Journal of Integer Sequences, Vol. 17 (2014),
Article 14.7.7

On the Asymptotic Behavior of Dedekind Sums

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Abstract

Let z be a real quadratic irrational. We compare the asymptotic behavior of Dedekind sums $S(p_k, q_k)$ belonging to convergents p_k/q_k of the *regular* continued fraction expansion of z with that of Dedekind sums $S(s_j/t_j)$ belonging to convergents s_j/t_j of the *negative regular* continued fraction expansion of z. Whereas the three main cases of this behavior are closely related, a more detailed study of the most interesting case (in which the Dedekind sums remain bounded) exhibits some marked differences, since the cluster points depend on the respective periods of these expansions. We show in which cases cluster points of $S(s_j, t_j)$ can coincide with cluster points of $S(p_k, q_k)$. An important tool for our purpose is a criterion that says which convergents s_j/t_j of z are convergents p_k/q_k.

1 Introduction and results

Let z be a real irrational number. We consider the *regular* continued fraction expansion

$$z = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}} = \lfloor a_0, a_1, a_2, \ldots \rfloor,$$

which is defined by the following algorithm:

$$z_0 = z,\ a_0 = \lfloor z_0 \rfloor,\ z_{j+1} = 1/(z_j - a_j),\ a_{j+1} = \lfloor z_{j+1} \rfloor,\ j \geq 0.$$

The *convergents* p_0/q_0, p_1/q_1, p_2/q_2, $\ldots$ of this expansion are given by

$$p_{-1} = 1, q_{-1} = 0, p_0 = a_0, q_0 = 1, p_{k+1} = a_{k+1}p_k + p_{k-1}, q_{k+1} = a_{k+1}q_k + q_{k-1}, k \geq 0.$$

For the sake of simplicity we call these convergents the $\lfloor\ \rfloor$-*convergents* of z.

We will compare the regular continued fraction expansion of z with the *negative regular* continued fraction expansion

$$z = c_0 - \cfrac{1}{c_1 - \cfrac{1}{c_2 - \cdots}} = \lceil c_0, c_1, c_2, \ldots \rceil,$$

which is defined by the algorithm

$$z_0 = z,\ c_0 = \lceil z_0 \rceil,\ z_{j+1} = 1/(c_j - z_j),\ c_{j+1} = \lceil z_{j+1} \rceil,\ j \geq 0.$$

The *convergents* s_0/t_0, s_1/t_1, s_2/t_2, $\ldots$ of this expansion are given by

$$s_{-1} = 1, t_{-1} = 0, s_0 = c_0, t_0 = 1, s_{j+1} = c_{j+1}s_j - s_{j-1}, t_{j+1} = c_{j+1}t_j - t_{j-1}, j \geq 0.$$

They are called the $\lceil\ \rceil$-*convergents* of z. Note that $c_j \geq 2$ for all $j \geq 1$, and $c_j \geq 3$ for infinitely many indices $j \geq 0$ (see [8]). Henceforth we call this continued fraction expansion simply the *negative regular expansion*. This expansion has aroused some interest due to the work of Hirzebruch and Zagier about class numbers of quadratic fields (see [6], [7, p. 136]).

It is well-known that $\lfloor\ \rfloor$-convergents of z have optimal approximation properties which $\lceil\ \rceil$-convergents have not, unless they happen to be $\lfloor\ \rfloor$-convergents (see [4, p. 44 ff.]). In general, $\lceil\ \rceil$-convergents are only *intercalary* fractions of $\lfloor\ \rfloor$-convergents: If

$$\frac{p_k}{q_k} \text{ and } \frac{p_{k+1}}{q_{k+1}} = \frac{a_{k+1}p_k + p_{k-1}}{a_{k+1}q_k + q_{k-1}}$$

are two adjacent $\lfloor\ \rfloor$-convergents of z, then the intercalary fractions are

$$\frac{p_k + p_{k-1}}{q_k + q_{k-1}}, \frac{2p_k + p_{k-1}}{2q_k + q_{k-1}}, \ldots, \frac{(a_{k+1} - 1)p_k + p_{k-1}}{(a_{k+1} - 1)q_k + q_{k-1}}.$$

They also approximate z successively, but not as well as $\lfloor\ \rfloor$-convergents do.

It is, therefore, desirable to be able to decide whether a $\lceil\ \rceil$-convergent is a $\lfloor\ \rfloor$-convergent. We give the following criterion.

