} & \sum_{j=0}^{k} p_{r+j-k} \tilde{F}^{(r)}_{n-1-j} = p_r \tilde{F}^{(r)}_{n-k-1} + p_{r-1} \tilde{F}^{(r)}_{n-k} + \cdots + p_{r-k} \tilde{F}^{(r)}_{n-1} \\ = \;& p_r \tilde{F}^{(r)}_{n-k-1} + \cdots + p_{r-k} \tilde{F}^{(r)}_{n-1} + p_{r-k-1} \tilde{F}^{(r)}_{n} + p_{r-k-2} \tilde{F}^{(r)}_{n+1} + \cdots + p_1 \tilde{F}^{(r)}_{n+r-k-2} \\ = \;& \tilde{F}^{(r)}_{n+r-k-1} = 0, \end{aligned}$$

where the inserted $r-k-1$ terms $\tilde{F}^{(r)}_j$, $j = n, n+1, \ldots n+r-k-2$, are zero due to the definition of $\tilde{F}^{(r)}_n$ and $0 \le n \le j \le n+r-k-2 < r-2$. The last two steps of the above equations are from the recurrence relation (1) and the assumptions of $0 \le k \le r-2$ and $1 \le n+1 \le n+r-k-1 < r-1$, respectively. This completes the proof of theorem. □

By using Theorems 2 and 3, we immediately obtain

Corollary 4. *Let A_r be the set of all linear recurring sequences defined by the homogeneous linear recurrence relation (1) with coefficient set $E_r = \{p_1, p_2, \ldots, p_r\}$, $p_r \ne 0$, and let $\alpha_1, \alpha_2, \ldots, \alpha_\ell$ be the zeros of characteristic polynomial, $p(t) = t^r - p_1 t^{r-1} - \cdots - p_r$, of the recurrence relation (1) with the multiplicities $m_1, m_2, \ldots, m_\ell$, respectively, and $m_1 + m_2 + \cdots + m_\ell = r$. Then, for any $\{a_n\} \in A_r$, we have*

$$a_n = a_{r-1} \sum_{j=1}^{\ell} \frac{\binom{n}{m_j-1} \alpha_j^{n-m_j+1}}{\prod_{k=1,k\ne j}^{\ell} (\alpha_j - \alpha_k)} + \sum_{j=0}^{r-2} \sum_{k=j}^{r-2} \sum_{j=1}^{\ell} a_k p_{r+j-k} \frac{\binom{n-j-1}{m_j-1} \alpha_j^{n-j-m_j}}{\prod_{k=1,k\ne j}^{\ell} (\alpha_j - \alpha_k)} \tag{14}$$

We now establish the relationship between a sequence in A_r and the IRS of A_r by using *Toeplitz matrices*. A Toeplitz matrix may be defined as a $n \times n$ matrix A with entries $a_{i,j} = c_{i-j}$, for constants $c_{1-n}, \ldots, c_{n-1}$. A $m \times n$ block Toeplitz matrix is a matrix that can be partitioned into $m \times n$ blocks and every block is a Toeplitz matrix.

Theorem 5. *Let A_r be the set of all linear recurring sequences defined by the homogeneous linear recurrence relation (1) with coefficient set $E_r = \{p_1, p_2, \ldots, p_r\}$, and let $\tilde{F}^{(r)}_n$ be the IRS of A_r. Then, for even r, we may write*

$$\tilde{F}^{(r)}_n = \sum_{j=r/2}^{r-1} c_j a_{n+r-j} + \sum_{j=r}^{(3r/2)-1} c_j a_{n+r-j-1}, \tag{15}$$

where the coefficients c_j, $r/2 \le j \le (3r/2) - 1$, satisfy

$$A_e\mathbf{c} := \begin{bmatrix} a_{r/2} & a_{(r/2)-1} & \cdots & a_1 & a_{-1} & a_{-2} & \cdots & a_{-r/2} \\ a_{(r/2)+1} & a_{r/2} & \cdots & a_2 & a_0 & a_{-1} & \cdots & a_{-(r/2)+1} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{r-1} & a_{r-2} & \cdots & a_{r/2} & a_{(r/2)-2} & a_{(r/2)-3} & \cdots & a_{-1} \\ a_r & a_{r-1} & \cdots & a_{(r/2)+1} & a_{(r/2)-1} & a_{(r/2)-2} & \cdots & a_0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{(3r/2)-1} & a_{(3r/2)-2} & \cdots & a_r & a_{r-2} & a_{r-3} & \cdots & a_{(r/2)-1} \end{bmatrix} \mathbf{c} = \mathbf{e_r} \tag{16}$$

with $\mathbf{c} = (c_{r/2}, c_{(r/2)-1}, \ldots, c_r, c_{r+1}, c_{r+2}, \ldots, c_{(3r/2)-1})^T$ *and* $\mathbf{e_r} = (0, 0, \ldots, 0, 1)^T \in \mathbb{R}^r$*. For odd* r*, we have*

$$\tilde{F}_n^{(r)} = \sum_{j=(r-1)/2}^{3(r-1)/2} c_j a_{n+r-j-1}, \tag{17}$$

where c_j*,* $(r-1)/2 \le j \le 3(r-1)/2$*, satisfy*

$$A_o\mathbf{c} := \begin{bmatrix} a_{(r-1)/2} & a_{((r-1)/2)-1} & \cdots & a_1 & a_0 & a_{-1} & \cdots & a_{-(r-1)/2} \\ a_{((r-1)/2)+1} & a_{(r-1)/2} & \cdots & a_2 & a_1 & a_0 & \cdots & a_{-(r-1)/2+1} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{(3(r-1)/2)} & a_{(3(r-1)/2)-1} & \cdots & a_r & a_{r-1} & a_{r-2} & \cdots & a_{(r-1)/2} \end{bmatrix} \mathbf{c} = \mathbf{e_r} \tag{18}$$

with $\mathbf{c} = (c_{(r-1)/2}, c_{((r-1)/2)-1}, \ldots, c_{r-1}, c_r, c_{r+1}, \ldots, c_{(3(r-1)/2)})^T$ *and* $\mathbf{e_r} = (0, 0, \ldots, 0, 1)^T \in \mathbb{R}^r$*. If the* 2×2 *block Toeplitz matrix* A_e *and Toeplitz matrix* A_o *are invertible, then we have unique expressions (15) and (17), respectively.*

