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A Counting Function Generalizing Binomial Coefficients and Some Other Classes of Integers

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Abstract

We define a counting function that is related to the binomial coefficients. For this function, we derive an explicit expression. In some particular cases, we prove simpler explicit formulae. We also derive a formula for the number of $(0, 1)$-matrices, having a fixed number of 1's, and having no zero rows and zero columns. Further, we show that our function satisfies several recurrence relations.

We then examine the relationship of our counting function with different classes of integers. These classes include: some figurate numbers, the number of points on the surface of a square pyramid, the magic constants, the truncated square numbers, the coefficients of the Chebyshev polynomials, the Catalan numbers, the Delannoy numbers, the Sulanke numbers, the numbers of the coordination sequences, and the number of the crystal ball sequences of a cubic lattice.

In the last part of the paper, we count several configurations by our function. Some of these are: the number of spanning subgraphs of the complete bipartite graph, the number of squares contained in a square, the number of colorings of points on a line, the number of divisors of some particular numbers, the number of all parts in the compositions of an integer, the numbers of the weak compositions of integers, and the number of particular lattice paths. We conclude by counting the number of possible moves of the rook, bishop, and queen on a chessboard.

For the most statements in the paper, we provide bijective proofs in terms of insets, which we define in the paper. Hence, using the same method, we count different configurations.

1 Introduction

For a set $Q = \{q_1, q_2, \ldots, q_n\}$ of positive integers and a nonnegative integer m, we consider the set X consisting of n blocks X_i, $(i = 1, 2, \ldots, n)$, X_i having q_i elements, and a block Y with m elements. We call X_i the main blocks, and Y the additional block of X.

Definition 1. By an $(n+k)$-*inset* of X, we mean an $(n+k)$-subset of X, intersecting each main block. We let $\binom{m,n}{k,Q}$ denote the number of $(n+k)$-insets of X.

In all what follows, elements of insets will always be written in terms of increasing indices.

Remark 2. Note that this function is first defined in Janjić's paper [5].

The case $n = 0$ also may be considered. In this case there are no main blocks, so that $\binom{m,0}{k,\emptyset} = \binom{m}{k}$. Also, when each main block has only one element, we have

$$\binom{m,n}{k,Q} = \binom{m}{k}.$$

Hence, the function $\binom{m,n}{k,Q}$ is a generalization of the ordinary binomial coefficients.

In the case $k = 0$, we obviously have

$$\binom{m,n}{0,Q} = q_1 \cdot q_2 \cdots q_n.$$

Thus, the product function is a particular case of our function.

Note that, when $q_1 = q_2 = \cdots = q_n = q$, we write $\binom{m,n}{k,q}$ instead of $\binom{m,n}{k,Q}$. In this case, we have

$$\binom{m,n}{0,q} = q^n.$$

Some powers may be obtained in a less obvious way.

Proposition 3. *For a nonnegative integer m, we have*

$$\binom{m,1}{2,2} = m^2.$$

Proof. Let $X_1 = \{x_1, x_2\}$ be the main and $Y = \{y_1, y_2, \ldots, y_m\}$ be the additional block of X. It is enough to define a bijection between 3-insets of X and the set of 2-tuples (s, t), where $s, t \in [m]$. A bijection goes as follows:

$$\begin{array}{ll} 1. & \{x_1, y_i, y_j\} \leftrightarrow (i, j), \\ 2. & \{x_2, y_i, y_j\} \leftrightarrow (j, i), \\ 3. & \{x_1, x_2, y_i\} \leftrightarrow (i, i). \end{array}$$

□

Proposition 4. *For a positive integer q, we have*

$$\binom{1,2}{1,q} = q^3.$$

Proof. Let $X_i = \{x_{i1}, x_{i2}, \ldots, x_{iq}\},\ (i = 1, 2)$ be the main blocks, and $Y = \{y\}$ the additional block of X. We need a bijection between 3-insets of X and 3-tuples (s, r, t), where $r, s, t \in [q]$. A bijection is defined in the following way:

$$\begin{array}{ll} 1. & \{x_{1s}, x_{1t}, x_{2r}\} \leftrightarrow (s, t, r), \\ 2. & \{x_{1s}, x_{2t}, x_{2r}\} \leftrightarrow (r, t, s), \\ 3. & \{x_{1s}, x_{2t}, y\} \leftrightarrow (s, s, t). \end{array}$$

□

Proposition 5. *Let n be a positive integer, and let k be a nonnegative integer. Then*

$$\binom{0,n}{k,2} = 2^{n-k}\binom{n}{k}. \tag{1}$$

Proof. We obtain $\binom{0,n}{k,2}$ by choosing both elements from arbitrary k main blocks, which may be done in $\binom{n}{k}$ ways, and one element from each of the remaining $n-k$ main blocks, which may be done in 2^{n-k} ways. □

The particular case $m = 0$ may be interpreted as numbers of 1's in a $(0,1)$-matrix. The following proposition is obvious:

Proposition 6. *Let n, q be positive integers, and let k be a nonnegative integer. Then the number $\binom{0,n}{k,q}$ equals the number of $(0,1)$-matrices of order $q \times n$ containing $n + k$ 1's, and which have no zero columns.*

Now, we count the number of $(0,1)$-matrices with a fixed number of 1's which have no zero rows and zero columns. Let $M(n, k, q)$ denote the number of such matrices of order $q \times n$, which have $n + k$ 1's.

