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On Highly Repetitive and Power Free Words

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Abstract

Answering a question of Richomme, Currie and Rampersad proved that 7/3 is the infimum of the real numbers $\alpha > 2$ such that there exists an infinite binary word that avoids α-powers but is highly 2-repetitive, i.e., contains arbitrarily large squares beginning at every position. In this paper, we prove similar statements about β-repetitive words, for some other β's, over the binary and the ternary alphabets.

1 Introduction

In this paper we study words that are non-repetitive, in the sense that they avoid α-powers for some α, but yet are highly repetitive, in the sense that for some β close to α, they contain infinitely many β-powers starting at every position. First we recall some basic definitions. A finite, non-empty word v can be written as $v = p^k e$, where $k \geq 1$, the word e is a prefix of p, and the length of p is minimal. We say that v has *period* p, *excess* e, and *exponent* $\mathrm{expo}(v) = |v|/|p|$. A word with exponent α is called an α-power. An α^+-*power* is a word that is a β-power for some $\beta > \alpha$. A 2-power is called a *square*; a 2^+-power is called an *overlap*. A word is α-*free* if none of its factors is a β-power for any $\beta \geq \alpha$. A word is α^+-*free* if none of its factors is an α^+-power.

Thue [TH2] proved that there exist infinite overlap-free binary words. Dekking [DEK] later showed that any infinite overlap-free binary word must contain arbitrarily large squares.

Currie, Rampersad, and Shallit [CRS] constructed 7/3-free words containing infinitely many overlaps. Their motivation came from the following result of Shur [SHU]: Any bi-infinite 7/3-free binary word is overlap-free. The analogue of this surprising result is not true for one-sided infinite words: There are infinite 7/3-free binary words that are not overlap-free. The result of Currie et al. shows that in fact the infinite 7/3-free binary words can be significantly different from the infinite overlap-free binary words.

Currie and Rampersad [CR] continued this line of study in order to respond to the following question of Richomme [RIC]: What is the infimum of the real numbers $\alpha > 2$ such that there exists an infinite word that avoids α-powers but contains arbitrarily large squares beginning at every position? They showed that over the binary alphabet the answer to Richomme's question is $\alpha = 7/3$. In this paper we show that if instead of requiring arbitrarily large squares at every position, we only ask for arbitrarily large β-powers at every position for some fixed $\beta < 2$, then the answer of 7/3 for Richomme's question can be replaced by 2. We also answer the analogous question for the ternary alphabet.

Specifically, we first define a *highly α-repetitive* word to be a word that contains infinitely many α-repetitions beginning at every position. Our main results are:

1. On the binary alphabet, for any real number $\beta < 2$, there exists a 2^+-free word that is highly β-repetitive.
2. On the ternary alphabet, for any real number $\beta < 7/4$, there exists a $7/4^+$-free word that is highly β-repetitive.

We also mention here the related work of Saari [SAA], who also studied infinite words containing squares (not necessarily arbitrarily large) beginning at every position. He called such words *squareful*, but he imposed the additional condition that the word contain only finitely many distinct minimal squares.

2 Preliminary definitions and results

Let α be a real number.

Definition 1. A finite word is an *α-repetition (resp. α^+*-repetition*) if it is a β-power for some $\beta \geq \alpha$ (resp. $\beta > \alpha$).*

Definition 2. A word (finite or infinite) is *highly α-repetitive* (resp. *highly α^+-repetitive*) if it contains infinitely many α-repetitions (resp. α^+-repetitions) beginning at every position.

Definition 3. A word (finite or infinite) is *α-free* (resp. *α^+-free*) if it contains no α-repetition (resp. α^+-repetition).

Definition 4. A morphism $\mu : A^* \to B^*$ is *α-free* (resp. *α^+-free*) if for any word $w \in A^*$, the following equivalence holds:

$$w \text{ is } \alpha\text{-free (resp. } \alpha^+\text{-free)} \Leftrightarrow \mu(w) \text{ is } \alpha\text{-free (resp. } \alpha^+\text{-free)}.$$

Definition 5. The *critical exponent* $E_c(w)$ of an infinite word w is the supremum of all rational numbers r such that w contains an r-power.

We now define some particular morphisms which we will use in the paper because of their α-freeness for certain α.

The Thue-Morse morphism is defined by

$$\mu_{TM} : \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 10. \end{cases}$$

Proposition 6 ([TH2]). *μ_{TM} is a 2^+-free morphism.*

Dejean's morphism is defined by

$$\mu_D : \begin{cases} a \mapsto abc\ acb\ cab\ c\ bac\ bca\ cba \\ b \mapsto \pi(\mu_D(a)) = bca\ bac\ abc\ a\ cba\ cab\ acb \\ c \mapsto \pi^2(\mu_D(a)) = cab\ cba\ bca\ b\ acb\ abc\ bac, \end{cases}$$

where π is the morphism that cyclically permutes the alphabet $\{a, b, c\}$.

Proposition 7 ([DEJ]). *μ_D is a $7/4^+$-free morphism.*

The following morphism was given by Brandenburg in [BRA]:

$$\mu_B : \begin{cases} a \mapsto abacbabcbac \\ b \mapsto abacbcacbac \\ c \mapsto abcbabcacbc. \end{cases}$$

Proposition 8 ([BRA]). *μ_B is a 2-free morphism.*

Finally, for any integers s, t such that $0 < s \leq t$, let us define the binary $(t+1)$-uniform morphism $\phi_{s,t}$ by

$$\phi_{s,t} : \begin{cases} 0 & \mapsto 01^t \\ 1 & \mapsto 01^{s-1}0^{t-s+1}. \end{cases}$$

Moreover, we define the rational numbers

$$q_{s,t} = \begin{cases} t + \frac{s}{t}, & \text{if } s > 1; \\ t(t+1) + 1 + \frac{1}{t}, & \text{if } s = 1. \end{cases}$$

These morphisms were introduced by Krieger, who proved that:

Proposition 9 ([KRI]). *The fixed point $\phi^{\omega}_{s,t}(0)$ of $\phi_{s,t}$ has critical exponent*

$$E_c(\phi^{\omega}_{s,t}(0)) = q_{s,t}.$$

In particular, the following remark will be useful:

Remark 10. For any factor v of the fixed point $\phi^{\omega}_{s,t}(0)$, the word $\phi_{s,t}(v)$ is $q^+_{s,t}$-free.

