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On the Diophantine Equation $x^4 + y^4 + z^4 + t^4 = w^2$

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Abstract

To our knowledge, only three parametric solutions to the equation $x^4+y^4+z^4+t^4 = w^2$ were previously known. In this paper, we study the equation $x^4 + y^4 + z^4 + t^4 = (x^2 + y^2 + z^2 - t^2)^2$. We prove that it is possible to obtain infinitely many parametric solutions by finding points on an elliptic curve over a field $\mathbb{Q}(m)$ and we give several new parametric solutions.

1 Introduction

Jacobi and Madden [3] considered the equation

$$x^4 + y^4 + z^4 + t^4 = (x + y + z + t)^4. \tag{1}$$

They showed the existence of infinitely many integral solutions to (1). This is a special case of the equation

$$x^4 + y^4 + z^4 + t^4 = w^4, \tag{2}$$

for which Elkies [1] found an infinite family of integral solutions when $t = 0$. In this paper, we consider a special case of a similar equation

$$x^4 + y^4 + z^4 + t^4 = w^2. \tag{3}$$

2 Background

Consider the equation (3). We say that a solution is *trivial* if at least three of the numbers x, y, z, t, w are zero, for instance $(x, y, z, t, w) = (x, 0, 0, 0, x^2)$. If two and only two of the numbers x, y, z, t are zero, the equation has no nontrivial solution since Fermat proved that the equation $x^4 + y^4 = w^2$ has no solution in nonzero integers.

The first known parametric solution is nontrivial but very elementary:

$$(x, y, z, t, w) = (a^2, ab, b^2, ab, a^4 + b^4).$$

In the next solution, found by Fauquembergue [2], one of the numbers x, y, z, t, w is zero, for instance $z = 0$:

$$(x, y, z, t, w) = (ac, bc, 0, ab, a^4 + a^2b^2 + b^4)$$

where $a^2 + b^2 = c^2$. The following solution was also found by Fauquembergue [2], again assuming $a^2 + b^2 = c^2$:

$$(x, y, z, t, w) = \big(2a^2bc^3, 2ab^2c^3, (a^2 - b^2)c^4, 2ab(a^4 + b^4), \\ (a^6 + 2a^5b + 3a^4b^2 + 3a^2b^4 + 2ab^5 + b^6)(a^6 - 2a^5b + 3a^4b^2 + 3a^2b^4 - 2ab^5 + b^6)\big).$$

These three parametric solutions yield nontrivial numerical solutions, unless $ab = 0$.

3 The Equation $x^4 + y^4 + z^4 + t^4 = (x^2 + y^2 + z^2 - t^2)^2$

While investigating solutions to equation (3), the second author noticed some interesting properties. After some numerical results, he considered the following three cases.

- If $w = x^2 + y^2 + z^2 + t^2$ then $x^2y^2 + x^2z^2 + z^2t^2 + y^2z^2 + y^2t^2 + z^2t^2 = 0$; hence, there are only trivial solutions.
- If $w = x^2 + y^2 - z^2 - t^2$ then $x^2y^2 - x^2z^2 - y^2z^2 = t^2(x^2 + y^2 - z^2)$. This is an interesting but complicated case. We leave this for future work.

- If $w = x^2+y^2+z^2-t^2$ then $x^2y^2+x^2z^2+y^2z^2 = t^2(x^2+y^2+z^2)$. This case also looked interesting and will be discussed below, beginning with the following proposition. We believe our analysis of this case is new.

Proposition 1. *If $x^2+y^2+z^2 \neq 0$, then (x,y,z,t) satisfies*

$$x^4+y^4+z^4+t^4 = (x^2+y^2+z^2-t^2)^2 \tag{4}$$

if and only if

$$(x^2+y^2+z^2)(y^2z^2+z^2x^2+x^2y^2) = \square \tag{5}$$

and

$$t^2 \;=\; \frac{y^2z^2+z^2x^2+x^2y^2}{x^2+y^2+z^2}. \tag{6}$$

Proof. We have

$$x^4+y^4+z^4+t^4-(x^2+y^2+z^2-t^2)^2 = 2\big((x^2+y^2+z^2)t^2-(y^2z^2+z^2x^2+x^2y^2)\big).$$

□

Since (4) is homogeneous of degree four, from now on we will write solutions to this equation as $(x:y:z:t)$. The equation represents a surface in $\mathbb{P}^3$,

$$\mathcal{S}' = \left\{(x:y:z:t) \in \mathbb{P}^3 \mid x^4+y^4+z^4+t^4 = (x^2+y^2+z^2-t^2)^2\right\}.$$

Assuming $t \neq 0$, we can view the surface in affine form by corresponding $(x,y,z) \leftrightarrow (x:y:z:1)$. This gives a rather interesting looking surface in three dimensions; see Figure 1.

