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A New Approach to Multivariate q-Euler Polynomials Using the Umbral Calculus

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Abstract

We derive numerous identities for multivariate q-Euler polynomials by using the umbral calculus.

1 Preliminaries

Throughout this paper, we use the following notation, where $\mathbb{C}$ denotes the set of complex numbers, $\mathcal{F}$ denotes the set of all formal power series in the variable t over $\mathbb{C}$ with $\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \mid a_k \in \mathbb{C} \right\}$, $\mathcal{P} = \mathbb{C}[x]$ and $\mathcal{P}^*$ denotes the vector space of all linear functional on $\mathcal{P}$, $\langle L \mid p(x) \rangle$ denotes the action of the linear functional L on the polynomial $p(x)$, and it is well-known that the vector space operation on $\mathcal{P}^*$ is defined by

$$\begin{aligned} \langle L + M \mid p(x) \rangle &= \langle L \mid p(x) \rangle + \langle M \mid p(x) \rangle , \\ \langle cL \mid p(x) \rangle &= c \langle L \mid p(x) \rangle , \end{aligned}$$

where c is some constant in $\mathbb{C}$ (for details, see [5, 6, 8, 11]).

The formal power series are known by the rule

$$f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \in \mathcal{F}$$

which defines a linear functional on $\mathcal{P}$ as $\langle f(t) \mid x^n \rangle = a_n$ for all $n \geq 0$ (for details, see [5, 6, 8, 11]]). Additionally,

$$\left\langle t^k \mid x^n \right\rangle = n! \delta_{n,k}, \tag{1}$$

where $\delta_{n,k}$ is the Kronecker symbol. When we take $f_L(t) = \sum_{k=0}^{\infty} \left\langle L \mid x^k \right\rangle \frac{t^k}{k!}$, then we obtain $\langle f_L(t) \mid x^n \rangle = \langle L \mid x^n \rangle$ and so as linear functionals $L = f_L(t)$ (see [5, 6, 8, 11]). Additionally, the map $L \to f_L(t)$ is a vector space isomorphism from $\mathcal{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ denotes both the algebra of the formal power series in t and the vector space of all linear functionals on $\mathcal{P}$, and so an element $f(t)$ of $\mathcal{F}$ can be thought of as both a formal power series and a linear functional. The algebra $\mathcal{F}$ is called the umbral algebra (see [5, 6, 8, 11]).

Also, the evaluation functional for y in $\mathbb{C}$ is defined to be power series e^{yt}. We can write that $\langle e^{yt} \mid x^n \rangle = y^n$ and so $\langle e^{yt} \mid p(x) \rangle = p(y)$ (see [5, 6, 8, 11]). We note that for all $f(t)$ in $\mathcal{F}$

$$f(t) = \sum_{k=0}^{\infty} \left\langle f(t) \mid x^k \right\rangle \frac{t^k}{k!} \tag{2}$$

and for all polynomials $p(x)$,

$$p(x) = \sum_{k=0}^{\infty} \left\langle t^k \mid p(x) \right\rangle \frac{x^k}{k!}, \tag{3}$$

(for details, see [5, 6, 8, 11]). The order $o(f(t))$ of the power series $f(t) \neq 0$ is the smallest integer k for which a_k does not vanish. It is considered $o(f(t)) = \infty$ if $f(t) = 0$. We see that $o(f(t)g(t)) = o(f(t)) + o(g(t))$ and $o(f(t) + g(t)) \geq \min \{o(f(t)), o(g(t))\}$. The series $f(t)$ has a multiplicative inverse, denoted by $f(t)^{-1}$ or $\frac{1}{f(t)}$, if and only if $o(f(t)) = 0$. Such series is called an invertible series. A series $f(t)$ for which $o(f(t))=1$ is called a delta series (see [5, 6, 8, 11]). For $f(t), g(t) \in \mathcal{F}$, we have $\langle f(t)g(t) \mid p(x)\rangle = \langle f(t) \mid g(t)p(x)\rangle$.