3 Highly repetitive binary words

The results of Currie and Rampersad [CR, Theorems 4 and 5] state that for any $\alpha > 7/3$, there exists an infinite binary word which is both α-free and highly 2-repetitive, and that such a word does not exist if $\alpha \leq 7/3$. The following proposition gives a similar result in the case of highly β-repetitive words, with $\beta < 2$: for any $\alpha > 2$ and any $\beta < 2$, there exists an infinite binary word which is both α-free and highly β-repetitive.

Theorem 11. *For any real number $\beta < 2$, there exists an infinite binary word which is both 2^+-free and highly β-repetitive.*

Proof. Let $\beta < 2$ be a real number. There exists an integer $m \geq 1$ such that $\beta \leq 2 - \frac{1}{2^m} < 2$. Let $r = 2 - \frac{1}{2^m}$, and let us construct a binary infinite word with infinitely many r-powers at every position, and which is 2^+-free. We recall that μ_{TM} is the Thue-Morse morphism.

We remark that $\mu_{TM}^m(11)$ is a factor of $\mu_{TM}^{m+3}(1)$. Indeed, $\mu_{TM}^3(1) = 10010110$, and so $\mu_{TM}^{m+3}(1) = \mu_{TM}^m(10010110) = u\mu_{TM}^m(11)\mu_{TM}^m(0)$, where $u = \mu_{TM}^m(10010)$. Let us define $q = m+3$. We define the following sequence of finite words:

$$\begin{aligned} A_0 &= \mu_{TM}^m(11) \\ A_1 &= (u)^{-1}\mu_{TM}^q(A_0) \\ &\vdots \\ A_n &= (u)^{-1}\mu_{TM}^q(A_{n-1}), \end{aligned}$$

where we denote by $(x)^{-1}y$ the word obtained by removing the prefix x from the word y (here, the word $\mu_{TM}^q(A_n)$ always has u as a prefix, since A_n always has 1 as a prefix). As 11 and μ_{TM} are 2^+-free, then for any $n \geq 0$, A_n is 2^+-free.

We show that for any $n \geq 0$, A_n is a prefix of A_{n+1}. This is clearly true for $n = 0$. If A_{n-1} is a prefix of A_n, then we can write $A_n = A_{n-1}S$, where S is a binary word. Then, $A_{n+1} = (u)^{-1}\mu_{TM}^q(A_{n-1}S) = A_n\mu_{TM}^q(S)$. Thus, the sequence A_n converges to an infinite limit word we denote by w, and this limit is 2^+-free. We now prove that w contains infinitely many r-powers beginning at every position. It can easily be seen that for any $n \geq 1$,

$$A_n = (u)^{-1}[\mu_{TM}^q(u)]^{-1}\cdots[\mu_{TM}^{q(n-1)}(u)]^{-1}\mu_{TM}^{qn}(A_0).$$

(Here the sequence of deletions indicated by the terms appearing with exponent -1 should be understood as being applied from right to left.) Let us denote by p_n the word

$$\mu_{TM}^{q(n-1)}(u)\cdots\mu_{TM}^q(u)u.$$

Then we have $A_n = (p_n)^{-1}\mu_{TM}^{qn}(A_0)$, since for any words u and v, $(uv)^{-1} = (v)^{-1}(u)^{-1}$. Moreover, we can see that p_n is a prefix of $\mu_{TM}^{qn}(1)$. Indeed,

$$\begin{aligned} |p_n| &= |u|\sum_{i=0}^{n-1}|\mu_{TM}(1)|^{qi} \\ &= |u|\sum_{i=0}^{n-1}2^{qi} \\ &< 2^{qn} = |\mu_{TM}^{qn}(1)|, \end{aligned}$$

since $|u| \leq 2^q - 1$. So, we can write $\mu_{TM}^{qn}(1) = p_n s_n$, where s_n is a binary word. For any $n \geq 1$, $A_n = (p_n)^{-1}\mu_{TM}^{qn}(A_0) = (p_n)^{-1}\mu_{TM}^{qn}(\mu_{TM}^{m}(11))$. We know that $\mu_{TM}^{m}(1)$ begins with the letter 1, and let us denote its last letter by $a \in \{0,1\}$. We can write $\mu_{TM}^{m}(1) = 1xa$, where $x \in \{0,1\}^*$. We can also write $\mu_{TM}^{qn}(a) = \tilde{p_n}\tilde{s_n}$, with $\tilde{p_n}$ and $\tilde{s_n}$ two binary words such that $|\tilde{p_n}| = |p_n|$ and $|\tilde{s_n}| = |s_n|$. Then,

$$A_n = s_n \mu_{TM}^{qn}(x)\tilde{p_n}\tilde{s_n}p_n s_n \mu_{TM}^{qn}(x)\tilde{p_n}\tilde{s_n}.$$

It follows that at all positions $0 \leq j \leq |s_n|$, there begins a repetition with period a conjugate of $s_n\mu_{TM}^{qn}(x)\tilde{p_n}\tilde{s_n}p_n$, and of length

$$\begin{aligned} & |s_n \mu_{TM}^{qn}(x)\tilde{p_n}\tilde{s_n}p_n s_n \mu_{TM}^{qn}(x)\tilde{p_n}| \\ = \;& 2 \cdot |x| \cdot 2^{qn} + 3 \cdot 2^{qn} \\ = \;& 2^{qn}(2^{m+1} - 1), \end{aligned}$$

since $|x| = |\mu_{TM}^{m}(1)| - 2 = 2^m - 2$. These repetitions have exponents $\frac{2^{qn}(2^{m+1}-1)}{2^{qn}|x|+2\cdot 2^{qn}} = \frac{2^{m+1}-1}{2^m} = r$. As n can be taken arbitrarily large, the result follows. □

Moreover, this result is optimal. Indeed, on the one hand, it is easy to see that for $\alpha \leq 2$, there is no infinite binary α-free word. On the other hand, Currie and Rampersad proved the following result.

