Journal of Integer Sequences, Vol. 16 (2013),
Article 13.9.5

Curious Relations and Identities Involving the Catalan Generating Function and Numbers

Asamoah Nkwanta
Department of Mathematics
Morgan State University
Baltimore, MD 21251
USA
Asamoah.Nkwanta@morgan.edu

Akalu Tefera
Department of Mathematics
Grand Valley State University
Allendale, MI 49401
USA
teferaa@gvsu.edu

Abstract

Riordan matrix methods and manipulation of various generating functions are used to find curious relations among the Catalan, central binomial, and RNA generating functions. In addition, the Wilf-Zeilberger method is used to find identities where the gamma function and Catalan numbers are expressed in terms of the Gauss hypergeometric function. As a consequence of the identities, new recurrence relations are obtained. In particular, a new recurrence relation is given for the RNA numbers. Furthermore, other representations of π and the Catalan numbers are obtained.

1 Introduction

In this paper we use Riordan matrix methods and manipulation of generating functions to find curious relations among the Catalan, central binomial, and RNA generating functions.

We also give relations between the Fibonacci and Catalan generating functions. In addition to finding relations involving generating functions, the Wilf-Zeilberger (WZ) method is used to find identities where the gamma function and Catalan numbers A000108 [13] are expressed in terms of the Gauss hypergeometric function. As a consequence of the identities, new recurrence relations are obtained. Furthermore, other representations of π and the Catalan numbers A000108 [13] are presented.

The results of this paper are proved by the WZ method, manipulation of generating functions, and Riordan matrix multiplication. The notion of a Riordan matrix is used to prove combinatorial sums and identities (Shapiro et al. [12] and Sprugnoli [14]). The WZ method is based on an algorithm that finds recurrence relations that are satisfied by certain sums (Wilf and Zeilberger [20] and Petkovsek et al. [10]). The recurrence relations given in this paper by Equations (15), (17), and (24) are automatically generated by the Maple package EKHAD, which can be downloaded from Zeilberger's website [23]. In addition, the SumTools package in Maple that implements Zeilberger's algorithm can also be used. For an excellent exposition of Zeilberger's algorithm, see Petkovsek et al. [10].

This paper is arranged as follows. Preliminary material on hypergeometric functions, the WZ method, the gamma function, generating functions, and Riordan matrices are given in Section 2. Readers familiar with these topics may skip this section. In Section 3, Riordan matrix and generating function methods are utilized to find curious relations among the Catalan, central binomial, and RNA generating functions. Curious relations involving the Fibonacci generating function are given in Section 4. The WZ method is used to find new recurrence relations that are presented in Section 5. The main result of the paper, Theorem 11, which is an identity involving the Catalan numbers A000108 [13], is also given in this section. In addition, we present in this section other representations of the irrational number π and the ubiquitous Catalan numbers. Some concluding comments are given in Section 6.

2 Preliminary material

Throughout this paper we denote the sets $\{1, 2, \ldots\}$ A000027 [13] and $\{0, 1, \ldots\}$ A001477 [13] by $\mathbb{N}$ and $\mathbb{N}_0$, respectively. Let $(x)_k$ denote the rising factorial defined by

$$(x)_k = x(x+1)\cdots(x+k-1)$$

for $k \in \mathbb{N}$ and $(x)_0 = 1$. The gamma function denoted by $\Gamma(z)$ is defined as

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt,$$

if $\operatorname{Re}(z) > 0$. For $z = n$ where $n \in \mathbb{N}_0$

$$\Gamma(n+1) = n! = (1)_n.$$

Some useful properties are $\Gamma\left(1/2\right)=\sqrt{\pi}$ and

$$(x)_k=\frac{(x+k-1)!}{(x-1)!}=\frac{\Gamma\left(k+x\right)}{\Gamma\left(x\right)}. \tag{1}$$

The Gauss hypergeometric function ${}_2F_1$ is defined by

$${}_2F_1(a,b;c;z)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}$$

where a, b and c are arbitrary complex constants. If we take $z=1$, then

$${}_2F_1(a,b;c;1)=\frac{\Gamma\left(c-a-b\right)\Gamma\left(c\right)}{\Gamma\left(c-a\right)\Gamma\left(c-b\right)}.$$

For more information on the gamma and hypergeometric functions, see Petkovsek et al. [10].