The main focus of this paper is to answer the following question.

Question 2. How many rational curves $m \mapsto (x(m),y(m),z(m))$ are on the surface $\mathcal{S}'$?

If $xyz \neq 0$, then (5) can be expressed as

$$(x^2+y^2+z^2)\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right) = \square.$$

This leads to the following lemma.

Lemma 3. *If $(x:y:z:t)$ is a solution to (4) such that $xyzt \neq 0$ then $\left(\frac{1}{x}:\frac{1}{y}:\frac{1}{z}:\frac{1}{t}\right)$ is also a solution to (4).*

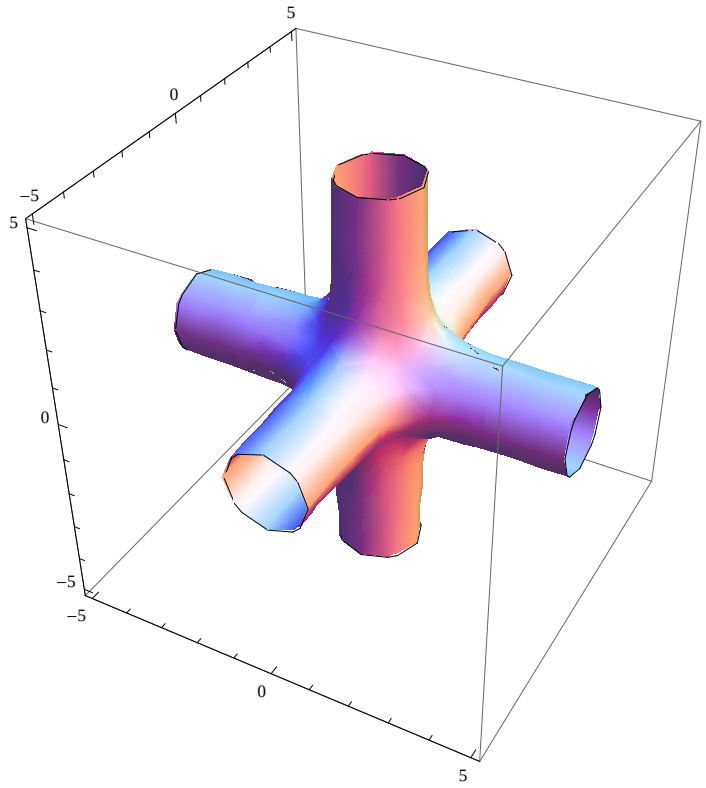

Figure 1: Plot of $\mathcal{S}'$ in affine space

Proof. Let $(X : Y : Z : T) = \left(\frac{1}{x} : \frac{1}{y} : \frac{1}{z} : \frac{1}{t}\right)$. Then

$$X^2 + Y^2 + Z^2 = \frac{1}{x^2y^2z^2}\ (y^2z^2 + z^2x^2 + x^2y^2) \text{ and}$$
$$Y^2Z^2 + Z^2X^2 + X^2Y^2 = \frac{1}{x^2y^2z^2}(x^2 + y^2 + z^2)$$

Hence

$$\frac{Y^2Z^2 + Z^2X^2 + X^2Y^2}{X^2 + Y^2 + Z^2} = \frac{x^2 + y^2 + z^2}{y^2z^2 + z^2x^2 + x^2y^2} = \frac{1}{t^2} = T^2$$

□

The following examples can be shown to be solutions to (4).

- The elementary solution $\mathcal{F}_0 = (a^2 : ab : b^2 : ab)$.
- The first solution of Fauquembergue $\mathcal{F}_1 = (ac : bc : 0 : ab)$ where $a^2 + b^2 = c^2$.

- The second solution of Fauquembergue $\mathcal{F}_2 = (2a^2bc^3 : 2ab^2c^3 : (a^2-b^2)c^4 : 2ab(a^4+b^4))$ where $a^2+b^2=c^2$.

An application of the previous lemma to the second solution of Fauquembergue, we deduce a new solution to (4).