Proposition 7. *Let n, q be positive integers, and let k be a nonnegative integer. Then*

$$M(n,k,q) = \sum_{i=0}^{q} (-1)^{q+i}\binom{q}{i}\binom{0,n}{k,i},\ (q > 1). \tag{2}$$

Proof. According to (1), we have $\binom{0,n}{k,q}$ $(0,1)$-matrices, which have $n+k$ 1's and no zero columns. Among them, there are $\binom{q}{i}M(n,k,q-i)$, $(i=0,1,2,\ldots,q)$ matrices having exactly i zero rows. It follows that

$$\binom{0,n}{k,q}=\sum_{i=0}^{q}\binom{q}{i}M(n,k,q-i),$$

and the proof follows from the inversion formula. $\square$

Obviously, the function $M(n,k,q)$ has the property:

$$M(n,k,q)=M(q,n+k-q,n).$$

Using (1) and (2), we obtain the binomial identity:

$$\binom{n}{k}=\frac{1}{2^{n-k}}\sum_{i=0}^{n}(-1)^{n+i}\binom{n}{i}\binom{0,\ 2}{n+k-2,\ i},\ (k>0).$$

2 Explicit formulae and recurrences

We first consider the particular case $m=k=1$, when the explicit formula for our function is easy to derive.

Proposition 8. *Let n be a positive integers, and let the set $Q=\{q_1,\ldots,q_n\}$ consists of positive integers. Then*

$$\binom{1,n}{1,Q}=q_1q_2\cdots q_n\binom{\sum_{i=1}^{n}q_i-n+2}{2}. \tag{3}$$

In particular, we have

$$\binom{1,2}{1,Q}=\frac{q_1q_2(q_1+q_2)}{2}. \tag{4}$$

Proof. If the element of the additional block is inserted into an $(n+1)$-inset, then each of the remaining elements must be chosen from different main blocks. For this, we have $q_1\cdot q_2\cdots q_n$ possibilities. If it is not inserted, we take two elements from one of the main blocks and one element from each of the remaining main blocks. For this, we have $\sum_{i=1}^{n}\binom{q_i}{2}q_1\cdots q_{i-1}\cdot q_{i+1}\cdots q_n$ possibilities. All in all, we have $q_1q_2\cdots q_n\binom{\sum_{i=1}^{n}q_i-n+2}{2}$ possibilities. $\square$

Using the inclusion-exclusion principle, we derive an explicit formula for $\binom{m,n}{k,Q}$.

Proposition 9. *Let n be a positive integer, let k, m be nonnegative integers, and let the set $Q = \{q_1, \dots, q_n\}$ consists of positive integers. Then*

$$\binom{m,n}{k,Q} = \sum_{I \subseteq [n]} (-1)^{|I|} \binom{|X| - \sum_{i \in I} q_i}{n+k}, \tag{5}$$

where the sum is taken over all subsets of $[n]$.

Proof. For $i = 1, 2, \dots, n$ and an $(n+k)$-subset Z of X, we define the following property:

The block X_i does not intersect Z.

By the inclusion-exclusion principle, we obtain

$$\binom{m,n}{k,Q} = \sum_{I \subseteq [n]} (-1)^{|I|} N(I),$$

where $N(I)$ is the number of $(n+k)$-subsets of X, which do not intersect main blocks X_i, $(i \in I)$. It is clear that there are

$$\binom{|X| - \sum_{i \in I} q_i}{n+k}$$

such subsets, and the formula is proved. □

In the particular cases $n = 1$ and $n = 2$, we obtain the following formulae:

$$\binom{m,1}{k,q} = \binom{q+m}{k+1} - \binom{m}{k+1}, \tag{6}$$

$$\binom{m,2}{k,Q} = \binom{q_1+q_2+m}{k+2} - \binom{q_1+m}{k+2} - \binom{q_2+m}{k+2} + \binom{m}{k+2}. \tag{7}$$

In the case $q_1 = q_2 = \dots = q_n = q$, formula (5) takes a simpler form:

$$\binom{m,n}{k,q} = \sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{nq+m-iq}{n+k}. \tag{8}$$

If $q = 1$, then $\binom{m,n}{k,1} = \binom{m}{k}$, so that formula (8) implies the well-known binomial identity

$$\binom{m}{k} = \sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{n+m-i}{n+k}.$$

Next, since we have $\binom{m,n}{0,q} = q^n$, equation (8) yields

$$q^n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{qn+m-qi}{n}.$$

Note that the left hand side does not depend on m, so we have here a family of identities.

Next, we derive a recurrence relation which stresses the similarity of our function and the binomial coefficients.

Proposition 10. *Let n be a positive integer, let k, m be nonnegative integers, and let the set $Q = \{q_1, \ldots, q_n\}$ consists of positive integers. Then*

$$\binom{m+1, n}{k+1, Q} = \binom{m, n}{k+1, Q} + \binom{m, n}{k, Q}. \tag{9}$$

Proof. Let $Y = \{y_1, y_2, \ldots, y_m, y_{m+1}\}$ be the additional block of X. We divide all $(n+k+1)$-insets of X into two classes. In the first class are the insets which do not contain the element y_{m+1}. There are $\binom{m,n}{k+1,Q}$ such insets. The second class consists of the remaining insets, namely those that contain y_{m+1}. There are $\binom{m,n}{k,Q}$ such insets. ◻

The next formula reduces the case of arbitrary m to the case $m = 0$.