The Wilf-Zeilberger algorithm was developed by Wilf and Zeilberger [20] as a method for certifying the truth of certain combinatorial identities. The identities involve certain hypergeometric functions. A discrete function $F\left(n,k\right)$ is called hypergeometric if

$$F\left(n+1,k\right)/F\left(n,k\right) \text{ and } F\left(n,k+1\right)/F\left(n,k\right)$$

are both rational functions in n and k. The WZ proof method is briefly described as follows. Suppose we want to prove the identity

$$\sum_{k=0}^{\infty}U\left(n,k\right)=r\left(n\right).$$

If $r\left(n\right)\neq 0$, then dividing through by $r\left(n\right)$ the given identity is equivalent to

$$\sum_{k=0}^{\infty}\left\{\frac{U\left(n,k\right)}{r\left(n\right)}\right\}=1.$$

Now set $F\left(n,k\right)=U\left(n,k\right)/r\left(n\right)$. We want to prove $\sum_{k=0}^{\infty}F\left(n,k\right)=1$. To certify the validity of this, it would suffice to find a function $G\left(n,k\right)$ such that

$$F\left(n+1,k\right)-F\left(n,k\right)=G\left(n,k+1\right)-G\left(n,k\right).$$

The production G comes as a result of using the WZ algorithm. The pair of functions F and G are called WZ pairs. See Wilf and Zeilberger [20], Petkovsek et al. [10], Wilf [21], and Gessel [4] for more information on the WZ algorithm and WZ pairs. See Tefera [17] for a quick review of the WZ method.

A formal power series of the form

$$a\left(z\right)=a_0+a_1z+a_2z^2+\cdots=\sum_{n\geq 0}a_nz^n$$

where z is an indeterminate is called the ordinary *generating function* of the sequence $\{a_0, a_1, \ldots\}$. Let

$$b(z) = \sum_{n\geq 0} b_n z^n = \frac{1}{\sqrt{1-4z}}$$

denote the central binomial generating function of the sequence $\{b_0, b_1, \ldots\}$. Then

$$\{b_n\}_{n\geq 0} = \{1, 2, 6, 20, \ldots\} = \left\{\binom{2n}{n}\right\}_{n\geq 0}$$

where b_n denotes the nth central binomial coefficient A00984 [13]. Let

$$c(z) = \sum_{n\geq 0} c_n z^n = \frac{1-\sqrt{1-4z}}{2z} \tag{2}$$

denote the Catalan generating function of the sequence $\{c_0, c_1, \ldots\}$. Then

$$\{c_n\}_{n\geq 0} = \{1, 1, 2, 5, \ldots\} = \left\{\frac{1}{1+n}\binom{2n}{n}\right\}_{n\geq 0} \tag{3}$$

where c_n denotes the nth Catalan number A000108 [13]. See Stanley [15, 16] for a number of combinatorial and analytical interpretations of the Catalan numbers A000108 [13]. Let

$$f(z) = \sum_{n\geq 0} f_n z^n = \frac{1}{1-z-z^2} \tag{4}$$

denote the Fibonacci generating function of the sequence $\{f_0, f_1, \ldots\}$. Then

$$\{f_n\}_{n\geq 0} = \{1, 1, 2, 3, \ldots\} = \left\{\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right)\right\}_{n\geq 0}$$

where f_n denotes the nth Fibonacci number A000045 [13]. We now let

$$s(z) = \sum_{n\geq 0} s_n z^n = \frac{1-z+z^2-\sqrt{1-2z-z^2-2z^3+z^4}}{2z^2} \tag{5}$$

denote the RNA generating function from discrete mathematical biology (Waterman [18, 19]). Then $s_0 = 0$, $s_1 = 1$,

$$s_n = \sum_{k\geq 1} \frac{1}{n-k}\binom{n-k}{k}\binom{n-k}{k-1} \ (k < n), \tag{6}$$

and $\{s_n\}_{n\geq 0} = \{0, 1, 1, 1, 2, 4, 8, \ldots\}$ where s_n denotes the nth RNA number (A004148 [13] and Waterman [19]). We now define the Fine generating function, which we denote by $\Phi(z)$. Let

$$\Phi(z) = \sum_{n\geq 0} \Phi_n z^n = \frac{1-\sqrt{1-4z}}{3-\sqrt{1-4z}} \frac{1}{z}.$$

Then $\{\Phi_n\}_{n\geq 0} = \{1, 0, 1, 2, 6, 18, \ldots\}$ where Φ_n denotes the nth Fine number A000957 [13]. See Deutsch and Shapiro [3] for more information on the Fine numbers. For more information on generating functions, hypergeometric functions, and the WZ method, see Petkovsek et al. [10].