Theorem 1. *Let $z = \lceil c_0, c_1, c_2, \ldots \rceil$ be an irrational number and s_j/t_j, $j \geq 1$, a $\lceil\ \rceil$-convergent of z. Then s_j/t_j is a $\lfloor\ \rfloor$-convergent of z iff $c_{j+1} \geq 3$. In particular, the sequence s_j/t_j, $j \geq 1$, contains infinitely many $\lfloor\ \rfloor$-convergents of z.*

Example 2. Let $z = e = 2.71828\cdots$ be Euler's number, whose regular continued fraction expansion is

$$\lfloor 2, \overline{1, 2k, 1}\rfloor_{k=1}^{\infty} = \lfloor 2, 1, 2, 1, 1, 4, 1, 1, 6, 1, \ldots\rfloor,$$

where we have used Perron's notation (see [4, p. 124]). By means of the transition formula

$$\lfloor a_0, a_1, a_2, a_3, a_4, \ldots\rfloor = \lceil a_0 + 1, 2^{(a_1-1)}, a_2 + 2, 2^{(a_3-1)}, a_4 + 2, \ldots\rceil \tag{1}$$

(see [1, p. 93]) one easily obtains

$$e = \lceil\overline{3, 4k, 3, 2^{(4k-1)}}\rceil_{k=1}^{\infty} = \lceil 3, 4, 3, 2^{(3)}, 3, 8, 3, 2^{(7)}, \ldots\rceil;$$

here $2^{(j)}$ stands for a sequence of j numbers 2. According to Theorem 1, the $\lceil\ \rceil$-convergents

$$\frac{s_1}{t_1} = \frac{11}{4}, \frac{s_5}{t_5} = \frac{87}{32}, \frac{s_6}{t_6} = \frac{193}{71}, \frac{s_7}{t_7} = \frac{1457}{536}, \frac{s_{15}}{t_{15}} = \frac{23225}{8544}, \frac{s_{16}}{t_{16}} = \frac{49171}{18089}, \frac{s_{17}}{t_{17}} = \frac{566827}{208524}$$

are $\lfloor\ \rfloor$-convergents of e. Indeed, if the latter are denoted by p_k/q_k, $k \geq 0$, we find that the above $\lceil\ \rceil$-convergents coincide with p_3/q_3, p_5/q_5, $\ldots$, p_{15}/q_{15}, respectively. As intercalary fractions we have, for instance, $s_2/t_2 = (p_4 + p_3)/(q_4 + q_3)$ or $s_8/t_8 = (p_{10} + p_9)/(q_{10} + q_9)$.

The main application of Theorem 1 in this paper is a comparative study of the asymptotic behavior of Dedekind sums with arguments near quadratic irrationals. For an integer a and a natural number b let

$$s(a, b) = \sum_{k=1}^{b} ((k/b))((ak/b))$$

be the classical Dedekind sum, where

$$((x)) = \begin{cases} x - \lfloor x\rfloor - 1/2, & \text{if } x \in \mathbb{R} \setminus \mathbb{Z}; \\ 0, & \text{if } x \in \mathbb{Z} \end{cases}$$

(see, for instance, [5, p. 1]). We work with $S(a, b) = 12s(a, b)$ instead of $s(a, b)$ and compare the asymptotic behavior of the Dedekind sums $S(p_k, q_k)$ of a quadratic irrational z with the asymptotic behavior of $S(s_j, t_j)$ when k and j tend to infinity. Theorem 1 has the effect that the asymptotic behavior is, roughly speaking, the same in both cases.

For a quadratic irrational z both expansions are periodic, i.e.,

$$z = \lfloor a_0, a_1, \ldots a_q, \overline{b_1, b_2, \ldots, b_l}\rfloor = \lceil c_0, c_1, \ldots c_r, \overline{d_1, \ldots, d_m}\rceil,$$

where $(b_1, \ldots, b_l)$ and $(d_1, \ldots, d_m)$ are the respective periods (see [8, Satz 14, Satz 15]). Here l and m are smallest possible. In addition, q and r are smallest possible for this choice of l and m. In the purely periodic case, we put $q = -1$ and $r = -1$. Further, let

$$L = \begin{cases} l, & \text{if } l \text{ is even;} \\ 2l, & \text{if } l \text{ is odd.} \end{cases}$$

Then $(b_1, \ldots, b_L)$ is also a period of the regular continued fraction expansion of z. If l is odd, it has the form $(b_1, \ldots, b_l, b_1, \ldots, b_l)$. The three cases of Theorem 3 (below) are connected with the values of

$$B = (-1)^q \sum_{j=1}^{L} (-1)^{j-1} b_j \quad \text{and} \quad D = \frac{1}{m} \sum_{j=1}^{m} d_j.$$

It should be noted that $B = 0$ if l is odd or if the period $(b_1, \ldots, b_l)$ is palindromic.