Proposition 11. *Let n be a positive integer, let k, m be nonnegative integers, and let the set $Q = \{q_1, \ldots, q_n\}$ consists of positive integers. Then*

$$\binom{m, n}{k, Q} = \sum_{i=0}^{m} \binom{m}{i} \binom{0, n}{k-i, Q}. \tag{10}$$

Proof. We may obtain all $(n+k)$-insets of X in the following way:

1. There are $\binom{0,n}{k,Q}$ $(n+k)$-insets not containing elements from Y.
2. The remaining $(n+k)$-insets of X, are a union of some $(n+k-i)$-inset of X, not intersecting Y, and some i-subset of Y, where $1 \le i \le m$. There are $\binom{m}{i}$ such insets.

◻

Particularly, we have

$$\binom{m, 1}{k, q} = \sum_{i=0}^{m} \binom{m}{i} \binom{0, 1}{k-i, q}.$$

According to (6), we have

$$\binom{0, 1}{k-i, q} = \binom{q}{k-i+1} - \binom{0}{k-i+1}, \quad \binom{m, 1}{k, q} = \binom{m+q}{k+1} - \binom{m}{k+1}.$$

As a consequence, we obtain the Vandermonde convolution:

$$\binom{q+m}{k+1} = \sum_{i=0}^{m} \binom{m}{i} \binom{q}{k+1-i}.$$

Using (1) and (10), we obtain another explicit formula for $\binom{m,n}{k,2}$:

$$\binom{m,n}{k,2} = 2^{n-k} \sum_{i=0}^{m} 2^i \binom{m}{i} \binom{n}{k-i}. \tag{11}$$

Finally, we derive two recurrence relations with respect to the number of main blocks:

Proposition 12. *Let n be a positive integer, let k, m be nonnegative integers, and let the set $Q = \{q_1, \ldots, q_n\}$ consists of positive integers. For a fixed $j \in [n]$ we have*

$$\binom{m,n}{k,Q} = \sum_{i=0}^{q_j-1} \binom{m+i, n-1}{k, Q \setminus \{q_j\}}, \tag{12}$$

$$\binom{m,n}{k,Q} = \sum_{i=1}^{q_j} \binom{q_j}{i} \binom{m,\ n-1}{k-i+1, Q \setminus \{q_j\}}. \tag{13}$$

Proof. Take $x_{jt} \in X_j$ arbitrarily. Consider the set Z_j, the main blocks of which are all the main blocks of X, except X_j. Let $U = Y \cup \{x_{j1}, \ldots, x_{j,t-1}\}$ be the additional block of Z_j. If T is a $(n+k-1)$-inset of Z, then $T \cup \{x_{jt}\}$ is the $(n+k)$-inset of X not containing elements of X_j, the second index of which is greater than t. The converse also holds. The assertion follows by summing over t, $(1 \le t \le q_j)$. Equation (12) is proved.

Omitting the jth main block of X, we obtain a set Z. Each $(n+k)$-inset of X may be obtained as a union of some $(n+k-i)$- inset of Z, $(1 \le i \le q_j)$ and some of $\binom{q_j}{i}$ i-subsets of the omitting main block, which proves (13). □

3 Connections with other classes of integers

We noted that our function is closely connected with the binomial coefficients. In this section, we establish its relation to some other classes of integers.

Proposition 13. *If $n \ge 0$, then*

$$\binom{n,\ 2}{n+2,\ 3} = \frac{(n+5)(n+6)}{2},$$

that is, $\binom{n,\ 2}{n+2,\ 3}$ equals the $(n+5)$th triangular number A000217.

Proof. The proof follows from (7). □

Proposition 14. *If $n \ge 2$, then*

$$\binom{n-1,\ 1}{1,\ n} = \frac{3n(n-1)}{2},$$

that is, $\binom{n-1,\ 1}{1,\ n}$ equals the $(n-1)$th triangular matchstick number A045943.

Proof. The proof follows from (6).

We also give a proof in terms of insets. Note first that $\binom{1,1}{1,2} = 3$ equals the first triangular matchstick number. Denote $TM_n = \binom{n,\,1}{1,\,n+1}$. We want to calculate the difference $TM_n - TM_{n-1}$. Consider two sets X and Z, both having one main block. Let $\{x_1, x_2, \ldots, x_n, x_{n+1}\}$, $\{x_1, x_2, \ldots, x_n\}$ be the main blocks of X and Z respectively, and let $\{y_1, y_2, \ldots, y_n\}$ and $\{y_1, y_2, \ldots, y_{n-1}\}$ be the additional blocks. The number $TM_n - TM_{n-1}$ equals the number of 2-insets of X, which are not insets of Z. Such an inset must contain either x_{n+1} or y_n. All insets of this form are

$$\{x_i, y_n\}, (i = 1, 2, \ldots, n+1), \}, \ \{x_{n+1}, y_i\}, \ (i = 1, 2, \ldots, n-1)\{x_i, x_{n+1}\}, (i = 1, 2, \ldots, n),$$

which are $3n$ in number. We conclude that

$$TM_n - TM_{n-1} = 3n,$$

which is the recurrence for the triangular matchstick numbers. □

Proposition 15. *Let n be a positive integer. Then*

$$\binom{n,1}{1,n} = \frac{n(3n-1)}{2},$$

that is, $\binom{n,1}{1,n}$ equals the nth pentagonal number A000326.

Proof. Firstly, a 2-inset may consists of pairs of elements from the main block, and there is $\binom{n}{2}$ such pairs. Secondly, it may consist of one element from the main and one element from the additional block. There is n^2 such insets. We thus have $\binom{n}{2} + n^2 = \frac{n(3n-1)}{2}$ 2-insets. □

Proposition 16. *Let n be a positive integer. Then*

$$\binom{n,2}{1,n} = (2n-1)n^2,$$

that is, $\binom{n,2}{1,n}$ equals the nth structured hexagonal prism number A015237.