Proof. Since the operator $L : \mathbb{Z} \times \mathbb{Z} \mapsto \mathbb{Z}$, $L(a_{n-1}, a_{n-2}, \ldots, a_{n-r}) := p_1 a_{n-1} + p_2 a_{n-2} + \cdots + p_r a_{n-r} = a_n$, is linear, sequence $\{a_n\}$ is uniquely determined by L from a given initial vector $(a_0, a_1, \ldots, a_{r-1})$. We may extend $\{a_n\}_{n\ge 0}$ to a sequence with negative indices using the technique we applied in Theorem 3. For instance, by defining a_{-1} from $a_{r-1} = p_r a_{-1} + p_{r-1}a_0 + p_{r-2}a_1 + \cdots + p_1 a_{r-2})$, we obtain

$$a_{-1} = \frac{1}{p_r}(a_{r-1} - p_{r-1}a_0 - p_{r-2}a_1 - \cdots - p_1 a_{r-2}), \quad p_r \ne 0.$$

Thus, the initial vector $(a_{-1}, a_0, \ldots, a_{r-2})$ generates $\{a_n\}_{n\ge -1}$ or $\{a_{n-1}\}_{n\ge 0}$ by using the operator L. Since both sequences $\{a_n\}$ and $\{\tilde{F}_n^{(r)}\}$ satisfy the recurrence relation (1), they are generated by the same operator L. Hence, for even r, we may write $\tilde{F}_n^{(r)}$ as a linear combination of a_{n+r-j}, $r/2 \le j \le r-1$, and $a_{n+r-j-1}$, $r \le j \le (3r/2)-1$, as shown in (15), provided the initial vectors of the sequences on the two sides are the same. Here, we mention again that the initial values with negative indices have been defined by using (1). Therefore, we substitute the initial values of $\{a_n\}$ with their negative extensions to the right-hand side of (15) and enforce the resulting linear combination equal to the initials of the left-hand side of (15), i.e., $\mathbf{e_r}$, which derives (16). Under the assumption of the invertibility of A_e, Toeplitz system (16) has a unique solution of $\mathbf{c} = (c_{r/2}, c_{(r/2)-1}, \ldots, c_r, c_{r+1}, c_{r+2}, \ldots, c_{(3r/2)-1})^T$, which shows the uniqueness of the expression of (15). (17) can be proved similarly. $\square$

Remark 6. It can be seen that not every Toeplitz matrix generated from the linear recurring sequence (1) is invertible. For example, consider the sequence $\{a_n\}$ generated by the linear recurrence relation $a_n = a_{n-1} + 3a_{n-2} + a_{n-3}$ with $a_0 = 1$, $a_1 = 0$, $a_2 = 1$, there hold $a_3 = a_0 + 3a_1 + a_2 = 2$ and $a_{-1} = a_2 - a_1 - 3a_0 = -2$. From (17) :

$$A_o = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ -2 & 1 & 0 \end{bmatrix},$$

which is not invertible. If $p_j = 1$, $1 \le j \le r$, and $r > 2$, the corresponding IRS is a higher-order Fibonacci sequence. For instance, $\tilde{F}_n^{(3)}$, $\tilde{F}_n^{(4)}$, $\tilde{F}_n^{(5)}$, $\tilde{F}_n^{(6)}$, $\tilde{F}_n^{(7)}$, etc. are Tribonacci numbers (A000073), Tetranacci numbers (A000078), Pentanacci numbers (A001591), Heptanacci numbers (A122189), Octanacci numbers (A079262), etc., respectively.

3 Examples

We now give some examples of IRS for $r = 2$ and 3. Let A_2 be the set of all linear recurring sequences defined by the homogeneous linear recurrence relation (2) with coefficient set $E_2 = \{p_1, p_2\}$. Hence, from Theorems 2 and 3 and the definition of the impulse response sequence of A_2 with respect to E_2, we obtain

Corollary 7. *Let A_2 be the set of all linear recurring sequences defined by the homogeneous linear recurrence relation (2) with coefficient set $E_2 = \{p_1, p_2\}$, and let $\{\tilde{F}_n^{(2)}\}$ be the IRS (i.e., the Lucas sequence) of A_2. Suppose α and β are two zeros of the characteristic polynomial of A_2, which do not need to be distinct. Then*

$$\tilde{F}_n^{(2)} = \begin{cases} \frac{\alpha^n - \beta^n}{\alpha - \beta}, & \text{if } \alpha \ne \beta; \\ n\alpha^{n-1}, & \text{if } \alpha = \beta. \end{cases} \tag{19}$$

In addition, every $\{a_n\} \in A_2$ can be written as

$$a_n = a_1 \tilde{F}_n^{(2)} - \alpha\beta a_0 \tilde{F}_{n-1}^{(2)}, \tag{20}$$

and a_n reduces to $a_1 \tilde{F}_n^{(2)} - \alpha^2 a_0 \tilde{F}_{n-1}^{(2)}$ when $\alpha = \beta$.

Conversely, there is an expression for $\tilde{F}_n^{(2)}$ in terms of $\{a_n\}$ as

$$\tilde{F}_n^{(2)} = c_1 a_{n+1} + c_2 a_{n-1}, \tag{21}$$

where

$$c_1 = \frac{a_1 - a_0 p_1}{p_1(a_1^2 - a_0 a_1 p_1 - a_0^2 p_2)}, \quad c_2 = -\frac{a_1 p_2}{p_1(a_1^2 - a_0 a_1 p_1 - a_0^2 p_2)}, \tag{22}$$

provided that $p_1 \neq 0$, and $a_1^2 - a_0 a_1 p_1 - a_0^2 p_2 \neq 0$.