Theorem 3. *In the above setting, $B > 0$, $B = 0$, $B < 0$ iff $D < 3$, $D = 3$, $D > 3$, respectively. Suppose that j and k tend to ∞. Then*
both $S(p_k, q_k)$ and $S(s_j, t_j)$ tend to ∞ if $B > 0$ (or $D < 3$),
these quantities remain bounded if $B = 0$ (or $D = 3$),
and they tend to $-\infty$ if $B < 0$ (or $D > 3$).

In the remainder of this section we assume $B = 0$, i.e., $D = 3$. In the paper [2] it has been shown that the Dedekind sums $S(p_k, q_k)$ accumulate near L cluster points U_h, $h = 1, \ldots, L$. They are given by

$$U_h = A + (-1)^q \sum_{k=1}^{h} (-1)^{k-1} b_k + \begin{cases} z + 1/u_h - 3, & \text{if } q + h \text{ is odd ;} \\ z - 1/u_h, & \text{if } q + h \text{ is even.} \end{cases} \tag{2}$$

Here $A = \sum_{k=0}^{q} (-1)^{k-1} a_k$ and $u_h = \lfloor \overline{b_h, b_{h-1}, \ldots, b_1, b_L, b_{L-1}, \ldots, b_{h+1}} \rfloor$ is a purely periodic quadratic irrational, $h = 1, \ldots, L$. On the other hand, we shall see that the Dedekind sums $S(s_j, t_j)$ accumulate near cluster points V_i, $i = 1, \ldots, m$. They are given by

$$V_i = C + \sum_{j=1}^{i} (3 - d_j) + z - 1/v_i - 3, \tag{3}$$

where $C = \sum_{j=0}^{r} (3 - c_j)$ and $v_i = \lceil \overline{d_i, d_{i-1}, \ldots, d_1, d_m, d_{m-1}, \ldots, d_{i+1}} \rceil$ is also a quadratic irrational, $i = 1, \ldots, m$. In the purely periodic cases, we have $A = 0$ and $C = 0$.

Theorem 4. *In the above setting, the cluster points U_h, $h = 1, \ldots, L$, are pairwise distinct, as are the cluster points $V_i, i = 1, \ldots, m$.*

The following theorem concerns cluster points V_i which coincide with cluster points U_h.

Theorem 5. *Let $(d_1, \ldots, d_m)$ be the period of the negative regular expansion of z. Let $i \in \{1, \ldots, m\}$ be such that $d_{i+1} \geq 3$ (if $i = m$ this means $d_1 \geq 3$). Then the cluster point V_i coincides with some cluster point U_h such that $q + h$ is odd. A cluster point U_h with $q + h$ even cannot coincide with a cluster point V_i.*

Remark 6. One can show that the period obtained from $(b_1, \ldots, b_L)$ by means of the transition formula (1) is shortest possible among the possible periods of the negative regular expansion of z — but we abstain from doing this here. Accordingly, there are $L/2$ indices $i \in \{1, \dots, m\}$ such that $d_{i+1} \geq 3$, the corresponding values of d_{i+1} being either $b_1+2, b_3+2, \dots, b_{L-1}+2$ or $b_2+2, b_4+2, \dots, b_L+2$. Hence there are $L/2$ coincidences of cluster points V_i with cluster points U_h. This means that *each* cluster point U_h, $q+h$ odd, coincides with such a V_i. So we have a fairly complete picture: The cluster points U_h, $q+h$ odd, coincide with the cluster points V_i, $d_{i+1} \geq 3$. Moreover, there are no coincidences of cluster points U_h, $q+h$ even, and cluster points V_i.