Proof. The proof follows from (7). □

Proposition 17. *Let m be a nonnegative integer, and let $Q = \{2, 3\}$. Then*

$$\binom{m,2}{2,Q} = 3(m+1)^2 + 2,$$

that is, $\binom{m,2}{2,Q}$ equals the number of points on the surface of a square pyramid A005918.

We give a short proof in terms of insets.

Proof. Let $X_1 = \{x_{11}, x_{12}\},\ X_2 = \{x_{21}, x_{22}, x_{23}\}$ be the main blocks, and $Y = \{y_1, y_2, \dots, y_n\}$ be the additional block of X. In the next table we write different types of 4-insets and its numbers.

4-insets	its number
$\{x_{11}, x_{12}, x_{2i}, x_{2j}\}$,	3,
$\{x_{11}, x_{12}, x_{2i}, y_j\}$,	$3m$,
$\{x_{1i}, x_{21}, x_{22}, x_{23}\}$,	2,
$\{x_{1i}, x_{2j}, x_{2k}, y_s\}$	$6m$,
$\{x_{1i}, x_{2j}, y_k, y_s\}$,	$6\binom{m}{2}$.

We have $3(m+1)^2 + 2$ 4-insets in total.

$\square$

It follows from (9) that the numbers $\binom{m,n}{k,Q}$ form a Pascal-like array, in which the first row $(m = 0)$ begins with $q_1 \cdot q_2 \cdots q_n$.

In the particular case $n = 1$, the first row is

$$\binom{q}{1}, \binom{q}{2}, \dots, \binom{q}{q}.$$

Hence, if $q = 2$, the first row is $2, 1$, so that we obtain the reverse Lucas triangle A029653. We note one property of the triangle connected with the *figurate numbers*. The third column consists of 2-dimensional square numbers, the forth column consists of 3-dimensional square numbers, and so on. We conclude from this that the following proposition is true:

Proposition 18. *For $m > 0,\ k > 2$, the number $\binom{m,1}{k,2}$ equals the mth k-dimensional square pyramidal number A000330.*

Proof. The proof follows from the preceding notes. We also give a short bijective proof. According to (7), we have

$$\binom{m,1}{k,2} = \binom{m}{k} + \binom{m+1}{k}.$$

Let $X_1 = \{x_1, x_2\}$ be the main, and $Y = \{y_1, y_2, \dots, y_m\}$ the additional block of X. Consider two disjoint sets $A = \{a_1, a_2, \dots, a_m\},\ B = \{b_1, b_2, \dots, b_{m+1}\}$. Let the set C consist of k-subsets of A and k- subsets of B. We need to define a bijection between the set of $(k+1)$-insets of X and the set C. A bijection goes as follows:

1. $\{x_1, x_2, y_{i_1}, y_{i_2}, \dots, y_{i_{k-1}}\} \leftrightarrow \{b_{i_1}, b_{i_2}, \dots, b_{i_{k-1}}, b_{m+1}\}$,
2. $\{x_1, y_{i_1}, y_{i_2}, \dots, y_{i_k}\} \leftrightarrow \{b_{i_1}, b_{i_2}, \dots, b_{i_{k-1}}, b_{i_k}\}$,
3. $\{x_2, y_{i_1}, y_{i_2}, \dots, y_{i_k}\} \leftrightarrow \{a_{i_1}, a_{i_2}, \dots, a_{i_{k-1}}, a_{i_k}\}$.

$\square$

The following result follows from the fact that, for $q = 3$, the third column

$$1, 4, 10, 19, 31, \dots$$

of the above array consists of the centered triangular numbers.

Proposition 19. *For $m > 0,\ k > 1$, the number $\binom{m,1}{k,3}$ equals the $(m+1)$th k-dimensional centered triangular number A047010.*

For $q = 4$, the array consists of k-dimensional centered tetrahedral numbers, and so on. Hence,

Proposition 20. *For $m > 0,\ k > 1$, the number $\binom{m,1}{k,4}$ equals the $(m+1)$th k-dimensional centered tetrahedral number A047030.*

The fourth column (omitting two first terms 1 and 5), in the case $q = 3$, consists of numbers $15, 34, 65, 111, \ldots$, which are of the form $\frac{m(m^2+1)}{2}$, $(m = 3, 4, \ldots)$. This fact connects our function with the *magic constants* A006003.

Proposition 21. *For $m > 2$, the number $\binom{m,1}{3,3}$ equals the magic constant for the standard $m \times m$ magic square.*

Proof. The proof follows from (7). We again add a short bijective proof. Let $X_1 = \{x_1, x_2, x_3\}$ be the main block of X, and let $Y = \{y_1, y_2, \ldots, y_m\}$ be the additional block. We have

$$\frac{m(m^2+1)}{2} = \binom{m+1}{2} + m\binom{m}{2}.$$

Consider the following two sets: $A = \{a_1, a_2, \ldots, a_{m+1}\}$ and $B = \{b_1, b_2, \ldots, b_m\}$. Let C be the union of the set of 2-subsets of A and $\{iB_2 | i \in \{1, 2, \ldots, m\}$, where B_2 runs over all 2-subsets of B. We define a bijection between sets X and C in the following way:

1. $\{x_1, x_2, x_3, y_i\} \leftrightarrow \{a_i, a_{m+1}\}$,
2. $\{x_1, x_2, y_i, y_j\} \leftrightarrow \{a_i, a_j\}$,
3. $\{x_2, x_3, y_i, y_j\} \leftrightarrow i\{b_i, b_j\}$,
4. $\{x_1, x_3, y_i, y_j\} \leftrightarrow j\{b_i, b_j\}$,
5. $\{x_1, y_i, y_j, y_k\} \leftrightarrow i\{b_j, b_k\}$,
6. $\{x_2, y_i, y_j, y_k\} \leftrightarrow j\{b_i, b_k\}$,
7. $\{x_3, y_i, y_j, y_k\} \leftrightarrow k\{b_i, b_j\}$.