Proposition 4. *If* $a^2+b^2=c^2$ *and*

$$\begin{aligned} x &= ac(a^2-b^2)(a^4+b^4) \\ y &= bc(a^2-b^2)(a^4+b^4) \\ z &= 2a^2b^2(a^4+b^4) \\ t &= ab(a^2+b^2)^2(a^2-b^2) \end{aligned}$$

then $(x:y:z:t)$ *is a solution to (4).*

We will label this solution $\mathcal{D}_1$. For example, if $(a:b:c)=(4:3:5)$ then $(x:y:z:t)=(47180:35385:97056:52500)$ is a solution to (4).

4 An Elliptic Curve over $\mathbb{Q}(m)$

We begin this section by providing some background on elliptic surfaces, which can be defined as a one-parameter algebraic family of elliptic curves. See Silverman [5, Chapter 3].

Let C be a curve defined over a field k. Consider a rational map $C \to \mathbb{P}^1$. The collection of all such maps is denoted by $K=k(C)$. For example, if $C: a^2+b^2=c^2$ is defined over $\mathbb{Q}$, we have an isomorphism $C \to \mathbb{P}^1$ given by

$$(a:b:c) \mapsto m = \frac{a}{c-b} \iff (a:b:c) = (2m : m^2-1 : m^2+1).$$

Hence, $K=\mathbb{Q}(C)=\mathbb{Q}(m)$. This is the field we will consider.

Consider a family of curves

$$E_m : x_2^2 = x_1^3 + A(m)x_1 + B(m)$$

with rational functions $A(m), B(m) \in K$. If we rewrite our equation in homogeneous form, we form the elliptic surface

$$\mathcal{E} = \left\{ ((x_1:x_2:x_3), m) \in \mathbb{P}^2 \times C | x_2^2x_3 = x_1^3 + A(m)x_1x_3^2 + B(m)x_3^3 \right\}.$$

We have a map $\pi : \mathcal{E} \to C$ defined by $((x_1:x_2:x_3),m) \mapsto m$. For $m \in \mathbb{Q}$ such that $4A(m)^3+27B(m)^2 \neq 0$, the *fiber*

$$\mathcal{E}_m = \pi^{-1}(m) = \left\{ (x_1:x_2:1) \in \mathbb{P}^2 | x_2^2x_3 = x_1^3 + A(m)x_1x_3^2 + B(m)x_3^3 \right\}$$

is the elliptic curve E_m over $\mathbb{Q}$.

We say that the elliptic surface is *non-split* if the j-invariant

$$j : C \to \mathbb{P}^1 \text{ defined by } j(m) = 1728\frac{4A(m)^3}{4A(m)^3 + 27B(m)^2}$$

is a non-constant function. A *parametrization of* $\mathcal{E}$ *by* C, or a *section to* π, is a map $\sigma : C \to \mathcal{E}$ such that the composition $\pi \circ \sigma : m \mapsto m$ is the identity map on C. There is always a trivial section on $\mathcal{E}$, namely the map $\sigma_0 : m \mapsto \mathcal{O}_m = ((0:1:0), m)$. In general, the collection $\mathcal{E}(C)$ of all sections is an abelian group, where we define

$$\begin{aligned} &\sigma_1(m) = (P(m), m) \\ &\qquad\qquad \Longrightarrow \qquad (\sigma_1 \oplus \sigma_2)(m) = (P(m) \oplus Q(m), m). \\ &\sigma_2(m) = (Q(m), m) \end{aligned}$$

We often abuse notation and write $E : x_2^2 = x_1^3 + Ax_1 + B$ as an elliptic curve over the function field $K = k(C)$. In fact, we have an isomorphism between the group of points of E over K and the group of sections of $\mathcal{E}$ over $\mathbb{Q}$.

$$\begin{array}{ccc} E(K) & \xrightarrow{\sim} & \mathcal{E}(C) \\ P(m) = (x_1(m) : x_2(m) : x_3(m)) & \mapsto & [\sigma : m \mapsto ((x_1(m) : x_2(m) : x_3(m)), m)] \end{array}$$

It will be helpful to consider $\mathcal{E}$ as a two-dimensional surface, where each section maps to a one-dimensional curve. The result [5, Theorem 6.1, Chapter 3] asserts that $E(K) \simeq \mathcal{E}(C)$ is a finitely generated abelian group whenever $\mathcal{E}$ is a non-split surface.