A delta series $f(t)$ has a compositional inverse $\overline{f}(t)$ such that $f\left(\overline{f}(t)\right) = \overline{f}\left(f(t)\right) = t$.

For $f(t), g(t) \in \mathcal{F}$, we have $\langle f(t)g(t) \mid p(x)\rangle = \langle f(t) \mid g(t)p(x)\rangle$. By (3), we have

$$p^{(k)}(x) = \frac{d^k p(x)}{dx^k} = \sum_{l=k}^{\infty} \frac{\left\langle t^l \mid p(x)\right\rangle}{l!} l\,(l-1)\cdots(l-k+1)\,x^{l-k}. \tag{4}$$

Thus, we see that

$$p^{(k)}(0) = \left\langle t^k \mid p(x)\right\rangle = \left\langle 1 \mid p^{(k)}(x)\right\rangle. \tag{5}$$

By (4), we get

$$t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k}. \tag{6}$$

So, we have

$$e^{yt}p(x) = p\,(x+y)\,. \tag{7}$$

Let $f(t)$ be a delta series and let $g(t)$ be an invertible series. Then there exists a unique sequence $S_n(x)$ of polynomials, with $\deg S_n(x) = n$, such that $\left\langle g(t)f(t)^k \mid S_n(x)\right\rangle = n!\delta_{n,k}$ for all $n, k \geq 0$. The sequence $S_n(x)$ is called the Sheffer sequence for $(g(t), f(t))$ or that $S_n(t)$ is Sheffer for $(g(t), f(t))$.

The Sheffer sequence for $(1, f(t))$ is called the associated sequence for $f(t)$; we also say $S_n(x)$ is associated with $f(t)$. The Sheffer sequence for $(g(t), t)$ is called the Appell sequence for $g(t)$; we also say $S_n(x)$ is Appell for $g(t)$.

Let $p(x) \in \mathcal{P}$. Then we have

$$\begin{aligned}
\left\langle \frac{e^{yt}-1}{t} \mid p(x) \right\rangle &= \int_0^y p(u)du, \\
\langle f(t) \mid xp(x)\rangle &= \langle \partial_t f(t) \mid p(x)\rangle = \langle f'(t) \mid p(x)\rangle\,, \\
\left\langle e^{yt}-1 \mid p(x)\right\rangle &= p(y)-p(0), \text{(see [5, 6, 8, 11])}.
\end{aligned} \tag{8}$$

Let $S_n(x)$ be Sheffer for $(g(t), f(t))$. Then the following results are known in [11]:

$$\begin{aligned}
h(t) &= \sum_{k=0}^{\infty} \frac{\langle h(t) \mid S_k(x)\rangle}{k!} g(t)\,f(t)^k\,,\ h(t) \in \mathcal{F} \\
p(x) &= \sum_{k=0}^{\infty} \frac{\left\langle g(t)\,f(t)^k \mid p(x)\right\rangle}{k!} S_k(x)\,,\ p(x) \in \mathcal{P}, \\
\frac{1}{g\left(\overline{f}(t)\right)} e^{y\overline{f}(t)} &= \sum_{k=0}^{\infty} S_k(y)\,\frac{t^k}{k!},\ \text{for all } y \in \mathbb{C}, \\
f(t)S_n(x) &= nS_{n-1}(x).
\end{aligned} \tag{9}$$

Let $a_1, \ldots, a_r, b_1, \ldots, b_r$ be positive integers. Kim and Rim [1] defined the generating function for multivariate q-Euler polynomials as follows:

$$F_q(t, x \mid a_1, \ldots, a_r; b_1, \ldots, b_r) = \sum_{n=0}^{\infty} E_{n,q}\left(x \mid a_1, \ldots, a_r; b_1, \ldots, b_r\right) \frac{t^n}{n!} \tag{10}$$
$$= \frac{2^r}{(q^{b_1} e^{a_1 t} + 1) \cdots (q^{b_r} e^{a_r t} + 1)} e^{xt}.$$