Proof. (19) is a special case of (5) and (6) and can be found from (5) and (6) by using the substitution $r = 2$, $a_0 = 0$ and $a_1 = 1$. Again from (12) and (14)

$$\begin{aligned} a_n &= a_1 \tilde{F}_n + a_0 p_2 \tilde{F}_{n-1} = a_1 \tilde{F}_n - \alpha\beta a_0 \tilde{F}_{n-1} \\ &= \begin{cases} a_1 \frac{\alpha^n - \beta^n}{\alpha - \beta} - \alpha\beta a_0 \frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta}, & \text{if } \alpha \neq \beta; \\ a_1(n\alpha^{n-1}) - \alpha^2((n-1)\alpha^{n-2}), & \text{if } \alpha = \beta \end{cases}. \end{aligned} \tag{23}$$

By using (15) and (16), we now prove (21) and (22), respectively. Denote by $L : \mathbb{Z}\times\mathbb{Z} \mapsto \mathbb{Z}$ the operator $L(a_{n-1}, a_{n-2}) := p_1 a_{n-1} + p_2 a_{n-2} = a_n$. As presented before, L is linear, and the sequence $\{a_n\}$ is uniquely determined by L from a given initial vector $(a_0,\, a_1)$. Define $a_{-1} = (a_1 - p_1 a_0)/p_2$, then (a_{-1}, a_0) is the initial vector that generates $\{a_{n-1}\}_{n\geq 0}$ by L. Similarly, the vector $(a_1, p_1 a_1 + p_2 a_0)$ generates sequence $\{a_{n+1}\}_{n\geq}$ by using L. Note the initial vector of $\tilde{F}_n^{(2)}$ is $(0, 1)$. Thus (21) holds if and only if the initial vectors on the two sides are equal:

$$(0, 1) = c_1(a_1, p_1 a_1 + p_2 a_0) + c_2 \left(\frac{a_1 - p_1 a_0}{p_2}, a_0 \right), \tag{24}$$

which yields the solutions (22) for c_1 and c_2 and completes the proof of the corollary. □

Recall that [8] presented the following result.

Proposition 8. *[8] Let $\{a_n\}$ be a sequence of order 2 satisfying linear recurrence relation (2), and let α and β be two zeros of the characteristic polynomial $x^2 - p_1 x - p_2 = 0$ of the relation (2). Then*

$$a_n = \begin{cases} \left(\frac{a_1 - \beta a_0}{\alpha - \beta}\right) \alpha^n - \left(\frac{a_1 - \alpha a_0}{\alpha - \beta}\right) \beta^n, & \text{if } \alpha \neq \beta; \\ n a_1 \alpha^{n-1} - (n-1) a_0 \alpha^n, & \text{if } \alpha = \beta. \end{cases} \tag{25}$$

The paper [8] also gives a method for finding the expressions of the linear recurring sequences of order 2 and the interrelationship among those sequences. However, [8] also pointed out "the method presented in this paper (i.e., Proposition 8) cannot be extended to the higher order setting." In Section 2, we have shown our method based on the IRS can be extended to the higher setting. Here, Proposition 8 is an application of our method in a particular case. In addition, it can be easily seen that (25) can be derived from (23).

Corollary 7 presents the interrelationship between a linear recurring sequence with respect to $E_2 = \{p_1, p_2)$ and its IRS, which can be used to transfer the identities of one sequence to the identities of other sequences in the same set.

Example 9. Let us consider A_2, the set of all linear recurring sequences defined by the homogeneous linear recurrence relation (2) with coefficient set $E_2 = \{p_1, p_2\}$. If $E_2 = \{1, 1\}$, then the corresponding characteristic polynomial has zeros $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$, and (21) gives the expression of the IRS of A_2, which is the Fibonacci sequence $\{F_n\}$:

$$F_n = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}.$$

The sequence in A_2 with the initial vector $(2,1)$ is the *Lucas number sequence* (which is different from the Lucas sequence) $\{L_n\}$. From (20) and (21) and noting $\alpha\beta = -1$, we have the well-known formulas (see, for example, [9]):

$$L_n = F_n + 2F_{n-1} = F_{n+1} + F_{n-1}, \quad F_n = \frac{1}{5}L_{n-1} + \frac{1}{5}L_{n+1}.$$

By using the above formulas, one may transfer identities of Fibonacci number sequence to those of Lucas number sequence and vice versa. For instance, the above relationship can be used to prove that the following two identities are equivalent:

$$\begin{aligned} &F_{n+1}F_{n+2} - F_{n-1}F_n = F_{2n+1} \\ &L_{n+1}^2 + L_n^2 = L_{2n} + L_{2n+2}. \end{aligned}$$

It is clear that both of the identities are equivalent to the Carlitz identity, $F_{n+1}L_{n+2} - F_{n+2}L_n = F_{2n+1}$, shown in [3].

Example 10. Let us consider A_2, the set of all linear recurring sequences defined by the homogeneous linear recurrence relation (2) with coefficient set $E_2 = \{p_1 = p, p_2 = 1\}$. Then (25) tell us that $\{a_n\} \in A_2$ satisfies

$$a_n = \frac{2a_1 - (p - \sqrt{4+p^2})a_0}{2\sqrt{4+p^2}}\alpha^n - \frac{2a_1 - (p + \sqrt{4+p^2})a_0}{2\sqrt{4+p^2}}\left(-\frac{1}{\alpha}\right)^n, \tag{26}$$

where α is defined by

$$\alpha = \frac{p + \sqrt{4+p^2}}{2} \text{ and } \beta = -\frac{1}{\alpha} = \frac{p - \sqrt{4+p^2}}{2}. \tag{27}$$

Similarly, let $E_2 = \{1, q\}$. Then

$$a_n = \begin{cases} \frac{2a_1 - (1-\sqrt{1+4q})a_0}{2\sqrt{1+4q}}\alpha_1^n - \frac{2a_1 - (1+\sqrt{1+4q})a_0}{2\sqrt{1+4q}}\alpha_2^n, & \text{if } q \neq -\frac{1}{4}; \\ \frac{1}{2^n}(2na_1 - (n-1)a_0), & \text{if } q = -\frac{1}{4}, \end{cases}$$

where $\alpha = \frac{1}{2}(1 + \sqrt{1+4q})$ and $\beta = \frac{1}{2}(1 - \sqrt{1+4q})$ are solutions of equation $x^2 - x - q = 0$. The first special case (26) was studied by Falbo [5]. If $p = 1$, the sequence is clearly the

Fibonacci sequence. If $p = 2$ (resp., $q = 1$), the corresponding sequence is the sequence of numerators (when two initial conditions are 1 and 3) or denominators (when two initial conditions are 1 and 2) of the convergent of a continued fraction to $\sqrt{2}$: $\{\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29} \ldots\}$, called the closest rational approximation sequence to $\sqrt{2}$. The second special case is for the case of $q = 2$ (resp., $p = 1$), the resulting $\{a_n\}$ is the Jacobsthal type sequences (See Bergum, Bennett, Horadam, and Moore [1]).