Let $m_0 \in \mathbb{Q}$ such that $E_0 = \pi^{-1}(m_0)$ is an elliptic curve over $k = \mathbb{Q}$. Silverman's "specialization theorem" [5, Theorem 11.4, Chapter 3] asserts that the map $\mathcal{E}(C) \to E_0(k)$ which sends a section $\sigma : m \mapsto (P(m), m)$ to the point $P_0 = P(m_0)$ is injective for all but finitely many $m_0 \in k$. In particular, if P_0 is a point of finite (infinite) order in $E_0(k)$ for some $m_0 \in k$, then $P(m)$ must be a point of finite (infinite) order in $E(K)$ as a function of m.

Let us return to the condition

$$(x^2 + y^2 + z^2)(y^2z^2 + z^2x^2 + x^2y^2) = \square.$$

If $a^2 + b^2 = c^2$ nonzero then we can express $(a : b : c) = (2m : m^2 - 1 : m^2 + 1)$. For the sake of space, we will leave our work below in terms of a, b, c. If we impose the condition $(x, y) = (a, b)$, we obtain

$$(c^2 + z^2)(c^2z^2 + a^2b^2) = \square.$$

Dividing by c^2, we can consider the following equation,

$$h^2 = z^4 + \frac{a^4 + 3a^2b^2 + b^4}{c^2}\, z^2 + a^2b^2 \tag{7}$$

From the preceding examples $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \mathcal{D}_1$, we know four solutions to (8),

$$\begin{aligned}
(z,h)_{\mathcal{F}_0} &= \left(\frac{b^2}{a}, \frac{b(a^4+a^2b^2+b^4)}{a^2c}\right);\\
(z,h)_{\mathcal{F}_1} &= (0, ab);\\
(z,h)_{\mathcal{F}_2} &= \left(\frac{(a^2-b^2)c}{2\ ab}, \frac{(a^4+b^4)c^2}{4\ a^2b^2}\right);\\
(z,h)_{\mathcal{D}_1} &= \left(\frac{2\ a^2b^2}{(a^2-b^2)c}, \frac{(a^4+b^4)ab}{(a^2-b^2)^2}\right).
\end{aligned}$$

Comment 5. If we impose the condition $(x,y) = (a,b) = (2m, m^2-1)$, we obtain

$$\left((m^2+1)^2+z^2\right)\left((m^2+1)^2z^2 + ((2m)(m^2-1))^2\right) = \square.$$

Dividing by $(m^2+1)^2$, we can consider the following equation,

$$h^2 = z^4 + \frac{m^8+8m^6-2m^4+8m^2+1}{(m^2+1)^2}z^2 + \left((2m)(m^2-1)\right)^2 \tag{8}$$

From the preceding examples, $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \mathcal{D}_1$, we know four solutions (z,h) to (8), imposing the condition $a^2+b^2=c^2$ and $(x,y)=(a,b)$. In terms of the parameter m we obtain

$$(z,h)_{\mathcal{F}_0} = \left(\frac{(m^2-1)^2}{2m}, \frac{(m^2-1)(m^4-2m^3+2m^2+2m+1)(m^4+2m^3+2m^2-2m+1)}{4m^2(m^2+1)}\right);$$

$$(z,h)_{\mathcal{F}_1} = (0, 4m(m^2-1));$$

$$(z,h)_{\mathcal{F}_2} = \left(\frac{-(m^2+1)(m^2+2m-1)(m^2-2m-1)}{4m(m^2-1)}, \frac{(m^2+1)^2(m^8-4m^6+22m^4-4m^2+1)}{16m^2(m^2-1)^2}\right);$$

$$(z,h)_{\mathcal{D}_1} = \left(\frac{8m^2(m^2-1)^2}{-(m^2+1)(m^2+2m-1)(m^2-2m-1)}, \frac{2m(m^2-1)(m^8-4m^6+22m^4-4m^2+1)}{((m^2+2m-1)(m^2-2m-1))^2}\right).$$

We will show that, in fact, there are infinitely many parametric solutions to equation (4) by showing there are infinitely many parametric solutions to (8).

Theorem 6. *Parametric solutions of the equation*

$$x^4+y^4+z^4+t^4 = (x^2+y^2+z^2-t^2)^2$$

may be obtained by finding points on an elliptic curve over the field $\mathbb{Q}(m)$.