$\square$

Take $Q = \{2, q\}$. In this case, formula (7) takes the following form:

$$\binom{m,2}{2,Q} = \frac{q^3}{3} + \left(m - \frac{1}{2}\right)q^2 + \left(m^2 - m + \frac{1}{6}\right)q.$$

This easily implies that

$$\binom{m,2}{2,Q} = m^2 + (m+1)^2 + \cdots + (m+q-1)^2. \tag{14}$$

Proposition 22. *The number* $\binom{m,2}{2,Q}$*, where* m *is a nonnegative integer, and* $Q = \{2, q\}$*, for a positive integer* q*, counts the truncated square pyramidal numbers A050409.*

There is a relationship of our function with coefficients of the Chebyshev polynomials of the second kind, which immediately follows from (1).

Proposition 23. *Let* $c(n, k)$ *denote the coefficient of* x^k *of the Chebyshev polynomial* $U_n(x)$ *A008312. Then*

$$c(n, k) = (-1)^{\frac{n-k}{2}} \binom{0, \frac{n+k}{2}}{\frac{n-k}{2}, 2},$$

if n *and* k *are of the same parity, otherwise* $c(n, k) = 0$.

Remark 24. In Janjić's paper [5], the preceding connection is used to define a generalization of the Chebyshev polynomials.

We now establish a connection of our function to the *Catalan numbers* A000108.

Proposition 25. *If* C_n *is the nth Catalan number, then*

$$C_n = \frac{1}{3n+2} \binom{2n,\ 1}{n,\ 2}.$$

Proof. Let $X = \{x_1, x_2\}$ be the main, and $Y = \{y_1, y_2, \ldots, y_{2n}\}$ be the additional block of X. In the next table we write different types of $(n + 1)$-insets of X and its numbers.

$(n+1)$-insets	its number
$\{x_1, x_2, y_{i_1}, y_{i_2}, \ldots, y_{i_{n-1}}\}$,	$\binom{2n}{n-1}$,
$\{x_1, y_{i_1}, y_{i_2}, \ldots, y_{i_{n-1}, y_{i_n}}\}$,	$\binom{2n}{n}$,
$\{x_2, y_{i_1}, y_{i_2}, \ldots, y_{i_{n-1}, y_{i_n}}\}$,	$\binom{2n}{n}$.

We thus have $\binom{2n}{n-1} + 2\binom{2n}{n} = \frac{3n+2}{n+1}\binom{2n}{n}$ $(n + 1)$-insets. □

Proposition 26. *If* F_q *is the Fibonacci number, and* $Q = \{F_q, F_{q+1}\}$*, then*

$$\binom{1, 2}{1, Q} = \binom{q+2}{3}_F,$$

where $\binom{q+2}{3}_F$ *is the Fibonomial coefficient A001655.*

Proof. The formula is an easy consequence of (4). □

Finally, we connect our function with Delannoy and Sulanke numbers.

The *Delannoy number* $D(m, n)$ A008288 is defined as the number of lattice paths from $(0, 0)$ to (m, n), using steps $(1, 0), (0, 1)$ and $(1, 1)$.

Proposition 27. *We have*

$$D(m,n) = \binom{m,n}{n,2}. \tag{15}$$

Proof. We obviously have

$$\binom{0,n}{n,2} = \binom{m,0}{0,2} = 1.$$

Furthermore, for $m,n \neq 0$, using (9), we obtain

$$\binom{m,n}{n,2} = \binom{m-1,n}{n,2} + \binom{m-1,n}{n-1,2}.$$

Applying (12), we have

$$\binom{m-1,n}{n-1,2} = \binom{m-1,n-1}{n-1,2} + \binom{m,n-1}{n-1,2}.$$

It follows that

$$\binom{m,n}{n,2} = \binom{m-1,n}{n,2} + \binom{m-1,n-1}{n-1,2} + \binom{m,n-1}{n-1,2}.$$

Hence, the numbers $\binom{m,n}{n,2}$ satisfy the same recurrence relation as do the Delannoy numbers. $\square$

Remark 28. In his paper [9], Sulanke gave the collection of 29 configurations counted by the central Delannoy numbers.

The *Sulanke numbers* $s_{n,k}$, $(n,k \geq 0)$ A064861 are defined in the following way:

$$s_{0,0} = 1, \quad s_{n,k} = 0, \quad \text{if } n < 0 \text{ or } k < 0,$$

and

$$s_{n,k} = \begin{cases} s_{n,k-1} + s_{n-1,k}, & \text{if } n+k \text{ is even;} \\ s_{n,k-1} + 2s_{n-1,k}, & \text{if } n+k \text{ is odd.} \end{cases}$$

Proposition 29. *Let n,k be nonnegative integers. Then*

$$s_{n,k} = \begin{cases} \binom{\frac{n+k}{2},\frac{n+k}{2}}{k,\,2}, & \textit{if } n+k \textit{ is even;} \\ \binom{\frac{n+k-1}{2},\frac{n+k+1}{2}}{k,\,2}, & \textit{if } n+k \textit{ is odd.} \end{cases} \tag{16}$$