Proposition 12 ([CR]). *If w is an infinite 2^+-free binary word, then there is a position i such that w does not contain a square beginning at position i.*

Our next result uses the morphisms $\phi_{s,t}$ defined in Section 2. We will first establish a few technical lemmas, which will be used to prove Proposition 16.

For any integers s, t and any integer N, we define

$$u_{s,t,N} = \begin{cases} \phi_{s,t}^{N-3}(0), & \text{if } s > 1; \\ \phi_{s,t}^{N-3}(01^t), & \text{if } s = 1, \end{cases}$$

and

$$r_{s,t} = \begin{cases} 1^t 0, & \text{if } s > 1; \\ 0^{t(t+1)+1}1, & \text{if } s = 1. \end{cases}$$

Lemma 13. *For any $N \geq 3$, $\phi_{s,t}^{N}(0)$ begins with the prefix $u_{s,t,N}\phi_{s,t}^{N-3}(r_{s,t})$.*

Proof. We prove this result by a simple induction. Suppose first that $s > 1$. It is clear that 01^t0 is a prefix of $\phi_{s,t}^{3}(0)$. Then, if we assume that $\phi_{s,t}^{N}(0) = u_{s,t,N}\phi_{s,t}^{N-3}(r_{s,t})s_{s,t,N}$, where $s_{s,t,N}$ denotes the corresponding suffix, then, we have:

$$\begin{aligned} \phi_{s,t}^{N+1}(0) &= \phi_{s,t}(\phi_{s,t}^{N-3}(01^t0)s_{s,t,N}) \\ &= u_{s,t,N+1}\phi_{s,t}^{N-2}(r_{s,t})\phi_{s,t}(s_{s,t,N}), \end{aligned}$$

which concludes the calculation. The case $s = 1$ is proved similarly. □

We denote by $P_{s,t,N}$ the longest common prefix of $\phi_{s,t}^N(0)$ and $\phi_{s,t}^N(1)$.

Lemma 14. *For any integer $N \geq 1$,*

$$P_{s,t,N} = \phi_{s,t}^{N-1}(01^{s-1})\phi_{s,t}^{N-2}(01^{s-1})\cdots\phi_{s,t}(01^{s-1})01^{s-1}.$$

Proof. We prove the result by induction. We suppose first that $s > 1$. It is clear that the longest common prefix of $\phi_{s,t}(0)$ and $\phi_{s,t}(1)$ is 01^{s-1}, since $s \leq t$. Now, assuming that $P_{s,t,N} = \phi_{s,t}^{N-1}(01^{s-1})\phi_{s,t}^{N-2}(01^{s-1})\cdots\phi_{s,t}(01^{s-1})01^{s-1}$, the longest common prefix of $\phi_{s,t}^{N+1}(0)$ and $\phi_{s,t}^{N+1}(1)$ is the longest common prefix of $\phi_{s,t}(P_{s,t,N}0)$ and $\phi_{s,t}(P_{s,t,N}1)$, i.e., the longest common prefix of

$$\phi_{s,t}^{N}(01^{s-1})\phi_{s,t}^{N}(01^{s-1})\cdots\phi_{s,t}(01^{s-1})\phi_{s,t}(0)$$

and

$$\phi_{s,t}^{N}(01^{s-1})\phi_{s,t}^{N}(01^{s-1})\cdots\phi_{s,t}(01^{s-1})\phi_{s,t}(1).$$

This prefix is obviously $\phi_{s,t}^{N}(01^{s-1})\phi_{s,t}^{N}(01^{s-1})\cdots\phi_{s,t}(01^{s-1})01^{s-1}$. The case $s = 1$ is proved with similar arguments. □

Lemma 15. *For any $N \geq 3$, $\phi_{s,t}^N(0)$ contains the repetition*

$$R_{s,t,N} = \begin{cases} (\phi_{s,t}^{N-3}(1))^t P_{s,t,N-3}, & \text{if } s > 1; \\ (\phi_{s,t}^{N-3}(0))^{t(t+1)+1} P_{s,t,N-3}, & \text{if } s = 1, \end{cases}$$

which has exponent

$$E(R_{s,t,N}) = \begin{cases} t + \frac{s}{t} - \frac{s}{t(t+1)^{N-3}}, & \text{if } s > 1; \\ t(t+1) + 1 + \frac{1}{t} - \frac{1}{t(t+1)^{N-3}}, & \text{if } s = 1. \end{cases}$$

More precisely, $\phi_{s,t}^N(0)$ begins with the prefix $u_{s,t,N}R_{s,t,N}$.

Proof. The fact that $\phi_{s,t}^N(0)$ contains the repetition $R_{s,t,N}$ is easily deduced from Lemmas 13 and 14. The exponent of this repetition is a simple calculation. Indeed, $R_{s,t,N}$ has period $\phi_{s,t}^{N-3}(1)$ or $\phi_{s,t}^{N-3}(0)$ (depending on the value of s), and excess $P_{s,t,N-3}$. As we have $|P_{s,t,N-3}| = \frac{s((t+1)^{N-3}-1)}{t}$, and $|\phi_{s,t}^{N-3}(1)| = |\phi_{s,t}^{N-3}(0)| = (t+1)^{N-3}$, the result follows. □

Proposition 16. *For any integers s, t, such that $0 < s \leq t$, and any real number $\beta < q_{s,t}$, there exists an infinite binary word w which is both $q_{s,t}^+$-free and highly β-repetitive.*

Proof. Let s, t be integers such that $0 < s \leq t$, and consider the morphism $\phi_{s,t}$ defined previously. Let $\beta < q_{s,t}$ be a real number. There exists integers N and p such that $\beta \leq q_{s,t} - \frac{1}{t(t+1)^{N-3}} - \frac{1}{(t+1)^{N-3+p}} < q_{s,t} - \frac{1}{t(t+1)^{N-3}}$. Let $r = q_{s,t} - \frac{1}{t(t+1)^{N-3}} - \frac{s}{(t+1)^{N-3+p}}$, and let us construct a binary infinite $q_{s,t}^+$-free word with infinitely many r-powers at every position.