The definition of a Riordan matrix and rules for multiplying Riordan matrices are presented.

Definition 1. (Shapiro et al. [12], Sprugnoli [14]) An infinite matrix

$$L = (l_{n,k})_{n,k\in\mathbb{N}_0}$$

with complex entries in $\mathbb{C}$ is called a *Riordan matrix* if the k^{th} column satisfies

$$\sum_{n\geq 0} l_{n,k}\ z^n = g(z)f(z)^k$$

where $g(z) = g_0 + g_1 z + g_2 z^2 + \cdots$ and $f(z) = f_1 z + f_2 z^2 + \cdots$ belong to the ring of formal power series $\mathbb{C}[[z]]$, and $f_1 \neq 0$ and $g_0 \neq 0$.

Riordan matrices are typically written in pair form as $(g(z), f(z))$ where $g(z)$ and $f(z)$ are ordinary generating functions. Note that Riordan matrices can also be defined for exponential generating functions. However, the Riordan matrices in this paper are defined by ordinary generating functions and are the proper Riordan arrays as given by Sprugnoli [14]. Pascal's triangle A007318 [13] written in lower-triangular form is an example of a Riordan matrix and is denoted by

$$P = (1/(1-z), z/(1-z)).$$

Let us denote by $L * N$, or by simple juxtaposition LN, the row-by column product of two Riordan matrices. If

$$L = (l_{n,k})_{n,k\in\mathbb{N}_0} = (g(z), f(z)) \text{ and } N = (\nu_{n,k})_{n,k\in\mathbb{N}_0} = (h(z), l(z))$$

are Riordan matrices, then

$$\begin{aligned} LN = \left(\sum_{j=0}^{n} l_{n,j}\nu_{j,k}\right)_{n,k\in\mathbb{N}_0} &= (g(z), f(z)) * (h(z), l(z)) \\ &= (g(z)h(f(z)), l(f(z))). \end{aligned}$$

For more information on Riordan matrices and arrays, see Shapiro et al. [12] and Sprugnoli [14].

3 Catalan, central binomial and RNA relations

Consider the following Riordan matrix relation

$$B = C_0 R^2 E \tag{7}$$

where $C_0 = (c(z^2), zc(z^2))$ is Catalan array A053121 [13], $R = (s(z), zs(z))$ is the RNA array A097724 [13] from mathematical biology, and $E = (1/(1-z), z)$ is the lower-triangular array A000012 [13] with entries with all ones on and below the main diagonal. Recall that the generating functions $c(z)$ and $s(z)$ are, respectively, given by Equations (2) and (5). The matrix B A111418 [13] can also be obtained from

$$B = P^2 C_0 E \tag{8}$$

where P is Pascal matrix A007318 [13] (Nkwanta [8, 9]). By Riordan matrix multiplication, the right side of Equation (8) gives

$$B = \left(c(z)/\sqrt{1-4z}, zc^2(z)\right) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 3 & 1 & 0 & 0 & 0 & \cdots \\ 10 & 5 & 1 & 0 & 0 & \cdots \\ 35 & 21 & 7 & 1 & 0 & \cdots \\ 126 & 84 & 36 & 9 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{9}$$

The generating function of the left component of B can also be expressed as

$$c(z)/\sqrt{1-4z} = c^2(z)/\left(1 - zc^2(z)\right) \tag{10}$$

and is well known as a special case of

$$B_k(z) = c^k(z)/\sqrt{1-4z} = \sum_{n\geq 0} \binom{2n+k}{n} z^n. \tag{11}$$

For more information on Equation (11), see Graham et al. [5], Riordan [11], and Wilf [21]. For more information and a combinatorial interpretation of B, see Nkwanta [8]. The inverse of B is given below

$$B^{-1} = \left((1-z)/(1+z)^2, z/(1+z)^2\right) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \ldots \\ -3 & 1 & 0 & 0 & 0 & \ldots \\ 5 & -5 & 1 & 0 & 0 & \ldots \\ -7 & 14 & -7 & 1 & 0 & \ldots \\ 9 & -30 & 27 & -9 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The entries of B^{-1} form monic orthogonal polynomials whose moments are certain binomial coefficients. The moments are associated with the the Chebyshev polynomials of the first kind. See Nkwanta and Barnes [7] for more on the connection of B to the Chebyshev polynomials.