Proof. According to (9), for even $n+k$, we have

$$\binom{\frac{n+k}{2},\ \frac{n+k}{2}}{k,\ 2} = \binom{\frac{n+k-2}{2},\ \frac{n+k}{2}}{k-1,\ 2} + \binom{\frac{n+k-2}{2},\ \frac{n+k}{2}}{k,\ 2}.$$

For odd $n+k$, using (13), we obtain

$$\binom{\frac{n+k-1}{2},\ \frac{n+k+1}{2}}{k,\ 2} = \binom{\frac{n+k-1}{2},\ \frac{n+k-1}{2}}{k-1,\ 2} + 2\binom{\frac{n+k-1}{2},\ \frac{n+k-1}{2}}{k,\ 2}.$$

We see that the numbers on the right side of (16) satisfy the same recurrence as do the Sulanke numbers. ☐

Equation (11) implies the following explicit formulae for the Sulanke numbers:

$$s_{n,k} = \sum_{i=0}^{\frac{n+k}{2}} 2^{\frac{n-k+2i}{2}} \binom{\frac{n+k}{2}}{i} \binom{\frac{n+k}{2}}{k-i},$$

if $n+k$ is even, and

$$s_{n,k} = \sum_{i=0}^{\frac{n+k-1}{2}} 2^{\frac{n+1-k+2i}{2}} \binom{\frac{n+k-1}{2}}{i} \binom{\frac{n+k+1}{2}}{k-i},$$

if $n+k$ is odd.

Remark 30. Using the method of the Z transform, Velasco [10] derived similar formulae for Sulanke numbers.

The following two results connect our function with the *coordination sequences* and the *crystal ball sequences* for cubic lattices.

Proposition 31. *Let m, n be positive integers.*

1. *The number $\binom{m,n}{n,2}$ equals the number of solutions of the Diophantine inequality*

$$|x_1| + |x_2| + \cdots + |x_n| \le m. \tag{17}$$

2. *The number $\binom{m-1,n}{n-1,2}$ equals the number of solutions of the Diophantine equation*

$$|x_1| + |x_2| + \cdots + |x_n| = m. \tag{18}$$

Proof. Each solution $(a_1, a_2, \ldots, a_n)$ of (17) corresponds to a $2n$-inset T of X as follows:

If $a_i = 0$, then both elements of the main block X_i are inserted in T. If $a_i \neq 0$, and its sign is $+$, then the first element from X_i is inserted into T. If the sign of a_i is $-$, then the second element of X_i is inserted into T. In this way, we insert elements from the main blocks into T.

Assume that $X_{i_1}, X_{i_2}, \ldots, X_{i_t},\ 1 < i_1 < i_2 < \cdots < i_t,\ (1 \le t \le n)$ are the main blocks from which, up until now, only one element is inserted into T. This means that $a_{i_1}, a_{i_2}, \ldots, a_{i_t}$ are all different from 0. Also, $|a_{i_1}| + |a_{i_2}| + \cdots + |a_{i_t}| \le m$. Now, we insert elements

$$y_{|a_{i_1}|}, y_{|a_{i_1}|+|a_{i_2}|}, \ldots, y_{|a_{i_1}|+|a_{i_2}|+\cdots+|a_{i_t}|}$$

from the additional block Y into T. In this way, we obtain a $2n$-inset T.

Now, we have to prove that the correspondence is bijective.

Let T be an arbitrary $2n$-inset of X. If there are no elements of Y in T, then T is obtained by the trivial solutions of (17). Assume that T contains the subset $\{y_{i_1}, y_{i_2}, \ldots, y_{i_s}\}$, $(1 \leq i_1 < i_2 < \cdots < i_s \leq m)$ of Y. We also have $s \leq n$, since a $2n$-inset of X has at most n elements from the additional block Y.

Form the solution $(b_1, b_2, \ldots, b_n)$ of (17) in the following way: Since there are $s - n$ main blocks X_t from which both elements are in T, we define $b_t = 0$. Let $X_{u_1}, X_{u_2}, \ldots, X_{u_s}$ be the remaining main blocks. We define $|b_{u_1}| = i_1$, and the sign of b_{u_1} is $+$, if the first element of the main block X_u is in T, and the sign $-$ otherwise. Next, we define $|b_{u_t}| = i_{u_t} - i_{u_{t-1}}$, $(t = 2, \ldots, s)$, choosing the sign of b_{u_t} in the same way as for b_{u_1}. It follows that $|b_{u_1}| + \cdots + |b_{u_s}| = i_s \leq m$. Hence, $(b_1, b_2, \ldots, b_n)$ is the solution of (17), which in the preceding correspondence produces the inset T. This means that the correspondence is surjective.

It is clear that no two different solutions may produce the same inset, which means that our correspondence is injective. This proves (17).

Using (9), we have

$$\binom{m-1, n}{n-1, 2} = \binom{m, n}{n, 2} - \binom{m-1, n}{n, 2},$$

which proves (18). $\square$

The referee offers the following more concise proof of Proposition 31, Part 2, which is based on an interpretation of the equation (11).