By using Corollary 7, for $E_2 = \{p, 1\}$, the IRS of A_2 with respect to E_2 is

$$\tilde{F}_n^{(2)} = \frac{1}{\sqrt{4+p^4}} \left\{ \left(\frac{p + \sqrt{4+p^2}}{2} \right)^n - \left(\frac{p - \sqrt{4+p^2}}{2} \right)^n \right\}.$$

In particular, the IRS for $E_2 = \{2, 1\}$ is the well-known Pell number sequence $\{P_n\} = \{0, 1, 2, 5, 12, 29, \ldots\}$ with the expression

$$P_n = \frac{1}{2\sqrt{2}} \left\{ (1+\sqrt{2})^n - (1-\sqrt{2})^n \right\}.$$

Similarly, for $E_2 = \{1, q\}$, the IRS of A_2 with respect to E_2 is

$$\tilde{F}_n^{(2)} = \frac{1}{\sqrt{1+4q}} \left\{ \left(\frac{1 + \sqrt{1+4q}}{2} \right)^n - \left(\frac{1 - \sqrt{1+4q}}{2} \right)^n \right\}.$$

In particular, the IRS for $E_2 = \{1, 2\}$ is the well-known Jacobsthal number sequence $\{J_n\} = \{0, 1, 1, 3, 5, 11, 21, \ldots\}$ with the expression

$$J_n = \frac{1}{3} \left(2^n - (-1)^n \right).$$

The Jacobsthal-Lucas numbers $\{j_n\}$ in A_2 with respect to $E_2 = \{1, 2\}$ satisfying $j_0 = 2$ and $j_1 = 1$ has the first few elements as $\{2, 1, 5, 7, 17, 31, \ldots\}$. From (20), one may deduce

$$j_n = J_n + 4J_{n-1} = 2^n + (-1)^n.$$

In addition, the above formula can transform all identities of the Jacobsthal-Lucas number sequence to those of Jacobsthal number sequence and vice-versa. For example, we have

$$\begin{aligned} &J_n^2 + 4J_{n-1}J_n = J_{2n}, \\ &J_m J_{n-1} - J_n J_{m-1} = (-1)^n 2^{n-1} J_{m-n}, \\ &J_m J_n + 2J_m J_{n-1} + 2J_n J_{m-1} = J_{m+n} \end{aligned}$$

from

$$\begin{aligned} &j_n J_n = J_{2n}, \\ &J_m j_n - J_n j_m = (-1)^n 2^{n+1} J_{m-n}, \\ &J_m j_n - J_n j_m = 2J_{m+n}, \end{aligned}$$

respectively. Similarly, we can show that the following two identities are equivalent:

$$j_n = J_{n+1} + 2J_{n-1}, \quad J_{n+1} = J_n + 2J_{n-1}.$$

Remark 11. Corollary 7 can be extended to the linear nonhomogeneous recurrence relations of order 2 with the form: $a_n = pa_{n-1} + qa_{n-2} + \ell$, $\ell \neq 0$, for $p + q \neq 1$. It can be seen that the above recurrence relation is equivalent to the homogeneous form (2) $b_n = pb_{n-1} + qb_{n-2}$, where $b_n = a_n - k$ and $k = \frac{\ell}{1-p-q}$.

Example 12. An obvious example of Remark 11 is the Mersenne number $M_n = 2^n - 1$, $n \geq 0$, which satisfies the linear recurrence relation of order 2: $M_n = 3M_{n-1} - 2M_{n-2}$ (with $M_0 = 0$ and $M_1 = 1$) and the non-homogeneous recurrence relation of order 1: $M_n = 2M_{n-1} + 1$ (with $M_0 = 0$). It is easy to check that sequence $M_n = (k^n - 1)/(k - 1)$ satisfies both the homogeneous recurrence relation of order 2, $M_n = (k + 1)M_{n-1} - kM_{n-2}$, and the non-homogeneous recurrence relation of order 1, $M_n = kM_{n-1} + 1$, where $M_0 = 0$ and $M_1 = 1$. Here, M_n is the IRS with respect to $E_2 = \{3, -2\}$. Another example is Pell number sequence that satisfies both homogeneous recurrence relation $P_n = 2P_{n-1} + P_{n-2}$ and the non-homogeneous relation $\bar{P}_n = 2\bar{P}_{n-1} + \bar{P}_{n-2} + 1$, where $P_n = \bar{P}_n + 1/2$.

Remark 13. Niven, Zuckerman, and Montgomery [12] studied some properties of two sequences, $\{G_n\}_{n\geq 0}$ and $\{H_n\}_{n\geq 0}$, defined respectively by the linear recurrence relations of order 2:

$$G_n = pG_{n-1} + qG_{n-2} \quad \textit{and} \quad H_n = pH_{n-1} + qH_{n-2}$$

with initial conditions $G_0 = 0$ and $G_1 = 1$ and $H_0 = 2$ and $H_1 = p$, respectively. Clearly, $G_n = \tilde{F}_n^{(2)}$, the IRS of A_2 with respect to $E_2 = \{p_1 = p, p_2 = q\}$. Using Corollary 7, we may rebuild the relationship between the sequences $\{G_n\}$ and $\{H_n\}$:

$$\begin{aligned} H_n &= pG_n + 2qG_{n-1}, \\ G_n &= \frac{q}{p^2 + 4q} H_{n-1} + \frac{1}{p^2 + 4q} H_{n+1}. \end{aligned}$$

We now give more examples of higher order IRS in A_3.