By multiplying the matrices on the right side of Equation (7) and equating generating function components of the matrix with Equation (9) , the left components give

$$c(z)/\sqrt{1-4z} = \left(c\left(z^2\right)\cdot r\left(z\right)\cdot t\left(z\right)\right)/\left(1-zc\left(z^2\right)\cdot r\left(z\right)\cdot t\left(z\right)\right) \tag{12}$$

where $r\left(z\right) = s\left(zc\left(z^2\right)\right)$ and

$$t\left(z\right) = s\left[zc\left(z^2\right)\cdot s\left(zc\left(z^2\right)\right)\right].$$

Recall that $s\left(z\right)$ is the RNA generating function. This leads to curious relations among the central binomial, Catalan and RNA generating functions given by Equation (12). Thus, we have the following theorem.

Theorem 2. *Let $r\left(z\right) = s\left(zc\left(z^2\right)\right)$ and $t\left(z\right) = s\left[zc\left(z^2\right)\cdot s\left(zc\left(z^2\right)\right)\right]$ where $c\left(z\right)$ and $s\left(z\right)$ are the Catalan and RNA generating functions, respectively. Then*

(a) $c\left(z\right) = \left(\left(\sqrt{1-4z}\right)\cdot c\left(z^2\right)\cdot r\left(z\right)\cdot t\left(z\right)\right)/\left(1-zc\left(z^2\right)\cdot r\left(z\right)\cdot t\left(z\right)\right)$.

(b) $b\left(z\right) = B_0\left(z\right) = \left(c\left(z^2\right)\cdot r\left(z\right)\cdot t\left(z\right)\right)/\left(c\left(z\right)\cdot\left(1-zc\left(z^2\right)\cdot r\left(z\right)\cdot t\left(z\right)\right)\right)$.

Proof. The theorem follows as a result of Riordan matrix multiplication and Equation (7). □

As a consequence of Theorem 2, the central binomial and Catalan generating functions are expressed in terms of the RNA generating function. An alternative expression of the RNA generating function is now given.

Proposition 3. *Let $w\left(z\right) = \left(s\left(z\right)-1\right)/z$ where $s\left(z\right)$ is the RNA generating function. Then*

$$w\left(z\right) = 1/\left(1-\left(z+z^2+z^3w\left(z\right)\right)\right) \text{ or} \tag{13}$$

$$s\left(z\right) = 1/\left(1-z-z^2\left(s\left(z\right)-1\right)\right).$$

Proof. The proof follows by using Equation (5) and simplifying. □

Note that the constant term of the formal power series of Equation (5) is removed from Equation (13). For more information on the derivation of Equation (13), see Nkwanta [9].

Remark 4. As a consequence of Equation (13) the RNA generating function $s\left(z\right)$ can be expressed as the following continued fraction expansion

$$s\left(z\right) = 1+\frac{z}{1-z-z^2-\frac{z^3}{1-z-z^2-\frac{z^3}{1-z-z^2-\cdots}}}.$$

4 In terms of the Fibonacci generating function

The following generating function

$$f(z) / \left(1 + z^2 f(z)\right) = 1/(1-z), \tag{14}$$

where $f(z)$ is the Fibonacci generating function given by Equation (4), leads to curious relations involving $1/(1-\alpha z)$ where $\alpha \neq 0$ and $\alpha \in \mathbb{R}$, binomial generating functions, the Catalan generating function $c(z)$, and the Fine generating function $\Phi(z)$.

Proposition 5. *(a)* $1/(1-\alpha z) = f(\alpha z) / (1 + \alpha^2 z^2 f(\alpha z))$.

(b) $B_k(z) = f(zc^2(z))\, c^k(z) / (1 + z^2 c^4(z) f(zc^2(z)))$.

(c) $c(z) = f(zc(z)) / (1 + z^2 c^2(z) f(zc(z)))$

(d) $\Phi(z) = f(z^2 c^2(z)) / (1 + z^4 c^4(z) f(z^2 c^2(z)))$.

Proof. (a) Use Equation (14) and some simplification.

(b) Use Equation (14) and multiply by $c^k(z)$. Then use Equation (11).

(c) For (c) use Equation (14). Then, use the fact from Deutsch and Shapiro [2] that

$$c(z) = 1/(1 - zc(z)).$$

(d) Use Equation (14). Then, use the fact from Deutsch and Shapiro [2] that

$$\Phi(z) = 1/\left(1 - z^2 c^2(z)\right).$$

□

We observe that for $k = 0$ and 1, respectively, the central binomial generating function $b(z)$ and Equation (10) are special cases of Proposition 5(b). As a result of Proposition 5, the binomial, Catalan, and Fine generating functions are expressed in terms of the Fibonacci generating function. For more information on the derivation of Equation (14), see Jean-Louis and Nkwanta [6]. See Deutsch and Shapiro [2, 3] for more information on the Fine numbers A000957 [13] and Fine generating function.