Proof. First, by (11) we have that

$$\binom{m-1, n}{n-1, 2} = \sum_{i=0}^{m} 2^{i+1} \binom{m-1}{i} \binom{n}{n-(i+1)} = \sum_{i \geq 1} 2^{i} \binom{m-1}{i-1} \binom{n}{n-i}.$$

Interpretation of the right-hand side gives that $\binom{m-1}{i-1}$ is the number $c(m, i)$ of integer of m with i positive parts, as is well-known. The term $\binom{n}{n-i}$ distributes $n - i$ zero parts among a total of n parts and the term 2^i assigns signs (± 1) to the i positive parts. In sum, $2^i \binom{m-1}{i-1} \binom{n}{n-i}$ counts the numbers of integral solutions $|x_1| + \cdots + |x_n| = m$ with exactly i non-zero parts and $n - i$ zero parts. Summing over all $i \geq 1$ yields Proposition 31, part 2. $\square$

Remark 32. Note that the number of solutions of Eq. (18) is the number of the coordination sequence, and the number of solutions of (17) is the number of the crystal ball sequence for the cubic lattice $\mathbb{Z}^n$. Also, the number of solutions of (17) equals the Delannoy number $D(m, n)$.

Remark 33. The formulae (17) and (18) concern the following sequences in OEIS [8]: A001105, A035597, A035598, A035599, A035600, A035601, A035602, A035603, A035604, A035605, A035605.

Comparing the results of the preceding proposition, and the formulae from Conway and Sloane [2, (16) and (17)], we obtain the following binomial identities:

$$\sum_{i=0}^{m} 2^i \binom{m}{i}\binom{n}{i} = \sum_{i=0}^{n} \binom{n}{i}\binom{m-i+n}{n}, \tag{19}$$

$$\sum_{i=0}^{m-1} 2^{i+1} \binom{m-1}{i}\binom{n}{i+1} = \sum_{i=0}^{n} \binom{n}{i}\binom{m-i+n-1}{n-1}. \tag{20}$$

4 Some configurations counted by $\binom{m,n}{k,Q}$.

In this section, we describe a number of configurations counted by our function. The first result concerns the complete bipartite graphs.

Proposition 34. *Let n, q be positive integers. Then the number $M(n, q-1, q)$ equals the number of spanning subgraphs of the complete bipartite graph $K(q, n)$, having $n+q-1$ edges with no isolated vertices.*

Proof. Let $A = (a_{ij})_{n\times n}$ be $(0,1)$-matrix which has $n+q-1$ ones, and has no zero rows and zero columns. This matrix corresponds to a spanning subgraph $S = (V(S), E(S))$ of the complete bipartite graph $K(n, q) = (V = \{v_1, v_2, \ldots, v_n\} \cup \{u_1, u_2, \ldots, u_q\}, E = \{v_i u_j : 1 \le i \le n,\ 1 \le j \le q\})$, having $n+q-1$ edges, in the following way:

1. $a(i, j) = 1,\ (1 \le i \le n,\ 1 \le j \le q)$ if and only if $v_i u_j \in E(S)$.
2. $a(i, j) = 0,\ (1 \le i \le n,\ 1 \le j \le q)$, if and only if $v_i u_j \notin E(S)$.

Note that the matrix A has $n+q-1$ ones if and only if $|E(S)| = n+q-1$, and that the matrix A has no zero rows and zero columns if and only if the subgraph S has no isolated vertices. $\square$

Remark 35. The function $M(n, q-1, q)$ produces the following sequences in [8]: A001787, A084485, A084486.

Proposition 36. *If n is a positive integer, then the number $\binom{n,\,1}{n-1,n}$ equals the number of square submatrices of an n by n matrix A030662.*

Proof. Let M be a square matrix of order n. If $X_1 = \{x_1, \ldots, x_n\}$ is the main block of X, and $Y = \{y_1, \ldots, y_n\}$ is the additional block, then each n-inset of X has the form

$$\{x_{i_1}, x_{i_2}, \ldots, x_{i_k}, y_{j_{k+1}}, \ldots, y_{j_n}\},\ (k \ge 1).$$

Every such inset corresponds to the square submatrix of M, of which the indices of rows are $i_1, i_2, \ldots, i_k$, and indices of columns belong to the set $\{1, 2, \ldots, n\} \setminus \{j_{k+1}, \ldots, j_n\}$. $\square$

Proposition 37. *If n is a positive integer, and if i is a nonnegative integer, then the number $\binom{n,\,1}{n+i-1,\,n}$ equals the number of lattice paths from $(0,0)$ to (n,n), with steps $E=(1,0)$ and $N=(0,1)$, which either touch or cross the line $x-y=i$.*

Proof. We may write arbitrary lattice path from $(0,0)$ to (n,n) in the form $P=P_1P_2\dots P_{2n}$, where each P_i is either E or N. Assume that s is the least index such that the end of P_s touches the line $x-y=i$, and let $(r,r-i),\ (i\le r\le n)$ be the touching point. It follows that $s=2r-i$.

Consider the lattice path $Q=Q_1Q_2\dots Q_sP_{s+1}\dots P_{2n}$, where P_t and Q_t are symmetric with respect to the line $y=x-i$. This path connects $(-i,i)$ and (n,n). Since every lattice path from $(i,-i)$ to (n,n) must cross the line $y=x-i$, the converse also holds. We thus have a bijection between the number of considered lattice paths and the number of all lattice paths from $(i,-i)$ to (n,n). The last lattice paths are of the form $L_1L_2\dots L_{2n}$, where $n-i$ L's equal E, and $n+i$ equal N. Hence, its number is $\binom{2n}{n+i}$, and the proof follows from (7).