As before, we define

$$r_{s,t} = \begin{cases} 1^t 0, & \text{if } s > 1; \\ 0^{t(t+1)+1}1, & \text{if } s = 1. \end{cases}$$

We can easily deduce from Lemma 13 that $\phi_{s,t}^{p+N-3}(r_{s,t})$ is a factor of $\phi_{s,t}^{p+N}(0)$. Indeed, by the Lemma, $\phi_{s,t}^{N}(0) = u_{s,t,N}\phi_{s,t}^{N-3}(r_{s,t})y$, with y a binary word, so

$$\phi_{s,t}^{p+N}(0) = \phi_{s,t}^{p}(u_{s,t,N})\phi_{s,t}^{p+N-3}(r_{s,t})\phi_{s,t}^{p}(y).$$

Let us denote by u the word $\phi_{s,t}^{p}(u_{s,t,N})$. Let us now set $q = p + N$, and define the following sequence of finite binary words:

$$\begin{aligned} A_0 &= \phi_{s,t}^{p+N-3}(r_{s,t}) \\ A_1 &= (u^{-1})\phi_{s,t}^{q}(A_0) \\ &\vdots \\ A_M &= (u^{-1})\phi_{s,t}^{q}(A_{M-1}). \end{aligned}$$

(Here, the word $\phi_{s,t}^{q}(A_M)$ always has u as a prefix, since A_M always has 0 as a prefix.) As $r_{s,t}$ is $q_{s,t}^{+}$-free and by Remark 10, we deduce that for any $M \geq 0$, A_M is $q_{s,t}^{+}$-free. We now show that for any $M \geq 0$, A_M is a prefix of A_{M+1}. This is clearly true for $M = 0$, by definition of u. Then, if A_{M-1} is a prefix of A_M, let us write $A_M = A_{M-1}S$, where $S \in \{0,1\}^*$. Then, $A_{M+1} = (u)^{-1}\phi_{s,t}^{q}(A_{M-1}S) = A_M\phi_{s,t}^{q}(S)$. Thus, the sequence of words A_M converges to an infinite binary limit word we denote by w, and this limit is $q_{s,t}^{+}$-free. Let us now prove that w contains infinitely many r-powers beginning at every position. It is easy to see that for any $M \geq 1$,

$$A_M = (u)^{-1}[\phi_{s,t}^{q}(u)]^{-1}\cdots[\phi_{s,t}^{q(M-1)}(u)]^{-1}\phi_{s,t}^{qM}(A_0).$$

We define $p_M = \phi_{s,t}^{q(M-1)}(u)\cdots\phi_{s,t}^{q}(u)u$, and then we have $A_M = (p_M)^{-1}\phi_{s,t}^{qM}(A_0)$, since for any words x, y, $(xy)^{-1} = y^{-1}x^{-1}$. Moreover, p_M is a prefix of $\phi_{s,t}^{qM}(0)$. Indeed, since $\phi_{s,t}$ is $(t+1)$-uniform,

$$\begin{aligned} |p_M| &= |u|\sum_{i=0}^{M-1}(t+1)^{qi} \\ &< (t+1)^{qN} = |\phi_{s,t}^{qN}(0)|, \end{aligned}$$

since $|u| \leq |\phi_{s,t}^{p+N-3}(01^t)| = (t+1)^{p+N-2} \leq (t+1)^q - 1$. So, we can write $\phi_{s,t}^{qM}(0) = p_M s_M$, where s_M is the corresponding suffix.

Now, by Lemma 14, we can write $\phi_{s,t}^{N-3}(1) = P_{s,t,N-3}P_{s,t,N-3}^{(1)}$ and $\phi_{s,t}^{N-3}(0) = P_{s,t,N-3}P_{s,t,N-3}^{(0)}$, where we recall that $P_{s,t,N-3}$ is the longest common prefix of $\phi_{s,t}^{N-3}(1)$ and $\phi_{s,t}^{N-3}(0)$, and where $P_{s,t,N-3}^{(0)}, P_{s,t,N-3}^{(1)}$ are the corresponding suffixes. Now, consider the word $\phi_{s,t}^{p}(P_{s,t,N-3})$. It begins with the letter 0. We can then write $\phi_{s,t}^{p}(P_{s,t,N-3}) = 0z$, with $z \in \{0,1\}^*$.

Now, let us suppose first that $s > 1$. Then,

$$\begin{aligned}
A_M &= p_M^{-1}\phi_{s,t}^{qM}(A_0) \\
&= p_M^{-1}\phi_{s,t}^{qM}(\phi_{s,t}^{p+N-3}(1^t 0)) \\
&= p_M^{-1}\phi_{s,t}^{qM}([0z\phi_{s,t}^{p}(P_{s,t,N-3}^{(1)})]^t[0z\phi_{s,t}^{p}(P_{s,t,N-3}^{(0)})]) \\
&= p_M^{-1}[p_M s_M\phi_{s,t}^{qM}(z)\phi_{s,t}^{p+qM}(P_{s,t,N-3}^{(1)})]^t[p_M s_M\phi_{s,t}^{qM}(z)\phi_{s,t}^{p+qM}(P_{s,t,N-3}^{(0)})].
\end{aligned}$$

We deduce that at every position $0 \leq j \leq |s_M|$, there begins a repetition with period a conjugate of $s_M\phi_{s,t}^{qM}(z)\phi_{s,t}^{p+qM}(P_{s,t,N-3}^{(1)})p_M$, and excess of length $\geq |(\phi_{s,t}^{qM}(z)| = |z|(t+1)^{qM}$. Thus, these repetitions have exponent at least

$$\begin{aligned}
& t + \frac{|z|(t+1)^{qM}}{(|z|+1)(t+1)^{qM} + |P_{s,t,N-3}^{(1)}|(t+1)^{p+qM}} \\
= \quad & t + \frac{|z|}{|z|+1+|P_{s,t,N-3}^{(1)}|(t+1)^{p}} \\
= \quad & t + \frac{|P_{s,t,N-3}|(t+1)^p - 1}{(|P_{s,t,N-3}| + |P_{s,t,N-3}^{(1)}|)(t+1)^p} \text{ (since } |z| = |\phi_{s,t}^{p}(P_{s,t,N-3})| - 1) \\
= \quad & t + \frac{|P_{s,t,N-3}|(t+1)^p - 1}{(t+1)^{N-3}(t+1)^p} \\
= \quad & t + \frac{s}{t} - \frac{s}{t(t+1)^{N-3}} - \frac{1}{(t+1)^{p+N-3}} = r.
\end{aligned}$$

As N can be taken arbitrarily large, the result follows.