Remark 6. By combining Theorem 2 and Proposition 5, we observe that the generating functions $b(z)$ and $c(z)$ can be simultaneously expressed in terms of the Fibonacci and RNA generating functions.

5 Finding recurrence relations

We now use the Wilf-Zeilberger algorithm to find various recurrence relations. Applying the WZ algorithm to Equation (6) gives the following recurrence relation for the RNA numbers A004148 [13].

Proposition 7. *For $n \geq 0$, s_n satisfies*

$$(n+5)\, s_{n+4} = (2n+7)\, s_{n+3} + (n+2)\, s_{n+2} + (2n+1)\, s_{n+1} - (n-1)\, s_n$$

where $s_0 = 0, s_1 = 1, s_2 = 1,$ and $s_3 = 1$.

Proof. Let $F(n,k)$ denote the summand of s_n, that is,

$$F(n,k) = \frac{1}{n-k}\binom{n-k}{k}\binom{n-k}{k-1}.$$

Now, applying the WZ algorithm on $F(n,k)$ we get

$$\begin{aligned}(n+5)\, F(n+4,k) - (2n+7)\, F(n+3,k) - (n+2)\, F(n+2,k)\\ - (2n+1)\, F(n+1,k) + (n-1)\, F(n,k) = G(n,k+1) - G(n,k)\end{aligned} \tag{15}$$

where $G(n,k) = F(n,k)\, R(n,k)$ and

$$R(n,k) = -\frac{(k-1)\, k\, (n+1-k)\, (n-k)}{(n+5-2k)(n+4-2k)^2(n+3-2k)^2(n+2-2k)^2(n+1-2k)} P(n,k)$$

where $P(n,k) = -\left(332 + 304n + 196n^2 + 107n^3 + 30n^4 + 3n^5\right)$

$$+\left(716 + 446n + 203n^2 + 81n^3 + 12n^4\right) k$$

$$-\left(567 + 183n + 47n^2 + 13n^3\right) k^2$$

$$-2\left(-98 - 2n + n^2\right) k^3 + (-25 + 7n)\, k^4$$

Summing both sides of Equation (15) for all values of k we get the result. □

We now find curious identities involving the Catalan numbers, π, hypergeometric functions, and the gamma function. Consider the following hypergeometric functions

a) $A(n) := {}_2F_1(-1/2, 2n+3; 2n+(5/2); -1)$

b) $M(n) := {}_2F_1(1/2, 2n+3; 2n+(5/2); -1)$.

Then we obtain the following propositions.

Proposition 8. *Let $D(n) = A(n) - 2M(n)$. Then*

(a) $D(n) = (4n+5)^{-1}\, {}_2F_1(1/2, 2n+3; 2n+(7/2); -1)$.

(b) $D(n) = \sum_{k=1}^{\infty} \frac{\left(-\frac{1}{2}\right)_k}{\left(2n+\frac{5}{2}\right)_k} \binom{2n+k+1}{k-1} (-1)^k$.

Proof. Use properties of hypergeometric functions. □

The next proposition follows as a result of the WZ algorithm.

Proposition 9. *$D(n)$ satisfies*

$$16(n+2)^2 D(n+1) - (4n+5)(4n+7) D(n) = 0. \tag{16}$$

Proof. Let $F(n,k)$ denote the summand of $D(n)$, that is,

$$F(n,k) = \frac{\left(-\frac{1}{2}\right)_k}{\left(2n+\frac{5}{2}\right)_k} \binom{2n+k+1}{k-1} (-1)^k.$$

Now, applying the WZ algorithm on $F(n,k)$ we get

$$16(n+2)^2 F(n+1,k) - (4n+5)(4n+7) F(n,k) = G(n,k+1) - G(n,k) \tag{17}$$

where $G(n,k) = F(n,k) R(n,k)$ and

$$R(n,k) = \frac{(2n+3)(4n+5+2k)}{(k-1)(4n+5)(4n+7)}.$$

Summing both sides of Equation (17) for all values of k we get Equation (16). □

Corollary 10. *Solving the recurrence relation given by Equation (16) gives*

$$D(n) = \frac{\sqrt{\pi}\Gamma(2n+5/2)}{2^{2n+4}(n+1)!^2} \tag{18}$$

where $D(0) = \frac{3\pi}{64}$.