Proof. By Proposition 1 and assuming $a^2+b^2=c^2$ nonzero, consider equation (8). Let $A=\dfrac{a^4+3a^2b^2+b^4}{4(a^2+b^2)}$ and $B=\dfrac{a^2b^2}{4}$, so that (8) can be expressed as

$$h^2=z^4+4Az^2+4B \tag{9}$$

If we have a rational solution to (9) then we get a rational solution to

$$v^2=u^3+\alpha u^2+\beta u \tag{10}$$

where $\alpha=-2A$ and $\beta=A^2-B$, by

$$u=\frac{1}{2}(z^2+2A-h) \quad \text{and} \quad v=\frac{1}{2}z(z^2+2A-h).$$

Conversely, assuming $(u,v)\neq(0,0)$, a rational solution to (10) leads to a rational solution to (9) by

$$z=\frac{v}{u} \quad \text{and} \quad h=\frac{v^2}{u^2}+2A-2u.$$

The discriminant of (10) is

$$(\alpha^2-4\beta)\beta^2=4B(A^2-B)^2=\frac{a^2b^2(a^4+a^2b^2+b^4)^4}{256(a^2+b^2)^4}$$

which is nonzero since at least one of a,b are nonzero. Hence (10) defines an elliptic curve over the field $\mathbb{Q}(m)$ and every point (u,v) on this elliptic curve yields a point (z,h) satisfying (8), which implies a solution (x,y,z,t) to (4). □

We will show how to obtain new solutions, by adding points on the elliptic curve. We write E for the elliptic curve (10) over $\mathbb{Q}(m)$, $+$ for the addition of points on the curve (10), and P_S will denote a point on E yielding a solution S to (4). The addition of points on an elliptic curve is described in Silverman [6]. Note that instead of writing $P+P$ we will write $2P$.

Example 7. The solutions $\mathcal{F}_0,\ \mathcal{F}_1,\ \mathcal{F}_2,$ and $\mathcal{D}_1$ to (4) are provided by the following points on (10):

$$P_{\mathcal{F}_0}=\left(\frac{(c-b)\ (a^2-ab+b^2)\ (a^2+ab+b^2)}{4\ (c+b)\ (a^2+b^2)}\ ,\frac{b^2\ (c-b)\ (a^2-ab+b^2)\ (a^2+ab+b^2)}{4\ a\ (c+b)\ (a^2+b^2)}\right)$$

$$P_{\mathcal{F}_1}=\left(\frac{(a^2-ab+b^2)^2}{4\ (a^2+b^2)}\ ,\ 0\right)$$

$$P_{\mathcal{F}_2}=\left(\frac{a^2b^2}{4\ (a^2+b^2)}\ ,\ \frac{ab\ (a^2-b^2)}{8c}\right)$$

$$P_{\mathcal{D}_1}=\left(\frac{(a-b)^2\ (a^2-ab+b^2)^2}{4\ (a+b)^2\ (a^2+b^2)}\ ,\ \frac{a^2b^2\ (a-b)\ (a^2+ab+b^2)^2}{2c\ (a+b)^3\ (a^2+b^2)}\right)$$

Let us remind the reader of the Lutz-Nagell theorem, which will be used in the proof of Theorem 8 along with the "specialization theorem".

Theorem 8. *Let E be given by $y^2 = x^3 + Ax + B$ with $A, B \in \mathbb{Z}$. Let $P = (x, y) \in E(\mathbb{Q})$. Suppose P has finite order. Then $x, y \in \mathbb{Z}$. If $y \neq 0$ then y^2 divides $4A^3 + 27B^2$.*

Theorem 9. *There exists infinitely many points on (10).*

Proof. Let $N = \dfrac{a^2 - ab + b^2}{2c}$ and $L = \dfrac{a^2 + ab + b^2}{2c}$. Then (10) can be expressed as

$$E: \quad v^2 = u\left(u - N^2\right)\left(u - L^2\right) \tag{11}$$

It can be shown that the rank of this elliptic curve over $\mathbb{Q}(m)$ is at least one. To show the rank is at least one, specialize at say, $(a, b) = (3, 4)$. Then the point $P = (36/25, 21/10)$ is on the curve

$$E_1: \quad v^2 = u^3 - \frac{769}{50}u^2 + \frac{231361}{10000}u.$$

In order to use the Lutz-Nagell theorem, we need to express E_1 in Weierstrass form with integral coefficients:

$$E_1': \quad y^2 = x^3 - 1538x^2 + 231361x.$$

The point P on E_1 corresponds to $P' = (144, 2100)$ on E_1', and thus

$$2P' = (70980625/28224, 389867877575/4741632).$$

Since $2P'$ is not an integral point, P' is not a point of finite order on E_1', so by the "Specialization Theorem" the rank of (11) is of positive rank, these are the same ideas as in Ulas [7]. Also note, calculations found at least 17 distinct points on E. The maximum number of torsion points is 16, so the rank must be at least one. □

Remark 10. On E, the torsion points are $(0, 0)$, $(N^2, 0)$, $(L^2, 0)$, and $\mathcal{O}$, the point at infinity.