Note that

$$E_{0,q}\left(x \mid a_1, \ldots, a_r; b_1, \ldots, b_r\right) = \frac{2^r}{[2]_{q^{b_1}} [2]_{q^{b_2}} \cdots [2]_{q^{b_r}}},$$

where $[x]_q$ is q-extension of x defined by

$$[x]_q = \frac{q^x - 1}{q - 1} = 1 + q + q^2 + \cdots + q^{x-1}.$$

We assume that $q \in \mathbb{C}$ with $|q| < 1$. Also, we note that $\lim_{q \to 1} [x]_q = x$ (see [1]–[11]). In the special case, $x = 0$, $E_{n,q}\left(0 \mid a_1, \ldots, a_r; b_1, \ldots, b_r\right) := E_{n,q}\left(a_1, \ldots, a_r; b_1, \ldots, b_r\right)$ are called multivariate q-Euler numbers. By (10), we obtain the following:

$$E_{n,q}\left(x \mid a_1, \ldots, a_r; b_1, \ldots, b_r\right) = \sum_{k=0}^{n} \binom{n}{k} x^k E_{n-k,q}\left(a_1, \ldots, a_r; b_1, \ldots, b_r\right). \tag{11}$$

Kim and Kim [5] studied some interesting identities for Frobenius-Euler polynomials arising from umbral calculus. They derived not only new but also fascinating identities in modern classical umbral calculus.

By the same motivation, we also get numerous identities for multivariate q-Euler polynomials by utilizing from the umbral calculus.

2 On the multivariate q-Euler polynomials arising from umbral calculus

Assume that $S_n(x)$ is an Appell sequence for $g(t)$. By (9), we have

$$\frac{1}{g(t)} x^n = S_n(x) \text{ if and only if } x^n = g(t) S_n(x), \ (n \geq 0). \tag{12}$$

Let us take

$$g\left(t \mid a_1, \ldots, a_r; b_1, \ldots, b_r\right) = \frac{\left(q^{b_1} e^{a_1 t} + 1\right) \cdots \left(q^{b_r} e^{a_r t} + 1\right)}{2^r} \in \mathcal{F}.$$

Then we readily see that $g\left(t \mid a_1, \ldots, a_r; b_1, \ldots, b_r\right)$ is an invertible series. By (12), we have

$$\sum_{n=0}^{\infty} E_{n,q}\left(x \mid a_1, \ldots, a_r; b_1, \ldots, b_r\right) \frac{t^n}{n!} = \frac{1}{g\left(t \mid a_1, \ldots, a_r; b_1, \ldots, b_r\right)} e^{xt}. \tag{13}$$

By (13), we obtain the following

$$\frac{1}{g\left(t \mid a_1, \ldots, a_r; b_1, \ldots, b_r\right)} x^n = E_{n,q}\left(x \mid a_1, \ldots, a_r; b_1, \ldots, b_r\right). \tag{14}$$

Also, by (6), we have

$$\begin{aligned} tE_{n,q}\left(x \mid a_1, \ldots, a_r; b_1, \ldots, b_r\right) &= E'_{n,q}\left(x \mid a_1, \ldots, a_r; b_1, \ldots, b_r\right) \\ &= nE_{n-1,q}\left(x \mid a_1, \ldots, a_r; b_1, \ldots, b_r\right). \end{aligned} \tag{15}$$

By (14) and (15), we have the following proposition.

Proposition 1. *For $n \geq 0$, $E_{n,q}\left(x \mid a_1, \ldots, a_r; b_1, \ldots, b_r\right)$ is an Appell sequence for*

$$g\left(t \mid a_1, \ldots, a_r; b_1, \ldots, b_r\right) = \frac{\left(q^{b_1} e^{a_1 t} + 1\right) \cdots \left(q^{b_r} e^{a_r t} + 1\right)}{2^r}.$$