Finally, let us suppose that $s = 1$. Then,

$$\begin{aligned}
A_M &= p_M^{-1}\phi_{s,t}^{qM}(A_0) \\
&= p_M^{-1}\phi_{s,t}^{qM}(\phi_{s,t}^{p+N-3}(0^{t(t+1)+1}1)) \\
&= p_M^{-1}\phi_{s,t}^{qM}([0z\phi_{s,t}^{p}(P_{s,t,N-3}^{(0)})]^{t(t+1)+1}[0z\phi_{s,t}^{p}(P_{s,t,N-3}^{(1)})]) \\
&= p_M^{-1}[p_M s_M\phi_{s,t}^{qM}(z)\phi_{s,t}^{p+qM}(P_{s,t,N-3}^{(0)})]^{t(t+1)+1}[p_M s_M\phi_{s,t}^{qM}(z)\phi_{s,t}^{p+qM}(P_{s,t,N-3}^{(1)})].
\end{aligned}$$

We deduce that at every position $0 \leq j \leq |s_M|$, there begins a repetition with period a conjugate of $s_M\phi_{s,t}^{qM}(z)\phi_{s,t}^{p+qM}(P_{s,t,N-3}^{(0)})p_M$, and excess of length $\geq |(\phi_{s,t}^{qM}(z)| = |z|(t+1)^{qM}$. Thus, these repetitions have exponent at least

$$
\begin{aligned}
& t(t+1)+1+\frac{|z|(t+1)^{qM}}{(|z|+1)(t+1)^{qM}+|P^{(0)}_{s,t,N-3}|(t+1)^{p+qM}} \\
= \;& t(t+1)+1+\frac{|z|}{|z|+1+|P^{(0)}_{s,t,N-3}|(t+1)^{p}} \\
= \;& t(t+1)+1+\frac{|P_{s,t,N-3}|(t+1)^{p}-1}{(|P_{s,t,N-3}|+|P^{(0)}_{s,t,N-3}|)(t+1)^{p}} \text{ (since } |z|=|\phi^{p}_{s,t}(P_{s,t,N-3})|-1) \\
= \;& t(t+1)+1+\frac{|P_{s,t,N-3}|(t+1)^{p}-1}{(t+1)^{N-3}(t+1)^{p}} \\
= \;& t(t+1)+1+\frac{s}{t}-\frac{s}{t(t+1)^{N-3}}-\frac{1}{(t+1)^{p+N-3}}=r.
\end{aligned}
$$

As N can be taken arbitrarily large, the result follows. □

The following proposition shows the optimality of the previous results.

Proposition 17. *Let $q \geq 2$ be a rational number. If an infinite word is q^+-free, then it is not highly q-repetitive.*

Proof. The proof uses an idea of Richomme mentioned in the conclusion of [CR]. Suppose to the contrary that x is an infinite q^+-free word that is highly q-repetitive. Consider two positions i and $i+1$ in x. Since x is highly q-repetitive, it contains infinitely many q-powers starting at position $i+1$, so in particular, there is a square ww at position $i+1$. Similarly, there are infinitely many squares starting at position i. We now apply the following result proved by Ilie [ILI, Lemma 2]: In any word, if vv and uu are two squares at position i and ww is a square at position $i+1$, then either $|w|=|u|$ or $|w|=|v|$ or $|w|\geq 2|v|$. Since there are infinitely many squares starting at position i, this implies that $|w|$ equals $|v|$ for some square vv occurring at position i. This means that the q-power w^n starting at position $i+1$ can be extended to the left by one letter to create a q^+-power. This is a contradiction, since x is q^+-free. □

4 Highly repetitive ternary words

In this part, we will state some similar results in the case of the ternary alphabet. We recall that the repetition threshold of the ternary alphabet is $7/4$, so for any $\alpha \leq 7/4$, there is no infinite α-free ternary word.

Theorem 18. *For any real number $\beta < 7/4$, there exists an infinite ternary word that is both $7/4^+$-free and highly β-repetitive.*

Proof. For any real number $\beta < 7/4$, there exists an integer $m \geq 1$ such that $\beta \leq 7/4 - \frac{1}{19^{m+2}} < 7/4$. Let $r = 7/4 - \frac{1}{19^{m+2}}$, and let us construct a $7/4^+$-free word which contains infinitely many r-powers at every position. We recall that μ_D is Dejean's morphism.

We remark that $\mu_D^m(abcbabc)$ is a factor of $\mu_D^{m+2}(a)$. Indeed,

$$\mu_D^2(a) = \mu_D(a)\mu_D(b)cabcbabcabacbabcbac\mu_D(acbcabcbacbcacba),$$

and so

$$\mu_D^{m+2}(a) = u\mu_D^m(abcbabc)\mu_D^m(abacbabcbac)\mu_D^{m+1}(acbcabcbacbcacba),$$

where $u = \mu_D^{m+1}(ab)\mu_D^m(c)$. Let us define $q = m+2$.

We define the following sequence of finite words:

$$\begin{aligned} A_0 &= \mu_D^m(abcbabc) \\ A_1 &= (u)^{-1}\mu_D^q(A_0) \\ &\vdots \\ A_n &= (u)^{-1}\mu_D^q(A_{n-1}). \end{aligned}$$

(Here, the word $\mu_D^q(A_n)$ always has u as a prefix, since A_n always has a as a prefix). As $abcbabc$ and μ_D are $7/4^+$-free, then for any $n \geq 0$, A_n is $7/4^+$-free.