Theorem 11. *Every solution to the equation*

$$x^4 + y^4 + z^4 + t^4 = (x^2 + y^2 + z^2 - t^2)^2 \qquad (4)$$

such that $(x, y) = (a, b)$, a and b nonzero, proceeds from exactly two points on E different from $(0, 0)$. If one of them is $P = (u, v)$, with $u \neq 0$, then the other one is $P' = (u', v') = \lambda(u, v)$, with $\lambda = \dfrac{\beta}{u^2}$.

Proof. Although E is defined over a function field, Figure 2 provides some intuition for our proof. Let $P = (u, v)$ be a point on E, with $u \neq 0$, which yields a solution to (4). If there exists a point $P' = (u', v')$ on E different from $(0, 0)$ yielding the same solution to (4) as

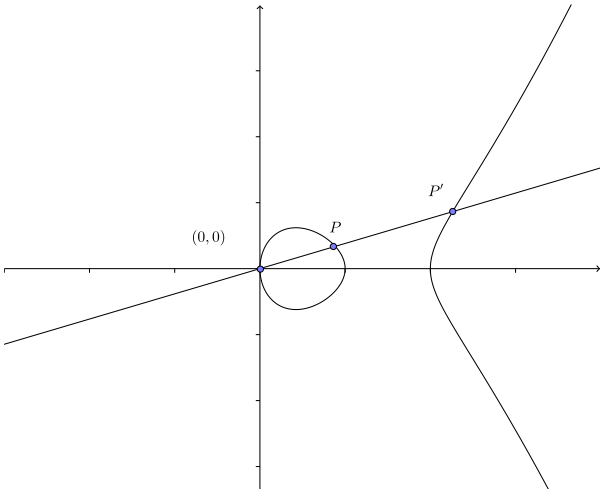

Figure 2: $E : v^2 = u^3 + \alpha u^2 + \beta u$

(u, v), then $z' = z$ so $u' \neq 0$ and $\dfrac{v'}{u'} = \dfrac{v}{u}$. Thus there exists a rational λ such that $u' = \lambda u$ and $v' = \lambda v$ and

$$0 = -v'^2 + u'^3 + \alpha u'^2 + \beta u' = -\lambda^2 v^2 + \lambda^3 u^3 + \alpha \lambda^2 u^2 + \beta \lambda u.$$

By substituting for v^2, we have

$$0 = -\lambda^2(u^3 + \alpha u^2 + \beta u) + \lambda^3 u^3 + \alpha^2 \lambda^2 u^2 + \beta \lambda u = \lambda\ (\lambda - 1)\ u\ (\lambda u^2 - \beta).$$

Since $u' \neq 0$, then $\lambda \neq 0$. Assuming $(u', v') \neq (u, v)$ implies $\lambda \neq 1$. Thus $\lambda = \dfrac{\beta}{u^2}$.

To show (u', v') is on E, notice

$$u'^3 + \alpha u'^2 + \beta u' = \frac{\beta^3}{u^3} + \alpha\ \frac{\beta^2}{u^2} + \frac{\beta^2}{u} = \frac{\beta^2}{u^4}\left(\beta u + \alpha u^2 + u^3\right) = \frac{\beta^2}{u^4} v^2 = \left(\frac{\beta}{u^2} v\right)^2 = v'^2.$$

Both of these points yield the same solution to (4) since $x = a, y = b$ and $z = z'$. This defines a solution $(x : y : z : t)$, except possibly for the sign of t. □

Remark 12. If $P_{\mathcal{F}_2} = (u, v)$, we find that $2P_{\mathcal{F}_0} = (u', v')$, both yielding the same solution $\mathcal{F}_2$.

From the previous two theorems we deduce the following corollary.

Corollary 13. *There exists infinitely many parametric solutions to* $x^4 + y^4 + z^4 + t^4 = (x^2 + y^2 + z^2 - t^2)^2$.

5 Obtaining new parametric solutions

Before obtaining new parametric solutions, we can interpret Lemma 3 in terms of points in $E(K)$ where $K = \mathbb{Q}(m)$ and E as described earlier.