By (10), we see that

$$\begin{aligned} \sum_{n=1}^{\infty} E_{n+1,q}\left(x \mid a_1, \ldots, a_r; b_1, \ldots, b_r\right) \frac{t^n}{n!} &= \frac{xge^{xt} - g'e^{xt}}{g^2} \\ &= \sum_{n=0}^{\infty} \left(x\frac{1}{g}x^n - \frac{g'}{g}\frac{1}{g}x^n\right) \frac{t^n}{n!} \end{aligned} \tag{16}$$

where we used $g := g\left(t \mid a_1, \ldots, a_r; b_1, \ldots, b_r\right)$. Because of (14) and (16), we discover the following:

$$\begin{aligned} &E_{n+1,q}\left(x \mid a_1, \ldots, a_r; b_1, \ldots, b_r\right) \\ &= xE_{n,q}\left(x \mid a_1, \ldots, a_r; b_1, \ldots, b_r\right) - \frac{g'}{g} E_{n,q}\left(x \mid a_1, \ldots, a_r; b_1, \ldots, b_r\right). \end{aligned} \tag{17}$$

Therefore, we deduce the following theorem.

Theorem 2. *Let $g := g\left(t \mid a_1, \ldots, a_r; b_1, \ldots, b_r\right) = \frac{\left(q^{b_1} e^{a_1 t}+1\right) \cdots \left(q^{b_r} e^{a_r t}+1\right)}{2^r} \in \mathcal{F}$. Then we have for $n \geq 0$:*

$$E_{n+1,q}\left(x \mid a_1, \ldots, a_r; b_1, \ldots, b_r\right) = \left(x - \frac{g'}{g}\right) E_{n,q}\left(x \mid a_1, \ldots, a_r; b_1, \ldots, b_r\right). \tag{18}$$

From (10), we derive that

$$\sum_{n=0}^{\infty}\left(q^{b_r}E_{n,q}\left(x+a_r\mid a_1,\ldots,a_r;b_1,\ldots,b_r\right)+E_{n,q}\left(x\mid a_1,\ldots,a_r;b_1,\ldots,b_r\right)\right)\frac{t^n}{n!} \tag{19}$$
$$=2\sum_{n=0}^{\infty}E_{n,q}\left(x\mid a_1,\ldots,a_{r-1};b_1,\ldots,b_{r-1}\right)\frac{t^n}{n!}.$$

By comparing the coefficients in the both sides of $\frac{t^n}{n!}$ on the above, we obtain the following

$$2E_{n,q}\left(x\mid a_1,\ldots,a_{r-1};b_1,\ldots,b_{r-1}\right)=q^{b_r}E_{n,q}\left(x+a_r\mid a_1,\ldots,a_r;b_1,\ldots,b_r\right) \tag{20}$$
$$+E_{n,q}\left(x\mid a_1,\ldots,a_r;b_1,\ldots,b_r\right).$$

From Theorem 2, we get the following equation

$$gE_{n+1,q}\left(x\mid a_1,\ldots,a_r;b_1,\ldots,b_r\right) \tag{21}$$
$$=gxE_{n,q}\left(x\mid a_1,\ldots,a_r;b_1,\ldots,b_r\right)-\acute{g}E_{n,q}\left(x\mid a_1,\ldots,a_r;b_1,\ldots,b_r\right).$$

By using (20) and (21), we arrive at the desired theorem.

Theorem 3. *For $n\geq 0$, we have*

$$2E_{n,q}\left(x\mid a_1,\ldots,a_{r-1};b_1,\ldots,b_{r-1}\right)=q^{b_r}E_{n,q}\left(x+a_r\mid a_1,\ldots,a_r;b_1,\ldots,b_r\right) \tag{22}$$
$$+E_{n,q}\left(x\mid a_1,\ldots,a_r;b_1,\ldots,b_r\right).$$