Proof. Using a symbolic algebra system (such as Mathematica or Maple) to solve Equation (16) gives the result. □

We note here that $D(n)$ appears below in the numerator of Equation (21). The motivation for the next theorem comes from evaluating the following integral

$$\int_0^1 \frac{2^{2n+5}\left(x^2\left(1-x^2\right)^{2n}\right)}{\pi\left(1+x^2\right)^{2n+3}} dx. \tag{19}$$

It is known by Dana-Picard [1] that the value of Expression (19) is the nth Catalan number c_n A000108 [13]. Using Maple to integrate Expression (19) gives

$$\int_0^1 \frac{2^{2n+5}\left(x^2\left(1-x^2\right)^{2n}\right)}{\pi\left(1+x^2\right)^{2n+3}} dx = \frac{\Gamma(n+1/2)4^n}{\sqrt{\pi}\Gamma(n+2)}$$

where by using properties of the gamma function the right side simplifies to c_n. Consequently,

$$\frac{c_n}{D(n)} = \left(\frac{(2n)!}{(n+1)\,n!^2}\right)\left(\frac{2^{2n+4}\,(n+1)!^2}{\sqrt{\pi}\Gamma(2n+5/2)}\right) = \frac{2^{2n+4}\,(n+1)\,\Gamma(2n+1)}{\sqrt{\pi}\Gamma(2n+5/2)}. \tag{20}$$

Thus, we find that

$$\int_0^1 \frac{2^{2n+5}\left(x^2\left(1-x^2\right)^{2n}\right)}{\pi\left(1+x^2\right)^{2n+3}}dx = \frac{2^{2n+4}\,(n+1)\,D(n)\,\Gamma(2n+1)}{\sqrt{\pi}\Gamma(2n+5/2)}. \tag{21}$$

The right side of Equation (21) is indeed c_n. This can also be directly obtained by using Equation (18) and some simplification. Given Equation (18) and rearranging and simplifying the right hand side of Equation (21) leads to the following simple identity connecting the Catalan numbers A000108 [13] and the gamma function. For $n \in \mathbb{N}_0$

$$c_n = \frac{\Gamma(2n+1)}{\Gamma(n+1)\,\Gamma(n+2)}.$$

We now note here that by using the Wolfram Alpha Widget: Definite Integral Calculator [22] to integrate Expression (19) gives

$$\frac{2^{2n+5}n\,(n+1)\,D(n)\,\Gamma(2n)}{\sqrt{\pi}}, \text{ if } \operatorname{Re}(n) > -1/2, \tag{22}$$

which does not simplify to c_n. As a consequence of this, we observe from Equation (21) that the Wolfram calculator does not include $\Gamma(2n+5/2)$ in the denominator of Expression (22). This has been reported to Wolfram by the authors. We have been informed by Wolfram that the appropriate computer technicians are investigating this issue with their calculator.

We now rearrange the right side of Equation (21) as Equation (23) in the theorem below. This theorem leads to curious relations where the Catalan numbers A000108 [13] are expressed in terms of π, hypergeometric functions, and the gamma function. The proof of the theorem follows as a result of the WZ algorithm.

Theorem 11. *For $n \in \mathbb{N}_0$,*

$$\sum_{k=1}^{\infty} \frac{2^{6n+6}\left(-\frac{1}{2}\right)_k}{(4n+3)\left(2n+\frac{5}{2}\right)_k}\binom{4n+2}{2n+2}^{-1}\binom{2n+k+1}{k-1}(-1)^k = \pi c_n. \tag{23}$$

Proof. Divide both sides of Equation (23) by πc_n and let $H(n)$ be the resulting left hand side and $F(n,k)$ be its summand, that is,

$$F(n,k) = \frac{2^{6n+6}\left(-\frac{1}{2}\right)_k}{\pi c_n\,(4n+3)\left(2n+\frac{5}{2}\right)_k}\binom{4n+2}{2n+2}^{-1}\binom{2n+k+1}{k-1}(-1)^k.$$