Proposition 14. *Let $(u, v) \in E(K)$ such that $(u, v) \notin \{\mathcal{O}, (0, 0), (N^2, 0), (L^2, 0)\}$, and let $(u', v'), (u'', v'') \in E(K)$ such that:*

$$(u, v) + (u', v') = (N^2, 0) \text{ and } (u, v) + (u'', v'') = (L^2, 0).$$

If (u, v) yields a solution $(x : y : z : t)$ to (4) such that $xyzt \neq 0$, then (u', v') and (u'', v'') both yield $\left(\frac{1}{y} : \frac{1}{x} : \frac{1}{z} : \frac{1}{t}\right)$, except perhaps the signs of $\frac{1}{z}$ and $\frac{1}{t}$.

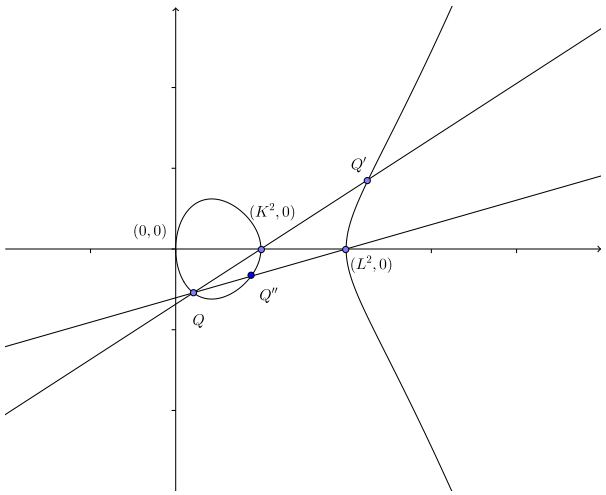

Figure 3: $E : v^2 = u^3 + \alpha u^2 + \beta u$

Proof. From the group law on equation (10), we deduce

$$(u', v') = \left(\frac{N^2(u - L^2)}{u - N^2}, \frac{N^2(N^2 - L^2)v}{(u - N^2)^2}\right), \quad (u'', v'') = \left(\frac{L^2(u - N^2)}{u - L^2}, \frac{L^2(L^2 - N^2)v}{(u - L^2)^2}\right).$$

If (u', v') yields $(x' : y' : z' : t')$ then $z' = \frac{v'}{u'} = \frac{(N^2 - L^2)v}{(u - N^2)(u - L^2)}$. From the relations $N^2 - L^2 = -ab$ and $v^2 = u(u - N^2)(u - L^2)$, we conclude

$$z' = -ab\frac{u}{v} = -\frac{ab}{z}.$$

Thus if (u, v) yields $(x : y : z : t) = (a : b : z : t)$, then (u', v') yields

$$(x' : y' : z' : t') = \left(a : b : \frac{ab}{z} : \frac{ab}{t}\right) = \left(\frac{1}{b} : \frac{1}{a} : \frac{1}{z} : \frac{1}{t}\right) = \left(\frac{1}{y} : \frac{1}{x} : -\frac{1}{z} : \frac{1}{t}\right),$$

where the signs of z' and t' may be positive or negative. The proof for z'' is similar. □

Next let $P_{\mathcal{D}_2} = P_{\mathcal{F}_0} + P_{\mathcal{D}_1}$. If $a^2 + b^2 = c^2$, we find $P_{\mathcal{D}_2} = (u, v)$ with

$$u = \frac{(c+a)(a^4 + a^2b^2 + b^4)(2a^3 - a^2c + b^2c)^2}{4(c-a)(a^2+b^2)(2a^3 + a^2c - b^2c)^2}$$
$$v = \frac{a^2b(a^4 + a^2b^2 + b^4)(2a^3 - a^2c + b^2c)(2b^3 - a^2c + b^2c)(2b^3 + a^2c - b^2c)}{4(c-a)^2(a^2+b^2)(2a^3 + a^2c - b^2c)^3}$$

By the same methods used in the proof of Theorem 6, we deduce the following solution $\mathcal{D}_2$:

Proposition 15. *If $a^2 + b^2 = c^2$ and if*

$$x = ab(2\ a^3 - a^2c + b^2c)(2\ a^3 + a^2c - b^2c)(2\ ab^2 + a^2c + b^2c)(-2\ ab^2 + a^2c + b^2c)$$
$$y = b^2(2\ a^3 - a^2c + b^2c)(2\ a^3 + a^2c - b^2c)(2\ ab^2 + a^2c + b^2c)(-2\ ab^2 + a^2c + b^2c)$$
$$z = a^2(2\ b^3 - a^2c + b^2c)(2\ b^3 + a^2c - b^2c)(2\ ab^2 + a^2c + b^2c)(-2\ ab^2 + a^2c + b^2c)$$
$$t = ab(2\ a^3 - a^2c + b^2c)(2\ a^3 + a^2c - b^2c)(2\ a^2b + a^2c + b^2c)(-2\ a^2b + a^2c + b^2c)$$

then $\mathcal{D}_2 = (x : y : z : t)$ is a solution to (4).