Example 14. Consider set A_3 of all linear recurring sequences defined by (1) with coefficient set $E_3 = \{p_1 = 1, p_2 = 1, p_3 = 1\}$. The IRS of A_3 is the tribonacci number sequence $\{0, 0, 1, 1, 2, 4, 7, 13, 24, 44, \ldots\}$ (A000073). From Proposition 1, the generating function of the tribonacci sequence is

$$\tilde{P}_3(t) = \frac{t^2}{1 - t - t^2 - t^3}.$$

Theorem 2 gives the expression of the general term of tribonacci sequence as

$$\tilde{F}_n^{(3)} = \frac{\alpha_1^n}{(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)} + \frac{\alpha_2^n}{(\alpha_2 - \alpha_1)(\alpha_2 - \alpha_3)} + \frac{\alpha_3^n}{(\alpha_3 - \alpha_1)(\alpha_3 - \alpha_2)},$$

where

$$\begin{aligned} \alpha_1 &= -\frac{1}{3} - \frac{C}{3} + \frac{2}{3C}, \\ \alpha_2 &= -\frac{1}{3} + \frac{C}{6}(1 + i\sqrt{3}) - \frac{1}{3C}(1 - i\sqrt{3}), \\ \alpha_3 &= -\frac{1}{3} + \frac{C}{6}(1 - i\sqrt{3}) - \frac{1}{3C}(1 + i\sqrt{3}), \end{aligned}$$

in which $C = ((6\sqrt{33}-34)/2)^{1/3}$. The tribonacci-like sequence $\{a_n\} = \{2, 1, 1, 4, 6, 11, 21, \ldots\}$ (A141036) is in the set A_3. From Theorem 3 and its corollary 4, a_n can be presented as

$$\begin{aligned} a_n &= a_2 \tilde{F}_n^{(3)} + \sum_{j=0}^{1} \sum_{k=j}^{1} a_k \tilde{F}_{n-j-1}^{(3)}. \\ &= a_2 \tilde{F}_n^{(3)} + (a_0 + a_1) \tilde{F}_{n-1}^{(3)} + a_1 \tilde{F}_{n-2}^{(3)}. \end{aligned} \tag{28}$$

Using (17) and (18) in Theorem 5, we may obtain the expression

$$\tilde{F}_n^{(3)} = \frac{6}{19} a_{n+1} - \frac{4}{19} a_n - \frac{1}{19} a_{n-1}, \quad n \geq 0, \tag{29}$$

where $a_{-1} := -2$ due to the linear recurrence relation $a_n = a_{n-1} + a_{n-2} + a_{n-3}$.

It is easy to see the identity

$$\tilde{F}_n^{(3)} = 2\tilde{F}_{n-1}^{(3)} - \tilde{F}_{n-4}^{(3)}, \quad n \geq 4,$$

because the right-hand side is equal to $\tilde{F}_{n-1}^{(3)} + \tilde{F}_{n-2}^{(3)} + \tilde{F}_{n-3}^{(3)}$ after substituting into $\tilde{F}_{n-1}^{(3)} = \tilde{F}_{n-2}^{(3)} + \tilde{F}_{n-3}^{(3)} + \tilde{F}_{n-4}^{(3)}$. Thus, by using (29) we have an identity for $\{a_n\}$:

$$6a_{n+1} - 16a_n + 7a_{n-1} + 2a_{n-2} + 6a_{n-3} - 4a_{n-4} - a_{n-5} = 0$$

for all $n \geq 5$. Similarly from the identity $a_n = 2a_{n-1} - a_{n-4}$, $n \geq 4$, and relation (28), we have the identity

$$\tilde{F}_n^{(3)} + \tilde{F}_{n-1}^{(3)} - 5\tilde{F}_{n-2}^{(3)} - 2\tilde{F}_{n-3}^{(3)} + \tilde{F}_{n-4}^{(3)} + 3\tilde{F}_{n-5}^{(3)} + \tilde{F}_{n-6}^{(3)} = 0$$

for all $n \geq 6$.

It is easy to see there exists r number sequences, denoted by $\{a_n^{(j)}\}$, $j = 1, 2, \ldots, r$, in A_r with respect to E_r such that for any $\{a_n\} \in A_r$ we have

$$a_n = \sum_{j=1}^{r} c_j a_n^{(j)}, \quad n \geq 0,$$

where $\mathbf{c} = (c_1, c_2, \ldots, c_r)$ can be found from the system consisting of the above equations for $n = 0, 1, \ldots, r-1$. In this sense, we may call $\{a_n^{(j)}\}$, $j = 1, 2, \ldots, r$, a basis of A_r with respect to E_r. For instance, for $r = 2$, let $\{a_n^{(1)}\}$ be $\{F_n\}$, the IRS of A_2 with respect to $E_2 = \{1, 1\}$ (i.e., the Fibonacci sequence), and let $\{a_n^{(2)}\}$ be the Lucas number sequence $\{L_n\}$. Then $\{\{F_n\}, \{L_n\}\}$ is a basis of A_2 because. Thus, for any $\{a_n\} \in A_2$ with respect to $E_2 = \{1, 1\}$ and the initial vector (a_0, a_1), we have

$$a_n = \left(a_1 - \frac{1}{2}a_0\right) F_n + \frac{1}{2} a_0 L_n,$$

where the coefficients of the above linear combination are found from the system

$$\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}.$$

Obviously, if $\{L_n\}$ is replaced by any sequence $\{a_n^{(2)}\}$ in A_2 with respect to $E_2 = \{1, 1\}$, provided the initial vector $(a_0^{(2)}, a_1^{(2)})$ of $\{a_n^{(2)}\}$ satisfies

$$\det \begin{bmatrix} 0 & a_0^{(2)} \\ 1 & a_1^{(2)} \end{bmatrix} \neq 0, \tag{30}$$