Then proving Equation (23) is equivalent to showing $H(n) = 1$ for $n \in \mathbb{N}_0$. Now, applying the WZ algorithm on $F(n,k)$ we get

$$F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k) \tag{24}$$

where $G(n,k) = F(n,k)\,R(n,k)$ and

$$R(n,k) = -\frac{k-1}{(2n+3)(5+2k+4n)}.$$

Summing both sides of Equation (24) for all values of k we get

$$H(n+1) - H(n) = 0.$$

Since $H(0) = 1$,we have $H(n) = 1$ for $n \in \mathbb{N}_0$. □

Corollary 12. *For $n \in \mathbb{N}_0$*

$$\frac{1}{c_n}\sum_{k=1}^{\infty} \frac{2^{6n+6}\left(-\frac{1}{2}\right)_k}{(4n+3)\left(2n+\frac{5}{2}\right)_k}\binom{4n+2}{2n+2}^{-1}\binom{2n+k+1}{k-1}(-1)^k = \pi.$$

Corollary 13. *For $n \in \mathbb{N}_0$*

$$\frac{1}{\pi}\sum_{k=1}^{\infty} \frac{2^{6n+6}\left(-\frac{1}{2}\right)_k}{(4n+3)\left(2n+\frac{5}{2}\right)_k}\binom{4n+2}{2n+2}^{-1}\binom{2n+k+1}{k-1}(-1)^k = c_n.$$

Theorem 11 can be rearranged and written in terms of the π, c_n, and $D(n)$ as the following identity.

Proposition 14. *For $n \in \mathbb{N}_0$*

$$\frac{2^{6n+6}D(n)}{(4n+3)\,\pi c_n} = \binom{4n+2}{2n+2} = \frac{4n+1}{n+1}\binom{4n}{2n}.$$

Proof. Using Equations (3) and (18), $\Gamma(1/2) = \sqrt{\pi}$ and some simplification gives

$$\begin{aligned}
\frac{2^{6n+6}D(n)}{(4n+3)\,\pi c_n} &= \frac{2^{6n+6}}{(4n+3)\,\pi}\cdot\frac{(n+1)\,n!n!}{(2n)!}\cdot\frac{\sqrt{\pi}\Gamma(2n+5/2)}{2^{2n+4}\,(n+1)!^2} \\
&= \frac{\Gamma(2n+5/2)}{\sqrt{\pi}}\cdot\frac{2^{4n+2}}{(4n+3)\,(n+1)\,(2n)!} \\
&= \frac{\Gamma(2n+5/2)}{\Gamma(1/2)}\cdot\frac{2^{4n+2}}{(4n+3)\,(n+1)\,(2n)!}.
\end{aligned} \tag{25}$$

Rewriting$\Gamma(2n+5/2)$ as $\Gamma(2(n+1)+1/2)$, applying Equation (1), and using

$$\frac{\Gamma(n+1/2)}{\Gamma(1/2)} = (1/2)_n = \frac{(2n)!}{n!4^n}$$

gives

$$\frac{\Gamma\left(2n+5/2\right)}{\Gamma\left(1/2\right)}=\frac{\Gamma\left(2\left(n+1\right)+1/2\right)}{\Gamma\left(1/2\right)}=\left(1/2\right)_{2n+2}=\frac{(4n+4)!}{(2n+2)!4^{2n+2}}. \tag{26}$$

Substituting Equation (26) into Equation (25) yields

$$\frac{2^{6n+6}D\left(n\right)}{\left(4n+3\right)\pi c_n}=\frac{(4n+4)!}{(2n+2)!4^{2n+2}}\cdot\frac{2^{4n+2}}{\left(4n+3\right)\left(n+1\right)\left(2n\right)!}.$$

Now apply properties of factorials and binomial coefficients and some simplification we obtain

$$\frac{2^{6n+6}D\left(n\right)}{\left(4n+3\right)\pi c_n}=\frac{(4n+2)!}{(2n+2)!\left(2n\right)!}=\binom{4n+2}{2n+2},$$

which proves the result. $\square$

We note that Equation (26) gives an exact formula for $\Gamma\left(2n+5/2\right)$. This can also be obtained by direct computation by using Proposition 14 and Equation (18).

6 Conclusion

We now give some concluding comments. The binomial, Catalan, and Fine generating functions are expressed in terms of the Fibonacci generating function by Proposition 5. This result may lead to new properties of these generating functions, which in turn may lead to new properties of the counting numbers associated the generating functions. The recurrence relations given by Propositions 7 and 9 are open for combinatorial interpretations. In particular, finding an RNA interpretation of Proposition 7 would be of interest to those who study discrete and combinatorial mathematical biology. The Wolfram calculator anomaly given by Expression (22) is what sparked the authors' interest in looking at relations among hypergeometric functions, the gamma function, and the Catalan numbers A000108 [13]. This generated further investigation and analysis of $D\left(n\right)$, which subsequently led to the derivation of Equation (21). As a result of the right side of Equation (21) and using the Wilf-Zeilberger algorithm we were able to find other representations of the irrational number π and the ubiquitous Catalan numbers c_n A000108 [13]. Thus, Theorem 11, which is the main result of the paper, demonstrates the roles in which symbolic computation, numerical computation, and experimental mathematics play in the discovery of new mathematical results.