Similar arguments as in the proof of Theorem 11 prove that the sequence A_n converges to an infinite limit word we denote by w, and this limit is $7/4^+$-free. We now prove that w contains infinitely many α-powers beginning at every position. Again with similar arguments as in the proof of Theorem 11, we can see that for any $n \geq 1$,

$$A_n = (p_n)^{-1}\mu_D^{qn}(A_0),$$

where p_n is the word $\mu_D^{q(n-1)}(u)\cdots\mu_D^q(u)u$. Moreover, we can see that p_n is a prefix of $\mu_D^{qn}(a)$. Indeed,

$$\begin{aligned} |p_n| &= |u|\sum_{i=0}^{n-1}|\mu_D(a)|^{qi} \\ &= |u|\sum_{i=0}^{n-1}19^{qi} \\ &< 19^{qn} = |\mu_D^{qn}(a)|. \end{aligned}$$

So, we can write $\mu_D^{qn}(a) = p_n s_n$, where s_n is a ternary word. For any $n \geq 1$, $A_n = (p_n)^{-1}\mu_D^{qn}(A_0) = (p_n)^{-1}\mu_D^{qn}(\mu_D^m(abcbabc))$. We know that $\mu_D^m(a)$ begins with the letter a, and let us denote its last letter by $x \in \{a,b,c\}$. We can write $\mu_D^m(a) = aXx$, where $X \in \{a,b,c\}^*$. Similarly, let us write $\mu_D^m(c) = cYy$, where $Y \in \{a,b,c\}^*$, and $y \in \{a,b,c\}$. We can also write $\mu_D^{qn}(y) = \tilde{p_n}\tilde{s_n}$, with $\tilde{p_n}$ and $\tilde{s_n}$ two ternary words such that $|\tilde{p_n}| = |p_n|$ and $|\tilde{s_n}| = |s_n|$. Then,

$$A_n = s_n\mu_D^{qn}(Xx\mu_D^m(b)cY)\tilde{p_n}\tilde{s_n}\mu_D^{qn+m}(b)p_n s_n\mu_D^{qn}(Xx\mu_D^m(b)cY)\tilde{p_n}\tilde{s_n}.$$

It follows that at all positions $0 \leq j \leq |s_n|$, there begins a repetition with period a conjugate of $s_n\mu_D^{qn}(Xx\mu_D^m(b)cY)\tilde{p_n}\tilde{s_n}\mu_D^{qn+m}(b)p_n$, and of length

$$\begin{aligned} l &= |s_n\mu_D^{qn}(Xx\mu_D^m(b)cY)\tilde{p_n}\tilde{s_n}\mu_D^{qn+m}(b)p_n s_n\mu_D^{qn}(Xx\mu_D^m(b)cY)\tilde{p_n}| \\ &= 2\cdot|Xx\mu_D^m(b)cY|\cdot 19^{qn} + 19^{qn+m} + 3\cdot 19^{qn} \\ &= 19^{qn}\cdot(7\cdot 19^m - 1), \end{aligned}$$

since $|X| = |Y| = |\mu_D^m(a)| - 2 = 19^m - 2$. These repetitions have exponent

$$\frac{19^{qn}(7 \cdot 19^m - 1)}{19^{qn}|Xx\mu_D^m(b)cY| + 19^{qn+m} + 2 \cdot 19^{qn}} = \frac{7 \cdot 19^m - 1}{4 \cdot 19^m} = \alpha.$$

As n can be taken arbitrarily large, the result follows. □

Now, we will consider the problem for highly 7/4-repetitive words. This would be Richomme's question for the ternary alphabet: what is the infimum of the real numbers α such that there exists an infinite ternary word which is both α-free and highly 7/4-repetitive?

Proposition 19. *Let $q \geq 7/4$ be a rational number. There is no q^+-free ternary word which is highly q-repetitive.*

Proof. For $q \geq 2$, the result follows from Proposition 17, so let us suppose that $q < 2$. Suppose there exists a word w over $\{a, b, c\}$ that is q^+-free and highly q-repetitive. Then, any factor of w is *left-special* (i.e., has two left extensions). Indeed, let u be a factor of w, and let i denote its position in w. w is *recurrent* (every factor occurs infinitely often) because it is highly repetitive, so we can suppose $i \geq 1$. There is a long enough q-repetition v beginning at position i. We denote its period by p. Then, u is repeated in the excess of v: pu is a factor of v. Now, let $x_1 = w_{i-1}$ and $x_2 = p_{|p|}$. It is clear that $x_1 \neq x_2$, since otherwise, $x_1 v$ would be a q^+-repetition, which is impossible since w is q^+-free. So u has two left extensions. Moreover, as the factors aa, bb and cc are clearly forbidden, we can see that w contains the factor $abab$ (by extending b on the left). But $abab$ is a square, which contradicts the fact that w is q^+-free. □

Theorem 20. *There exists an infinite ternary word which is both 2-free and highly 7/4-repetitive.*

Proof. We use the morphism of Brandenburg, which is a square-free morphism:

$$\mu_B : \begin{cases} a \mapsto abacbabcbac \\ b \mapsto abacbcacbac \\ c \mapsto abcbabcacbc. \end{cases}$$

We use it to give a similar construction as in the proofs of Theorems 11 and 18. We consider the sequence:

$$\begin{aligned} A_0 &= cbac\mu_B(cbc)abac = cbacabcbabcacbcabacbcacbacabcbabcacbcabac \\ A_1 &= (u)^{-1}\mu_B^3(A_0) \\ &\vdots \\ A_n &= (u)^{-1}\mu_B^3(A_{n-1}), \end{aligned}$$

where $u = \mu_B^2(a)\mu_B(ab)abacbab$. The sequence A_n converges to a limit we denote by w, which is a square-free word because μ_B is square-free. For any $n \geq 1$, if we denote by p_n

the word $\mu_B^{3(n-1)}(u)\cdots\mu_B^3(u)u$, we can see that $A_n = (p_n)^{-1}\mu_B^{3n}(A_0)$. But p_n is a prefix of $\mu_B^{3n}(c)$, since

$$\begin{aligned} |p_n| &= |u|\sum_{i=0}^{n-1}|\mu_B(a)|^{3i} \\ &= |u|\sum_{i=0}^{n-1}11^{3i} \\ &< 11^{3n} = |\mu_B^{3n}(c)|. \end{aligned}$$