Now, we consider that

$$\begin{aligned}
&\int_x^{x+y}E_{n,q}\left(u\mid a_1,\ldots,a_r;b_1,\ldots,b_r\right)du\\
=\;&\frac{1}{n+1}\left(E_{n+1,q}\left(x+y\mid a_1,\ldots,a_r;b_1,\ldots,b_r\right)-E_{n+1,q}\left(x\mid a_1,\ldots,a_r;b_1,\ldots,b_r\right)\right)\\
=\;&\frac{1}{n+1}\sum_{j=1}^{\infty}\binom{n+1}{j}E_{n+1-j,q}\left(x\mid a_1,\ldots,a_r;b_1,\ldots,b_r\right)y^j\\
=\;&\sum_{j=1}^{\infty}\frac{n\left(n-1\right)\left(n-2\right)\cdots\left(n-j+2\right)}{j!}E_{n+1-j,q}\left(x\mid a_1,\ldots,a_r;b_1,\ldots,b_r\right)y^j\\
=\;&\frac{1}{t}\left(\sum_{j=0}^{\infty}\frac{y^jt^j}{j!}-1\right)E_{n,q}\left(x\mid a_1,\ldots,a_r;b_1,\ldots,b_r\right)\\
=\;&\frac{e^{yt}-1}{t}E_{n,q}\left(x\mid a_1,\ldots,a_r;b_1,\ldots,b_r\right).
\end{aligned}$$

Therefore, we discover the following theorem:

Theorem 4. *For $n \geq 0$, we have*

$$\int_{x}^{x+y} E_{n,q}\left(u \mid a_1, \ldots, a_r; b_1, \ldots, b_r\right) du = \frac{e^{yt}-1}{t} E_{n,q}\left(x \mid a_1, \ldots, a_r; b_1, \ldots, b_r\right). \tag{23}$$

By (15) and Proposition 1, we have

$$t\left\{\frac{1}{n+1} E_{n+1,q}\left(x \mid a_1, \ldots, a_r; b_1, \ldots, b_r\right)\right\} = E_{n,q}\left(x \mid a_1, \ldots, a_r; b_1, \ldots, b_r\right). \tag{24}$$

Thanks to (24), we readily derive the following:

$$\begin{aligned} &\left\langle e^{yt}-1 \mid \frac{E_{n+1,q}\left(x \mid a_1, \ldots, a_r; b_1, \ldots, b_r\right)}{n+1}\right\rangle \qquad (25) \\ = &\left\langle \frac{e^{yt}-1}{t} \mid t\left\{\frac{E_{n+1,q}\left(x \mid a_1, \ldots, a_r; b_1, \ldots, b_r\right)}{n+1}\right\}\right\rangle \\ = &\left\langle \frac{e^{yt}-1}{t} \mid E_{n,q}\left(x \mid a_1, \ldots, a_r; b_1, \ldots, b_r\right)\right\rangle. \end{aligned}$$

On account of (8) and (24), we get

$$\begin{aligned} \left\langle \frac{e^{yt}-1}{t} \mid E_{n,q}\left(x \mid a_1, \ldots, a_r; b_1, \ldots, b_r\right)\right\rangle &= \left\langle e^{yt}-1 \mid \frac{E_{n+1,q}\left(x \mid a_1, \ldots, a_r; b_1, \ldots, b_r\right)}{n+1}\right\rangle \\ &= \frac{1}{n+1}\left\{E_{n+1,q}\left(y \mid a_1, \ldots, a_r; b_1, \ldots, b_r\right) - E_{n+1,q}\left(a_1, \ldots, a_r; b_1, \ldots, b_r\right)\right\} \\ &= \int_0^y E_{n,q}\left(u \mid a_1, \ldots, a_r; b_1, \ldots, b_r\right) du. \end{aligned}$$

Consequently, we obtain the following theorem.

Theorem 5. *For $n \geq 0$, we have*

$$\left\langle \frac{e^{yt}-1}{t} \mid E_{n,q}\left(x \mid a_1, \ldots, a_r; b_1, \ldots, b_r\right)\right\rangle = \int_0^y E_{n,q}\left(u \mid a_1, \ldots, a_r; b_1, \ldots, b_r\right) du. \tag{26}$$

Assume that

$$\mathcal{P}\left(q \mid a_1, \ldots, a_r; b_1, \ldots, b_r\right) = \left\{p(x) \in Q\left(q \mid a_1, \ldots, a_r; b_1, \ldots, b_r\right)[x] \mid \deg p(x) \leq n\right\}$$

is a vector space over $Q\left(q \mid a_1, \ldots, a_r; b_1, \ldots, b_r\right)$ which are the space of all polynomials including coefficients $q, a_1, \ldots, a_r, b_1, \ldots, b_r$.