Example 7. Let $z = 1/\sqrt{53} = \lfloor 0, 7, \overline{3, 1, 1, 3, 14} \rfloor = \lceil 1, 2^{(6)}, \overline{5, 3, 2, 2, 16, 2, 2, 3, 5, 2^{(13)}} \rceil$. Since $l = 5$ is odd, the sequences $S(p_k, q_k)$ and $S(s_j, t_j)$ remain bounded. Because we have five $(= L/2)$ entries ≥ 3 in the period of the negative regular expansion of z, we have five common cluster points, namely $V_1 = U_2 = (636 + 60\sqrt{53})/371 = 2.89166\cdots$, $V_4 = U_4 = (-159 + 54\sqrt{53})/53 = 4.41747\cdots$, $V_7 = U_6 = (-2862 + 60\sqrt{53})/371 = -6.53690\cdots$, $V_8 = U_8 = (-1749 + 57\sqrt{53})/212 = -6.29261\cdots$, and $V_{22} = U_{10} = (477 + 57\sqrt{53})/212 = 4.20738\cdots$. As we expect, there are no further coincidences between the remaining 17 cluster point V_i and the remaining 5 cluster points U_h, although these points may lie close together, like $V_{12} = (-13833 + 97\sqrt{53})/2332 = -5.62900\cdots$ and $U_7 = (-1749 - 46\sqrt{53})/371 = -5.61694\cdots$.

2 Proofs

Proof of Theorem 1. Let z_{j+1} denote the $(j+1)$th complete quotient of the negative regular expansion of z. Accordingly,

$$z = \frac{z_{j+1}s_j - s_{j-1}}{z_{j+1}t_j - t_{j-1}}.$$

This gives

$$z - \frac{s_j}{t_j} = \frac{s_j t_{j-1} - s_{j-1}t_j}{t_j(z_{j+1}t_j - t_{j-1})}.$$

Here we use $s_j t_{j-1} - s_{j-1} t_j = -1$ (see [8, formula (4)]) and obtain

$$\left| z - \frac{s_j}{t_j} \right| = \frac{1}{t_j(z_{j+1}t_j - t_{j-1})} = \frac{1}{t_j^2} \cdot \frac{t_j}{z_{j+1}t_j - t_{j-1}}. \tag{4}$$

We need the regular continued fraction expansion $s_j/t_j = \lfloor a_0, \dots, a_n \rfloor$. However, we do not require $a_n \geq 2$; hence we may assume that n is odd. Let p_0/q_0, $\dots$, p_n/q_n be the convergents of this expansion, in particular, $s_j/t_j = p_n/q_n$. Now we can apply a criterion of Legendre (see [4, p. 39]) to the identity (4); thereby, we see that s_j/t_j is a $\lfloor\ \rfloor$-convergent of z, iff

$$\frac{t_j}{z_{j+1}t_j - t_{j-1}} < \frac{q_n}{q_n + q_{n-1}}. \tag{5}$$

Let $s^* \in \{1, \dots, t_j - 1\}$ denote the inverse of s_j mod t_j, i.e., $s_j s^* \equiv 1$ mod t_j (observe $j \geq 1$). Since $s_j t_{j-1} - s_{j-1} t_j = -1$, we see that $t_{j-1} \equiv -s^*$ mod t_j, and $1 \leq t_{j-1} < t_j$ implies

$t_{j-1} = t_j - s^*$. On the other hand, $s_j = p_n$ and $t_j = q_n$, which yields

$$s_j q_{n-1} - p_{n-1} t_j = (-1)^{n-1} = 1$$

(see [4, p. 25]). Accordingly, $q_{n-1} = s^*$. Therefore, the condition (5) is equivalent to

$$z_{j+1} t_j - (t_j - s^*) > t_j + s^*, \quad \text{i.e., to} \quad z_{j+1} > 2.$$

Recall that $z_{j+1} > 2$ iff $c_{j+1} \geq 3$. Recall, further, that $c_{j+1} \geq 3$ holds for infinitely many indices j. Thereby, we obtain the desired result. $\square$

Proof of Theorem 3. Let $z = \lfloor a_0, a_1, \dots a_q, \overline{b_1, b_2, \dots, b_l} \rfloor = \lceil c_0, c_1, \dots c_r, \overline{d_1, \dots, d_m} \rceil$ be as above. In [2] we studied the asymptotic behavior of $S(p_k, q_k)$ for the $\lfloor\ \rfloor$-convergents p_k/q_k of z. Indeed, if $k = q + nL + h$, $n \geq 0$, $h \in \{1, \dots, L\}$,

$$S(p_k, q_k) = A + nB + (-1)^q \sum_{j=1}^{h} (-1)^{j-1} b_j + \begin{cases} (p_k + q_{k-1})/q_k - 3, & \text{if } k \text{ is odd ;} \\ (p_k - q_{k-1})/q_k, & \text{if } k \text{ is even.} \end{cases} \tag{6}$$