So, we can write $\mu_B^{3n}(c) = p_n s_n$, where s_n is a ternary word. Then, for any $n \geq 1$,

$$\begin{aligned} A_n &= (p_n)^{-1}\mu_B^{3n}(A_0) \\ &= s_n\mu_B^{3n}(bacabcbabcacbcaba)p_n s_n\mu_B^{3n}(bca)p_n s_n\mu_B^{3n}(bacabcbabcacbcaba)p_n s_n. \end{aligned}$$

It follows that at all positions $0 \leq j \leq |s_n|$, there begins a repetition with period a conjugate of $s_n\mu_B^{3n}(bacabcbabcacbcaba)p_n s_n\mu_B^{3n}(bca)p_n$, and of length

$$\begin{aligned} l &= |s_n\mu_B^{3n}(bacabcbabcacbcaba)p_n s_n\mu_B^{3n}(bca)p_n s_n\mu_B^{3n}(bacabcbabcacbcaba)p_n| \\ &= 40 \cdot 11^{3n}. \end{aligned}$$

These repetitions have exponents $40/22 = 20/11 \geq 7/4$. As n can be taken arbitrarily large, the result follows. $\square$

Remark 21. In fact, with this proof, we have the result: There exists an infinite 2-free ternary word which is highly 20/11-repetitive.

So the problem of finding the infimum of the real numbers α such that there exists an infinite ternary α^+-free word which is highly 7/4-repetitive is open. We proved that this infimum is between 7/4 and 2.

5 Weaker results on ternary words

In the following section, we will prove some weaker results for words over the ternary alphabet. First, instead of considering highly repetitive words, we consider words with powers at every position. Then, we will discuss words containing infinitely many powers, but not necessary at every position. These questions were asked at the end of [CR] by Currie and Rampersad.

The following results concern Dejean's morphism μ_D and its behaviour regarding repetitions and synchronizing properties, and were proved in [VAS]. If v is a word, we denote by δ the application that removes the first letter of v (and so, $\delta^t(v)$ removes the first t letters of v). Moreover, we denote by $\mathfrak{F}_D$ the set $\mathrm{Fact}(\mu_D(\{a,b,c\}^*))$ of factors of $\mu_D(\{a,b,c\}^*)$.

Lemma 22 ([VAS]). *Let $\alpha \in]7/4, 2[$ be given. Let s, t be natural numbers such that $\mu_D^s(b) = xabcbabcy$, with $|x| = t$. Let $\beta = 2 - \frac{t}{4\cdot 19^s}$. Suppose that $7/4 < \beta < \alpha$, and that $abcbabcv \in \mathfrak{F}_D$ is α-free. Consider the word $w = \delta^t\mu_D^s(babcbabcv)$. Then, we have:*

1. *w has a prefix with exponent β.*

2. *If $abcbabcv$ has a factor with exponent γ and period p, then, w has a factor with exponent γ and a period of length $19^s|p|$.*

3. *w is α-free.*

Definition 23 ([VAS])**.** A real number $\beta < \alpha$ is said to be *obtainable* if β can be written as $\beta = 2 - \frac{t}{4 \cdot 19^s}$, where the natural numbers s and t satisfy:

* $s \geq 3$

* the word $\delta^t(\mu_D^s(b))$ begins with *abcbabc*.

We note that for any given $s \geq 3$, it is possible to choose t such that

* $7/4 < \beta = 2 - \frac{t}{4 \cdot 19^s} < \alpha$

* $|\alpha - \beta| \leq \frac{19^2}{4 \cdot 19^s}$.

Indeed, $\mu_D^2(a)$, $\mu_D^2(b)$, and $\mu_D^2(c)$ have length 19^2, and each has *abcbabc* as a factor. Therefore, choosing a large enough s, we can always find some obtainable real numbers β arbitrarily close to α.

Theorem 24. *For every real number $\alpha > 7/4$, there exists an infinite α-free ternary word that contains $7/4$-powers beginning at every position.*

Proof. Theorem 3.4 in [CR] established the result for any $\alpha > 2$. Let us consider $7/4 < \alpha \leq 2$. Let β be an obtainable number (see Definition 23), that is, it can be written as:

$$\beta = 2 - \frac{t}{4 \cdot 19^s},$$

where $s \geq 3$, and $\delta^t \mu_D^s(b)$ begins with *abcbabc*.

For any word v in $\mathfrak{F}_D$, we denote by $\Phi(v)$ the word $\delta^t \mu_D^s(bv)$ (we recall that $\delta(u)$, for some word u, removes the first letter of u) and we consider the sequence:

$$\begin{array}{rcl} v_1 & = & \Phi(abcbabc) = \delta^t \mu_D^s(babcbabc) \\ v_2 & = & \Phi(v_1) = \delta^t \mu_D^s(b \delta^t \mu_D^s(babcbabc)) \\ & \vdots & \\ v_n & = & \Phi(v_{n-1}) \\ & \vdots & \end{array}$$

Let $w = \lim v_n$ (possible because each v_i is clearly a prefix of v_{i+1}). By iteration of Lemma 22, as *abcbabc* is α-free, each v_i is α-free, and then w is α-free.