Example 16. Since $(a : b : c) = (2m : m^2 - 1 : m^2 + 1)$, if $m = 2$, then

$$x = 1\ 899\ 301\ 428$$
$$y = 1\ 424\ 476\ 071$$
$$z = 282\ 491\ 696$$
$$t = 1\ 165\ 848\ 372$$

Next let $P_{\mathcal{D}_3} = 2P_{\mathcal{D}_1}$. We find that $P_{\mathcal{D}_3} = (u, v)$ with:

$$u = \frac{(a^2+b^2)(a^4+b^4)^2}{16a^2b^2(a-b)^2(a+b)^2}, \quad v = \frac{PQRS(a^4+b^4)}{64a^3b^3c(a-b)^3(a+b)^3}$$

where

$$P = -a^3 + a^2b + ab^2 + b^3, \qquad Q = a^3 - a^2b + ab^2 + b^3$$
$$R = a^3 + a^2b - ab^2 + b^3, \qquad S = a^3 + a^2b + ab^2 - b^3.$$

From this we deduce $\mathcal{D}_3$:

Proposition 17. *If* $a^2+b^2=c^2$ *and if*

$$\begin{aligned} x &= a^2bc(a^2-b^2)(a^2+b^2)(a^4+b^4)G \\ y &= ab^2c(a^2-b^2)(a^2+b^2)(a^4+b^4)G \\ z &= PQRSG \\ t &= 4ab(a^2-b^2)(a^2+b^2)^2(a^4+b^4)\ H \end{aligned}$$

with P,Q,R,S *as above, and*

$$\begin{aligned} G &= a^{12}+6a^{10}b^2-a^8b^4+4a^6b^6-a^4b^8+6a^2b^{10}+b^{12} \\ H &= a^{12}-2a^{10}b^2+7a^8b^4+4a^6b^6+7a^4b^8-2a^2b^{10}+b^{12}, \end{aligned}$$

then $\mathcal{D}_2=(x:y:z:t)$ *is a solution to (4).*

Now put $P_{\mathcal{D}_4}=P_{\mathcal{D}_3}+P_{\mathcal{F}_1}$. We find $P_{\mathcal{D}_4}=(u,v)$, with

$$u=\frac{(a^2-ab+b^2)^2P^2S^2}{4(a^2+b^2)^2Q^2R^2},\quad v=\frac{a^2b^2c(a^2-b^2)(a^2-ab+b^2)^2(a^4+b^4)PS}{Q^3R^3}$$

From this we deduce the following solution $\mathcal{D}_4$:

Proposition 18. *If* $a^2+b^2=c^2$ *and if*

$$\begin{aligned} x &= acPQRSH \\ y &= bcPQRSH \\ z &= 4a^2b^2(a^2-b^2)(a^2+b^2)^2(a^4+b^4)H \\ t &= abPQRSG \end{aligned}$$

with the same P,Q,R,S,G,H *as in Proposition 17, then* $\mathcal{D}_4=(x:y:z:t)$ *is a solution to (4).*

The degree of this solution is 26 in $(a:b:c)$, and hence 52 in the homogeneous coordinates $(m:n)$ if we express $(a:b:c)=(2mn:m^2-n^2:m^2+n^2)$ for nonzero integers m,n.

Remark 19. The values z for $P_{\mathcal{D}_3}$ and z' for $P_{\mathcal{D}_4}$ satisfy $zz'=ab$, so except perhaps exchanging (x',y') and (y',x'), $\mathcal{D}_4$ is deduced from $\mathcal{D}_3$ by replacing (x,y,z) by $\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right)$.

In summary, parametric solutions $(x:y:z:t)$ to (4) with their degree are shown in Table 1.

solution	$\mathcal{F}_0$	$\mathcal{F}_1$	$\mathcal{F}_2$	$\mathcal{D}_1$	$\mathcal{D}_2$	$\mathcal{D}_3$	$\mathcal{D}_4$
degree	4	4	12	16	28	48	52

Table 1: Degree of parametric solutions

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