7 Acknowledgement

The authors would like to thank the anonymous referee for providing useful comments and suggestions. The authors would also like to thank Guoping Zhang for useful comments and discussions on early drafts of the manuscript.

References

[1] T. Dana-Picard, Integral presentations of Catalan numbers and Wallis formula, *Int. J. Math. Educ. in Science and Technology* **42** (2011) 122–129.

[2] E. Deutsch and L. W. Shapiro, Seventeen Catalan identities, *Bulletin of the ICA* **31** (2001) 31–38.

[3] E. Deutsch and L. W. Shapiro, A survey of the Fine numbers, *Discrete Math.* **241** (2001) 241–265.

[4] I. M. Gessel, Finding identities with the WZ method, *J. Symbolic Computation* **20** (1995) 537–566.

[5] R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics*, Addison-Wesley, 1989.

[6] C. Jean-Louis and A. Nkwanta, Some algebraic structure of the Riordan group, *Linear Algebra Appl.* **438** (2013) 2018–2035.

[7] A. Nkwanta and E. R. Barnes, Two Catalan-type Riordan arrays and their connections to the Chebyshev polynomials of the first kind, *J. Integer Sequences* **15** (2012), Article 12.3.3.

[8] A. Nkwanta, Riordan matrices and higher-dimensional lattice walks, *J. Statist. Plann. Inference* **140** (2010) 2321–2334.

[9] A. Nkwanta, Lattice paths, generating functions, and the Riordan group, Ph. D. Dissertation, Howard University, Washington, DC, 1997.

[10] M. Petkovsek, H. S. Wilf, and D. Zeilberger, $A = B$, AK Peters, 1996.

[11] J. Riordan, *Combinatorial Identities*, Wiley, 1968.

[12] L. W. Shapiro, S. Getu, W. Woan, and L. Woodson, The Riordan group, *Discrete Appl. Math.* **34** (1991) 229–239.

[13] N. J. A. Sloane, *The On-line Encyclopedia of Integer Sequences*, published online at `http://oeis.org`, 2013.

[14] R. Sprugnoli, Riordan arrays and combinatorial sums, *Discrete Math.***132** (1994) 267–290.

[15] R. P. Stanley, *Enumerative Combinatorics*, Vol. 2, Cambridge Studies in Advanced Mathematics **62**, Cambridge University Press, 1999.

[16] R. P. Stanley, *Enumerative Combinatorics*, Vol. 2, Catalan Addendum, published online at `http://www-math.mit.edu/~rstan/ec/catadd.pdf`, October 22 2011.

[17] A. Tefera, What is a Wilf-Zeilberger pair? *Notices Amer. Math. Soc.* **57** (2010) 508–509.

[18] M. S. Waterman, Secondary structure of single-stranded nucleic acids, *Studies in Foundations & Combinatorics, Advances in Mathematics Supplementary* **1**, (1978) 167–212.

[19] M. S. Waterman, *Introduction to Computational Biology: Maps, Sequences and Genomes*, Chapman & Hall/CRC Press, 2000.

[20] H. Wilf and D. Zeilberger, Rational functions certify combinatorial identities, *J. Amer. Math. Soc.* 3 (1990) 77–83.

[21] H. Wilf, *generatingfunctionology*, Academic Press, San Diego, 1990.

[22] Wolfram Alpha Widget, published online at `http://www.wolfram.com/widgets`, 2013.

[23] Doron Zeilberger, published online at `http://www.math.rutgers.edu/~zeilberg/`, 2013.

2010 *Mathematics Subject Classification*: Primary 05A15; Secondary 05A19.
Keywords: WZ algorithm, WZ method, Riordan array, gamma function, generating function, Catalan number.

(Concerned with sequences A000012, A000027, A000045, A000108, A000957, A001477, A004148, A007318, A039598, A053121, A097724, and A111418.)

Received July 14 2013; revised version received October 6 2013. Published in *Journal of Integer Sequences*, October 13 2013.

Return to Journal of Integer Sequences home page.