For any $n \geq 0$, w begins with a prefix of the form

$$\delta^t \mu_D^s(b) \mu_D^s(\delta^t \mu_D^s(b)) \cdots \mu_D^{ns}(\delta^t \mu_D^s(b)) \mu_D^{(n+1)s}(abcbabc).$$

Thus, w contains the word $\mu_D^{ns}(\delta^t(\mu_D^s(b))\mu_D^{(n+1)s}(abcbabc)$ at position

$$\begin{aligned}
p_n &= |\delta^t\mu_D^s(b)\mu_D^s(\delta^t\mu_D^s(b))\cdots\mu_D^{(n-1)s}(\delta^t\mu_D^s(b))| \\
&= \sum_{i=1}^{n}|\mu_D^{is}(b)| - \sum_{i=0}^{n-1} t\cdot 19^{is} \\
&= (19^s - t)\sum_{i=0}^{n-1} 19^{is} \\
&= (19^s - t)\frac{19^{ns}-1}{19^s-1}
\end{aligned}$$

Moreover, the word $\mu_D^{ns}(\delta^t\mu_D^s(b))\mu_D^{(n+1)s}(abcbabc)$ contains 7/4-powers at every position between 1 and $q_n = |\mu_D^{ns}(\delta^t\mu_D^s(b))| = (19^s - t)\cdot 19^{ns}$. So, in w, there is a 7/4-repetition starting at every position j, for every $j \in [p_n, p_n + q_n - 1]$. Since $p_{n+1} = p_n + q_n$, every position j in $\mathbb{N}$ is reached. □

Theorem 25. *For every real number $\alpha \geq 7/4$ there exists a real number β, with $7/4 \leq \beta < \alpha$, arbitrarily close to α, such that there is an infinite β^+-free ternary word containing infinitely many β-powers.*

Proof. Theorem 3.4 in [CR] established the result for any $\alpha > 2$. Let us consider $7/4 < \alpha \leq 2$. Let β be an obtainable number, that is, it can be written as:

$$\beta = 2 - \frac{t}{4\cdot 19^s},$$

where $s \geq 3$, and $\delta^t\mu_D^s(b)$ begins with $abcbabc$.

Similarly to the proof of Theorem 24, we consider the sequence:

$$\begin{aligned}
v_1 &= \Phi(abcbabc) = \delta^t\mu_D^s(babcbabc) \\
v_2 &= \Phi(v_1) = \delta^t\mu_D^s(b\delta^t\mu_D^s(babcbabc)) \\
&\vdots \\
v_n &= \Phi(v_{n-1}) \\
&\vdots
\end{aligned}$$

Let $w = \lim v_n$. The word w is α-free, and it is β^+-free, since $\beta < \alpha$. Moreover, for any $n \geq 1$, w contains the word $\mu_D^{ns}(\delta^t\mu_D^s(babcbabc))$, which has length $19^{ns}(8\cdot 19^s - t)$ and exponent β. □

6 Conclusion

We propose the following definition. For an integer k, consider the set

$$\mathfrak{S}_k = \{(\alpha,\beta)\in\mathbb{R}^2 : \text{there exists a word in } \{0,1,\ldots,k-1\}^\omega \text{ which is } \alpha\text{-free and highly } \beta\text{-repetitive}\}.$$

The combined results of Currie and Rampersad [CR], and of Sections 3 and 4 of this paper, give a partial desciption of $\mathfrak{S}_k$ for $k = 2$ and $k = 3$. We believe it is reasonable to expect the sets $\mathfrak{S}_k$, for $k = 2$ and $k \geq 3$, to be as described respectively in Figure 1 and 2. Proving this would be an interesting continuation of this work, as it is obviously related to important open problems about critical exponents of morphic words.

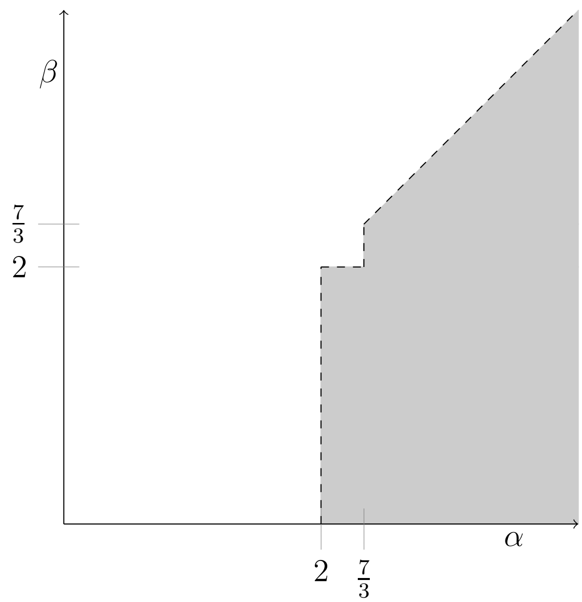

Figure 1: Conjecture for $k = 2$ ($\mathfrak{S}_2$ in grey.)

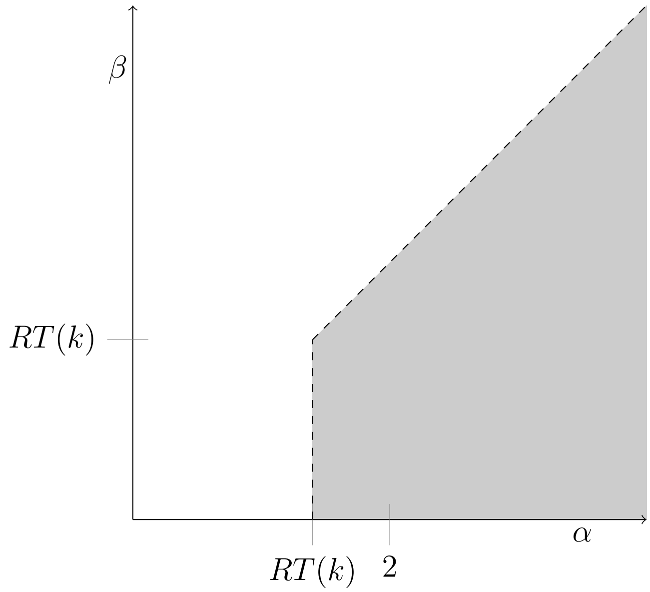

Figure 2: Conjecture for $k \geq 3$ ($\mathfrak{S}_k$ in grey.)

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