then $\{\{F_n\}, \{a_n^{(2)}\}\}$ is a basis of A_2 with respect to $E_2 = \{1, 1\}$. However, if $a_0^{(2)} = 1$ and $a_1^{(2)} = 0$, then the corresponding $a_n^{(2)} = F_{n-1}$, and the corresponding basis $\{\{F_n\}, \{a_n^{(2)} = F_{n-1}\}\}$ is called a trivial basis. Thus, for A_2 with respect to $E_2 = \{p_1, p_2\}$, $\{\{\tilde{F}_n^{(2)}\}, \{a_n^{(2)}\}\}$ forms a non-trivial basis, if the initial vector $(a_0^{(2)}, a_1^{(2)})$ of $\{a_n^{(2)}\}$ satisfies (30) and $a_0^{(2)} \neq 1/p_2$ when $a_1^{(2)} = 0$. The second condition guarantees that the basis is not trivial, otherwise $a_n^{(2)} = \tilde{F}_{n-1}^{(2)}$.

4 Applications

In this section, we will give the applications of IRS to the Stirling numbers of the second kind, Wythoff array, and the Boustrophedon transform. Let A_r be the set of all linear recurring sequences defined by the homogeneous linear recurrence relation (1) with coefficient set $E_r = \{p_1, p_2, \ldots, p_r\}$, and let $\tilde{F}_n^{(r)}$ be the IRS of A_r. Theorems 3 gives expressions of $\{a_n\} \in A_r$ and the IRS of A_r in terms of each other. We will give a different expression of $\{a_n\} \in A_r$ in terms of $\tilde{F}_n^{(r)}$ using the generating functions of $\{a_n\}$ and $\{\tilde{F}_n^{(r)}\}$ shown in Proposition 1. More precisely, let $\{a_n\} \in A_r$. Then Proposition 1 shows that its generating function $P(t)$ can be written as (3). In particular, the generating function for the IRS with respect to $\{p_j\}$ is presented as (4). Comparing the coefficients of $P(t)$ and $\tilde{P}(t)$, one finds

Proposition 15. *Let $\{a_n\}$ be an element of A_r, and let $\{\tilde{F}_n^{(r)}\}$ be the IRS of A_r. Then*

$$\tilde{F}_n^{(r)} = [t^{n-r+1}] \frac{1}{1 - \sum_{j=1}^{r} p_j t^j} \tag{31}$$

$$a_n = a_0 \tilde{F}_{n+r-1}^{(r)} + \sum_{k=1}^{r-1} (a_k - \sum_{j=1}^{k} p_j a_{k-j}) \tilde{F}_{n+r-k-1}^{(r)}. \tag{32}$$

Proof. Eq. (31) comes from

$$\tilde{F}_n^{(r)} = [t^n] \tilde{P}(t) = [t^{n-r+1}] \frac{1}{1 - \sum_{j=1}^{r} p_j t^j}.$$

Hence,

$$a_n = [t^n]P(t) = a_0[t^n]\frac{1}{1-\sum_{j=1}^r p_j t^j} + [t^n]\frac{\sum_{k=1}^{r-1}\left(a_k - \sum_{j=1}^k p_j a_{k-j}\right)t^k}{1-\sum_{j=1}^r p_j t^j}$$

$$= a_0\tilde{F}_{n+r-1}^{(r)} + \sum_{k=1}^{r-1}\left(a_k - \sum_{j=1}^{k} p_j a_{k-j}\right)[t^{n-k}]\frac{1}{1-\sum_{j=1}^r p_j t^j},$$

which implies (31). □

Proposition 16. *If*

$$g(t) = \frac{t^{r-1}}{1-\sum_{j=1}^r p_j t^j} \tag{33}$$

is the generating function of a sequence $\{b_n\}$*, then* $\{b_n\}$ *satisfies*

$$b_n = \begin{cases} 0, & \text{if } n = 0, 1, \ldots, r-2; \\ 1, & \text{if } n = r-1; \\ \sum_{j=1}^r p_j b_{n-j}, & \text{if } n \geq r. \end{cases}$$

Proof. It is clear that

$$b_j = [t^j]g(t) = [t^{j-r+1}]\frac{1}{1-\sum_{j=1}^r p_j t^j}$$

$$= [t^{j-r+1}]\sum_{k\geq 0}\left(\sum_{j=1}^r p_j t^j\right)^k = \begin{cases} 0 & \text{if } 0 \leq j \leq r-2, \\ 1 & \text{if } j = r-1. \end{cases}$$

For $n \geq r$,

$$\sum_{j=1}^r p_j b_{n-j} = \sum_{j=1}^r p_j [t^{n-j}]g(t)$$

$$= [t^n]\sum_{j=1}^r t^{r-1}\frac{p_j t^j}{1-\sum_{i=1}^r p_i t^i}$$

$$= [t^n]t^{r-1}\left(\frac{1}{1-\sum_{i=1}^r p_i t^i} - 1\right)$$

$$= [t^n]g(t) - [t^n]t^{r-1} = b_n,$$

which completes the proof. □

The Stirling numbers of the second kind $S(n,k)$ count the number of ways to partition a set of n labelled objects into k unlabeled subsets. We now prove that the sequence $\{b_n = S(n+1,k)\}_{n\geq 0}$ is the IRS of A_k for any $k \in \mathbb{N}_0$ by using the structure of the triangle of the Stirling numbers of the second kind shown in Table 1.