Here A and B are as in Section 1. Since $p_k/q_k \to z$ for $k \to \infty$ and $0 \leq q_{k-1}/q_k \leq 1$ for $k \geq 0$, this gives

$$\begin{gathered} S(p_k, q_k) \to \infty \text{ if } B > 0, \\ S(p_k, q_k) \text{ remains bounded if } B = 0, \text{ and} \\ S(p_k, q_k) \to -\infty \text{ if } B < 0. \end{gathered} \tag{7}$$

A similar asymptotic behavior takes place for the Dedekind sums $S(s_j, t_j)$ belonging to the $\lceil\ \rceil$-convergents s_j/t_j of z. Indeed, a formula of Hirzebruch, Zagier and Myerson (see [3]) says

$$S(s_j, t_j) = \sum_{k=0}^{j} (3 - c_k) + (s_j - t_{j-1})/t_j - 3,$$

where we have written $z = \lceil c_0, c_1, c_2, \dots \rfloor$ disregarding the period. Hence we have, for $j = r + nm + i$, $n \geq 0$, $i \in \{1, \dots, m\}$,

$$S(s_j, t_j) = C + nm(3 - D) + \sum_{k=0}^{i} (3 - c_k) + (s_j - t_{j-1})/t_j - 3, \tag{8}$$

where C and D are as in Section 1. Observe $s_j/t_j \to z$ for $j \to \infty$ and $0 \leq t_{j-1}/t_j \leq 1$ for $j \geq 0$ (see [8, Satz 4, Satz 1]). Then we obtain

$$\begin{gathered} S(s_j, t_j) \to \infty \text{ if } D < 3, \\ S(s_j, t_j) \text{ remains bounded if } D = 3, \text{ and} \\ S(s_j, t_j) \to -\infty \text{ if } D > 3. \end{gathered} \tag{9}$$

By Theorem 1, the sequence s_j/t_j contains infinitely many $\lfloor\ \rfloor$-convergents p_k/q_k of z. Therefore, the asymptotic behavior of $S(p_k, q_k)$ for these convergents of z in the sense of (7) must be the same as in the sense of (9). This, however, yields Theorem 3, in particular, the connection between B and D. $\square$

Remark 8. The relation between the quantities B and D can also be proved in a simple and direct way by means of the transition formula (1).

Proof of Theorem 4. In the paper [2] we deduced formula (2) from (6). In exactly the same way one can deduce (3) from (8). First we show that the cluster points V_i, $i = 1, \ldots, m$, are pairwise distinct. Suppose that $V_i = V_j$ for some $i, j \in \{1, \ldots, m\}$. This implies

$$1/v_i = 1/v_j + a, \quad \text{for some } a \in \mathbb{Z}.$$

Since v_i, v_j are both > 1, we have $0 < 1/v_i, 1/v_j < 1$, whence $a = 0$ and $v_i = v_j$ follows. Since m is smallest possible, v_i and v_j have different negative regular expansions if $i \neq j$. Therefore, $i = j$.

In the case of the cluster points U_h, $h = 1, \ldots, L$, the proof is slightly more subtle. Suppose, first, that l is odd. Then the values $1/u_h$, $h = 1, 3, \ldots, l$, and $1/u_{l+h} = 1/u_h$, $h = 2, 4, \ldots, l-1$, appear with the same sign in (2), whereas $1/u_h$, $h = 2, 4, \ldots, l-1$, and $1/u_{l+h} = 1/u_h$, $h = 1, 3, \ldots, l$, appear with the opposite sign. If $1/u_h$ and $1/u_k$ have the same sign, one can argue as in the case of the cluster points V_i, since u_h, u_k are both > 1. So we are left with the case

$$1/u_h = -1/u_k + a, a \in \mathbb{Z}.$$

Since $0 < 1/u_h, 1/u_k < 1$, this is only possible with $a = 1$. Accordingly, we obtain

$$u_h = (1 - 1/u_k)^{-1} = 1 + 1/(u_k - 1).$$

Now $u_k = \lfloor \overline{b_k, b_{k-1}, \ldots, b_1, b_l, b_{l-1}, \ldots, b_{k+1}} \rfloor$. If $b_k > 1$, we obtain

$$1 + 1/(u_k - 1) = \lfloor 1, b_k - 1, \overline{b_{k-1}, \ldots, b_1, b_l, \ldots, b_k} \rfloor = u_h = \lfloor \overline{b_h, \ldots, b_1, b_l, \ldots, b_{h+1}} \rfloor.$$