For $p(x) \in \mathcal{P}\left(q \mid a_1, \ldots, a_r; b_1, \ldots, b_r\right)$, let us consider

$$p(x) = \sum_{k=0}^{n} b_k E_{k,q}\left(x \mid a_1, \ldots, a_r; b_1, \ldots, b_r\right). \tag{27}$$

By Proposition 1, $E_{n,q}(u \mid a_1,\ldots,a_r;b_1,\ldots,b_r)$ is an Appell sequence for

$$g := g(t \mid a_1,\ldots,a_r;b_1,\ldots,b_r) = \frac{\left(q^{b_1}e^{a_1 t}+1\right)\cdots\left(q^{b_r}e^{a_r t}+1\right)}{2^r}.$$

Thus we have

$$\left\langle g(t \mid a_1,\ldots,a_r;b_1,\ldots,b_r)\,t^k \mid E_{n,q}(x \mid a_1,\ldots,a_r;b_1,\ldots,b_r)\right\rangle = n!\delta_{n,k}. \tag{28}$$

From (27) and (28), we compute

$$\begin{aligned}\left\langle g(t \mid a_1,\ldots,a_r;b_1,\ldots,b_r)\,t^k \mid p(x)\right\rangle &= \sum_{l=0}^{n} b_l \left\langle gt^k \mid E_{l,q}(x \mid a_1,\ldots,a_r;b_1,\ldots,b_r)\right\rangle \\ &= \sum_{l=0}^{n} b_l l!\delta_{l,k} = k!b_k.\end{aligned} \tag{29}$$

Thus, by (29), we derive

$$\begin{aligned} b_k &= \frac{1}{k!}\left\langle gt^k \mid p(x)\right\rangle \\ &= \frac{1}{2^r k!}\left\langle \left(q^{b_1}e^{a_1 t}+1\right)\cdots\left(q^{b_r}e^{a_r t}+1\right) \mid p^{(k)}(x)\right\rangle. \end{aligned} \tag{30}$$

It is not difficult to show the following

$$\left(q^{b_1}e^{a_1 t}+1\right)\cdots\left(q^{b_r}e^{a_r t}+1\right) = \sum_{\substack{k_1,\ldots,k_r\geq 0\\ k_1+k_2+\ldots+k_r=1}} q^{\sum_{l=1}^{r} b_l k_l} e^{t\sum_{j=1}^{r} a_j k_j}. \tag{31}$$

Via the results (30) and (31), we easily see that

$$\begin{aligned} b_k &= \frac{1}{2^r k!}\sum_{\substack{k_1,\ldots,k_r\geq 0\\ k_1+k_2+\ldots+k_r=1}} q^{\sum_{l=1}^{r} b_l k_l}\left\langle e^{t\sum_{j=1}^{r} a_j k_j} \mid p^{(k)}(x)\right\rangle \\ &= \frac{1}{2^r k!}\sum_{\substack{k_1,\ldots,k_r\geq 0\\ k_1+k_2+\ldots+k_r=1}} q^{\sum_{l=1}^{r} b_l k_l} p^{(k)}\left(\sum_{j=1}^{r} a_j k_j\right). \end{aligned}$$

As a result, we state the following theorem.

Theorem 6. *For $p(x) \in \mathcal{P}(q \mid a_1,\ldots,a_r;b_1,\ldots,b_r)$, when we consider*

$$p(x) = \sum_{k=0}^{n} b_k E_{k,q}(x \mid a_1,\ldots,a_r;b_1,\ldots,b_r),$$

we obtain

$$b_k = \frac{1}{2^r k!}\sum_{\substack{k_1,\ldots,k_r\geq 0\\ k_1+k_2+\ldots+k_r=1}} q^{\sum_{l=1}^{r} b_l k_l} p^{(k)}\left(\sum_{j=1}^{r} a_j k_j\right).$$

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