$n\backslash k$	0	1	2	3	4	5	6
0	1						
1	0	1					
2	0	1	1				
3	0	1	3	1			
4	0	1	7	6	1		
5	0	1	15	25	10	1	
6	0	1	31	90	65	15	1

Table 1. Triangle of the Stirling numbers of the second kind

Theorem 17. *Let $S(n,k)$ be the Stirling numbers of the second kind. Then, for any fixed integer $k>0$, $S(n+1,k)$ is the IRS of A_k, i.e.,*

$$S(n+1,k)=\tilde{F}_n^{(k)}, \tag{34}$$

where A_k is the set of linear recurring sequences $\{a_n\}$ generated by

$$a_n=\sum_{j=1}^{k} p_j a_{n-j}, \tag{35}$$

which has characteristic polynomial

$$P(t)=t^k-\sum_{j=1}^{k} p_j t^{k-j}=\prod_{j=1}^{k}(1-jt). \tag{36}$$

Thus, the generating function of $\{S(n+1,k)\}$ is

$$\tilde{P}(t)=\frac{t^k}{\prod_{j=1}^{k}(1-jt)}. \tag{37}$$

Proof. $\{S(n,k)\}_{0\le k\le n}$ obeys the recurrence relation

$$S(n+1,k)=kS(n,k)+S(n,k-1), \quad 0\le k\le n, \tag{38}$$

which can be explained by using the definition of $S(n,k)$ shown above. A partition of the $n+1$ objects into k nonempty subsets either contains the subset $\{n+1\}$ or it does not. The number of ways that $\{n+1\}$ is one of the subsets is given by $S(n,k-1)$, since we must partition the remaining n objects into the available $k-1$ subsets. The number of ways that $\{n+1\}$ is not one of the subsets is given by $kS(n,k)$ because we partition all elements other than $n+1$ into k subsets with k choices for inserting the element $n+1$. Summing these two values yields (38). Write (38) as

$$S(n,k-1)=S(n+1,k)-kS(n,k), \quad 0<k\le n.$$

Denote by $g_{k-1}(t)$ and $g_k(t)$ the generating functions of $k-1$st and kth columns of the Stirling triangle shown in Table 2. Then the above equation means

$$[t^n]g_{k-1}(t) = S(n,k-1) = S(n+1,k) - kS(n,k) = [t^{n+1}]g_k(t) - k[t^n]g_k(t) = [t^{n+1}](1-kt)g_k(t)$$

for $0 < k \leq n$. Therefore, we have

$$g_k(t) = \frac{t}{1-kt} g_{k-1}(t).$$

Since $g_1(t) = t/(1-t)$, we obtain the generating function of $\{S(n,k)\}$ for $k > 0$

$$g_k(t) = \frac{t^k}{\prod_{j=1}^{k}(1-jt)}. \tag{39}$$

Denote $b_n = S(n+1,k)$ and the generating function of $\{b_n\}$ by $B(t)$. Then

$$b_n = [t^n]B(t) = S(n+1,k) = [t^{n+1}]g_k(t), \quad 0 < k \leq n,$$

which implies the generating function of $\{b_n\}$ is

$$B(t) = \frac{1}{t} g_k(t) = \frac{t^{k-1}}{\prod_{j=1}^{k}(1-jt)}. \tag{40}$$

Using Proposition 16, we immediately know that $\{S(n+1,k)\}$ is the IRS of A_k that has the generating function (37), where A_k is the set of linear recurring sequences $\{a_n\}$ generated by recurrence relation (35) with the characteristic polynomial (36). $\square$

Example 18. Since the second column of the Stirling triangle, $\{S(n+1,2)\}$, has the generating function

$$\tilde{P}_2(t) = \frac{t}{(1-t)(1-2t)} = \frac{1}{1-3t+2t^2},$$

we have the recurrence relation

$$S(n+1,2) = 3S(n,2) - 2S(n-1,2)$$

with the initial $S(1,2) = 0$ and $S(2,2) = 1$. Thus we have $S(3,2) = 3$, $S(4,2) = 7$, $S(5,2) = 15$, $S(6,2) = 31$, etc.

Similarly, $\{S(n+1,3)\}$ is a linear recurring sequence generated by

$$S(n+1,3) = 6S(n,3) - 11S(n,2) + 6S(n,1)$$

with initials $S(1,3) = S(2,3) = 0$ and $S(3,3) = 1$. From the recurrence relation one may find $S(4,3) = 6$, $S(5,3) = 25$, $S(6,3) = 90$, etc.

If we transfer the entries in the array shown in Table 1 to $d_{n,k} = k!S(n+1, k+1)/n!$, then the array $(d_{n,k})_{0\leq k\leq n}$ is a Riordan array. The structure and computation of the Riordan array of the generalized Stirling numbers can be found from [6], while the sequence characterization of the Riordan array is shown in [7].

We now consider the Wythoff array shown in Table 2, in which the two columns to the left of the vertical line consist respectively of the nonnegative integers n, and the lower Wythoff sequence (A201), whose nth term is $[(n+1)\alpha]$, where $\alpha = (1+\sqrt{5})/2$. The rows are linear recurring sequences generated by the Fibonacci rule that each term is the sum of the two previous terms.

0	1	1	2	3	5	8	13	$\cdots$
1	3	4	7	11	18	29	47	$\cdots$
2	4	6	10	16	26	42	68	$\cdots$
3	6	9	15	24	39	63	102	$\cdots$
4	8	12	20	32	52	84	136	$\cdots$
5	9	14	23	37	60	97	157	$\cdots$
6	11	17	28	45	73	118	191	$\cdots$
7	12	19	31	50	81	131	212	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Table 2. the Wythoff array

The first row sequence in Table 2 is Fibonacci sequence, i.e., the IRS in A_2 with respect to $E_2 = \{1, 1\}$. The jth row sequence $\{a_n^{(j)}\}_{n\geq 0}$ are linear recurring sequences in A_2 with respect to $E_2 = \{1, 1\}$ with initials $(j, [(j+1)\alpha])$, $j = 1, 2, \ldots$, where $\alpha = (1+\sqrt{5})/2$. From (23) or (25), the expression of $a_n^{(j)}$ can be written as

$$\begin{aligned} a_n^{(j)} &= \left(\frac{\left[(j+1)\frac{1+\sqrt{5}}{2}\right] - j\frac{1-\sqrt{5}}{2}}{\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n \\ &\quad - \left(\frac{\left[(j+1)\frac{1+\sqrt{5}}{2}\right] - j\frac{1+\sqrt{5}}{2}}{\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n . \end{aligned} \tag{41}$$

Since every integer in $\mathbb{N}_0$ appears exactly once (see [14]), the collection $\{a_n^{(j)}\}_{n\geq 0}$, $j = 0, 1, \ldots$, gives a partition of $\mathbb{N}_0$. $[(j+1)\alpha]$, $\alpha = (1+\sqrt{5})/2$, can be considered as the representative of the equivalence class $\{a_n^{(j)} : n = 0, 1, \ldots\}$.