However, u_h is purely periodic with period length l, and so must be $1 + 1/(u_k - 1)$. But if we compare the second entry $b_k - 1$ with the second entry that follows the period, we get the contradiction $b_k - 1 = b_k$. If $b_k = 1$, we obtain

$$1 + 1/(u_k - 1) = \lfloor b_{k-1} + 1, \overline{b_{k-2}, \ldots, b_1, b_l, \ldots, b_{k-1}} \rfloor.$$

Again, $1+1/(u_k-1)$ must be purely periodic with period length l, which gives the impossible relation $b_{k-1} + 1 = b_{k-1}$.

If $l = L$ is even, we proceed in a similar way: We have the same sign in (2) for $1/u_h$, $h = 1, 3, \ldots, L-1$, and the opposite sign for $1/u_h$, $h = 2, 4, \ldots, L$ and, thus, can rule out the corresponding identities if the signs are equal. In the case $1/u_h = -1/u_k + a$, $a \in \mathbb{Z}$, we argue as above. $\square$

Proof of Theorem 5. Let $h \in \{1, \ldots, L\}$. We say that the $\lfloor\ \rfloor$-convergent p_k/q_k belongs to the $\lfloor\ \rfloor$-*class* h, if $k = q + nL + h$ for some $n \geq 0$. For the $\lfloor\ \rfloor$-convergents p_k/q_k belonging to this class, the corresponding Dedekind sums $S(p_k, q_k)$ converge against the cluster point U_h — as we have shown in the paper [2]. In the same way we say that the $\lceil\ \rceil$-convergent s_j/t_j belongs to the $\lceil\ \rceil$-*class* i, $i \in \{1, \ldots, m\}$, if $j = r + nm + i$ for some $n \geq 0$. For $\lceil\ \rceil$-convergents s_j/t_j belonging to this class, the corresponding Dedekind sums $S(s_j, t_j)$ converge against the cluster point V_i.

Let $i \in \{1, \ldots, m\}$ be such that $d_{i+1} \geq 3$ (if $i = m$, $d_1 \geq 3$). By Theorem 1, each convergent s_j/t_j of the $\lceil\ \rceil$-class i equals some $\lfloor\ \rfloor$-convergent p_k/q_k. Since $s_j/t_j > z$, k must be odd. Up to finitely many exceptions, these convergents s_j/t_j must belong to exactly one $\lfloor\ \rfloor$-class h — otherwise the convergent sequence $S(s_j, t_j)$ would have more than one cluster point. Accordingly, $S(s_j, t_j)$ converges against $U_h = V_i$ for this h. Because $q + h \equiv k \bmod 2$ for all p_k/q_k in the $\lfloor\ \rfloor$-class h, we see that $q + h$ is odd.

Finally, suppose that $q + h$ is even and $U_h = V_i$ for some $i \in \{1, \ldots, m\}$. By (2) and (3), $-1/u_h = -1/v_i + a$, $a \in \mathbb{Z}$. As in the proof of Theorem 4, we conclude $u_h = v_i$. Now

$$u_h = \lfloor \overline{b_h, b_{h-1}, \ldots, b_1, b_L, \ldots b_{h+1}} \rfloor = \lceil b_h + 1, \overline{2^{(b_{h-1}-1)}, b_{h-2} + 2, \ldots, 2^{(b_{h+1}-1)}, b_h + 2} \rceil.$$

Since $v_i = \lceil \overline{d_i, d_{i-1}, \ldots, d_1, d_m, \ldots d_{i+1}} \rceil$ is purely periodic, we see that $b_h + 1$ must be equal to $b_h + 2$, which is impossible. $\square$

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2010 *Mathematics Subject Classification*: Primary 11F20; Secondary 11A55.
Keywords: Asymptotic behavior of Dedekind sums, continued fraction expansion of a quadratic irrational.

Received March 14 2014; revised version received July 3 2014. Published in *Journal of Integer Sequences*, July 11 2014.

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