0	1	2	5	12	29	70	169	$\cdots$
1	3	7	17	41	99	239	577	$\cdots$
2	6	14	34	82	198	470	1154	$\cdots$
3	8	19	46	111	268	647	1562	$\cdots$
4	11	26	63	152	367	886	2139	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Table 3. the Pell-Wythoff array

$$\begin{array}{llll}
a_0 = b_0 & & & \\
a_1 \rightarrow & b_1 = a_0 + a_1 & & \\
b_2 = a_1 + a_2 + b_1 & \leftarrow a_2 + b_1 & \leftarrow a_2 & \\
a_3 \rightarrow & a_3 + b_2 \rightarrow & a_2 + a_3 + b_1 + b_2 \rightarrow & b_3 = 2a_2 + a_3 + b_1 + b_2
\end{array}$$

Table 4. Triangle of the boustrophedon transform

The Wythoff array can be extended to similar arrays associated with different linear recurrence relations. For instance, one may define the Pell-Wythoff array shown in Table 3.

The first row sequence in Table 2 is the Pell sequence, which is also the IRS in A_2 with respect to $E_2 = \{2, 1\}$. The jth row sequence $\{b_n^{(j)}\}_{n\geq 0}$ are linear recurring sequences in A_2 with respect to $E_2 = \{2, 1\}$ with initials $(j, [(j+1)r] - 1)$, $j = 1, 2, \ldots$, where $r = 1 + \sqrt{2}$.

Noting (23) or (25), we obtain the expression of $b_n^{(j)}$ as

$$\begin{aligned}
b_n^{(j)} &= \left(\frac{\left[(j+1)(1+\sqrt{2})\right] - j(1-\sqrt{2}) - 1}{2\sqrt{2}}\right)\left(1+\sqrt{2}\right)^n \\
&\quad - \left(\frac{\left[(j+1)(1+\sqrt{2})\right] - j(1+\sqrt{2}) - 1}{2\sqrt{2}}\right)\left(1-\sqrt{2}\right)^n . \qquad (42)
\end{aligned}$$

Millar, Sloane, and Young [11] showed that the boustrophedon transform of a given sequence $\{a_n\}_{n\geq 0}$ is the sequence produced by the triangle shown in Table 4.

Formally, the entries $T_{n,k}$, $n \geq k \geq 0$, of the above triangle are defined by

$$\begin{aligned}
&T_{n,0} = a_n \quad n \geq 0, \\
&T_{n+1,k+1} = T_{n+1,k} + T_{n,n-k} \quad n \geq k \geq 0, \qquad (43)
\end{aligned}$$

and then $b_n = T_{n,n}$ for all $n \geq 0$. Denote the (exponential) generating functions of sequences $\{a_n\}$ and $\{b_n\}$ by $A(t)$ and $B(t)$. Then Theorem 1 of [11] gives

$$B(t) = (\sec t + \tan t)A(t),$$

which implies a relationship of the generating function $\tilde{P}_r(t)$ of the IRS $\{\tilde{F}_n^{(r)}\}$ of A_r and the generating function $(BP_r)(t)$ of the boustrophedon transform sequence of $\{\tilde{F}_n^{(r)}\}$:

$$(BP_r)(t) = (\sec t + \tan t)P_r(t) = (\sec t + \tan t)\frac{t^{r-1}}{1 - \sum_{j=1}^{r} p_j t^j}. \tag{44}$$

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References

[1] G. E. Bergum, L. Bennett, A. F. Horadam, and S. D. Moore, Jacobsthal polynomials and a conjecture concerning Fibonacci-like matrices, *Fibonacci Quart.* **23** (1985), 240–248.

[2] R. L. Burden and J. D. Faires, *Numerical Methods*, 8th Edition, Brooks/Cole Publishing Co., 2005.

[3] L. Carlitz, Problem B-110, *Fibonacci Quart.* **5** (1967), 469–470.

[4] L. Comtet, *Advanced Combinatorics*, D. Reidel Publishing Comp., 1974.

[5] C. Falbo, The golden ratio — a contrary viewpoint, *College Math. J.* **36** (2005), 123–134.

[6] T. X. He, Expression and computation of generalized Stirling numbers, *J. Combin. Math. Combin. Comput.*, **86** (2013), 239–268.

[7] T. X. He and R. Sprugnoli, Sequence characterization of Riordan arrays, *Discrete Math.*, **309** (2009), 3962–3974.

[8] T. X. He and P. J.-S. Shiue, On sequences of numbers and polynomials defined by second order recurrence relations, *Int. J. Math. Math. Sci.*, **2009** (2009), Article ID 709386, 1–21.

[9] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Pure and Applied Mathematics, Wiley-Interscience, 2001.

[10] R. Lidl and H. Niederreiter, *Finite Fields*, With a foreword by P. M. Cohn, 2nd edition, Encyclopedia of Mathematics and its Applications, Vol. 20, Cambridge University Press, Cambridge, 1997.

[11] J. Millar, N. J. A. Sloane, and N. E. Young, A new operation on sequences: the boustrophedon transform, *J. Combin. Theory, Series A*, **76** (1996), 44–54.

[12] I. Niven, H. S. Zuckerman, and H. L. Montgomery, *An Introduction to the Theory of Numbers*, 5th edition, John Wiley & Sons, Inc., 1991.

[13] W. Rudin, *Principles of Mathematical Analysis*, 3rd edition, International Series in Pure and Applied Mathematics. McGraw-Hill Book Co., 1976.

[14] N. J. A. Sloane, My favorite integer sequences, *AT&T Lab Research Preprint*, http://www2.research.att.com/~njas/doc/sg.pdf.

[15] G. Strang, *Linear Algebra and Its Applications*, 2nd edition, Academic Press, 1980.

[16] H. S. Wilf, *Generatingfunctionology*, 2nd edition, Academic Press, 1990.

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