Again, we add a short bijective proof. Let X be a set which have one main block $X_1=\{x_1,x_2,\dots,x_n\}$, and the additional block $Y=\{y_1,y_2,\dots,y_n\}$. We need to define a bijection between all lattice paths from $(i,-i)$ to (n,n) and $(n+i)$-insets of X. Let $\{x_{i_1},\dots,x_{i_u},y_{j_1},\dots,y_{j_v}\},\ (u+v=n+i)$ be an $(n+i)$-inset of X. Define the path $L_1L_2\dots L_{2n}$ in the following way.

$$L_p=\begin{cases}N, & \text{if } p\in\{i_1,\dots,i_u,j_1,\dots,j_v\};\\ E, & \text{otherwise.}\end{cases}$$

It is clear that this correspondence is bijective. □

Note that, in this case, all $(n+i)$-subsets of X are $(n+i)$-insets.

Remark 38. This proposition concerns the following sequences in OEIS [8]: A001791, A002694, A004310, A004311 A004312, A004313, A004314, A004315, A004316, A004317, A004318.

We now give a combinatorial interpretation of the formula (14).

Consider the square S, the vertices of which are $(1,1),(1,m+q),(m+q,1)$, and $(m+q,m+q)$. Let X be the set of squares, whose vertices are $(u,v),(u+w,v),(u,v+w),(u+w,v+w),\ (1\le w\le q)$, and which are contained in S.

Proposition 39. *If $Q=\{2,q\}$, and if m is a nonnegative integer, then the number $\binom{m,2}{2,Q}$ equals $|X|$.*

Proof. Let $X_1=\{x_{11},x_{12}\}$ and $X_2=\{x_{21},x_{22},\dots,x_{2q}\}$ be the main blocks of X, and let $Y=\{y_1,\dots,y_m\}$ be the additional block.

We need to define a bijection between 4-insets of X and the set S. If U is a 4-inset of X, then it must contain an element from X_2. The length of the side of the corresponding square will be the minimal i such that $x_{2i}\in U$. In the next correspondence, it will be denoted by d. We now define a correspondence between 4-insets and pairs (i,j), which represent the upper right corner of the square.

$$\begin{array}{rc}
1. & \{x_{11}, x_{2d}, x_{2i}, x_{2j}\} \leftrightarrow (i,j),\\
2. & \{x_{12}, x_{2d}, x_{2i}, x_{2j}\} \leftrightarrow (j,i),\\
3. & \{x_{11}, x_{12}, x_{2d}, x_{2i}\} \leftrightarrow (i,i),\\
4. & \{x_{11}, x_{12}, x_{2d}, y_i\} \leftrightarrow (q+i,q+i),\\
5. & \{x_{11}, x_{2d}, x_{2i}, y_j\} \leftrightarrow (i,q+j),\\
6. & \{x_{12}, x_{2d}, x_{2i}, y_j\} \leftrightarrow (j+q,i),\\
7. & \{x_{11}, x_{2d}, y_i, y_j\} \leftrightarrow (q+i,q+j),\\
8. & \{x_{12}, x_{2d}, y_i, y_j\} \leftrightarrow (q+j,q+i).
\end{array}$$

It is easy to see that the correspondence is bijective. ☐

Proposition 40. *Let $p_1 < p_2 < p_3$ be prime numbers. If we denote $s = p_1p_2p_3^2$, then $\binom{n,2}{1,n}$ equals the number of divisors of s^{n-1} A015237.*

Proof. Let $X_i = \{x_{i1}, x_{i2}, \ldots, x_{in}\}$, $(i = 1, 2)$ be the main blocks of X, and let $Y = \{y_1, y_2, \ldots, y_n\}$ be the additional block. It is enough to define a bijection between 3-insets of X, and 3-tuples (i, j, k), such that $0 \le i, j \le n-1,\ 0 \le k \le 2n-2$. A bijection goes as follows:

$$\begin{array}{rc}
1. & \{x_{1i}, x_{1j}, x_{2k}\} \leftrightarrow (i-1, j-1, k-1),\\
2. & \{x_{1k}, x_{2j}, x_{2i}\} \leftrightarrow (i-1, j-1, k-1),\\
3. & \{x_{1i}, x_{2j}, y_k\} \leftrightarrow (i-1, j-1, n+k-2),\ (1 < k),\\
4. & \{x_{1i}, x_{2j}, y_1\} \leftrightarrow (i-1, i-1, j-1).
\end{array}$$

☐

Proposition 41. *Let n be a positive integer. Then the number $\binom{1,n}{1,2}$ equals the number of parts in all compositions of $n+1$ A001792.*

Proof. Let $X_i = \{x_{i1}, x_{i2}\}$, $(i = 1, 2, \ldots, n)$ be the main blocks of X, and let $Y = \{y\}$ be the additional block. For a fixed k, $(k = 0, 1, \ldots, n)$, we prove that $(n+1)$-insets of X, in which exactly k elements of the form x_{i1} are not chosen, count the number of parts in all compositions of $n+1$ into $n-k+1$ parts. Take $1 \le i_1 < i_2 < \ldots < i_k \le n$, and consider $(n+1)$-inset U of X not containing elements $x_{i_1,1}, x_{i_2,1}, \ldots, x_{i_k,1}$, but containing the remaining $n-k$ elements of the form x_{i1}. The remaining k of $k+1$ elements of U must be $x_{i_1,2}, x_{i_2,2}, \ldots, x_{i_k,2}$. For the remaining element, therefore, either y or one of x_{j2}, $(j \ne i_t,\ (t = 1, \ldots, k))$ must be chosen. For this, we have $(n-k+1)$ possibilities. Since $i_1, \ldots, i_k$ may be chosen in $\binom{n}{k}$ ways, we have $(n-k+1)\binom{n}{n-k}$ insets containing $(n+1)$ elements, but not containing exactly k elements of the form x_{i1}. On the other hand, the number of the compositions of $n+1$ with $n-k+1$ parts equals $\binom{n}{n-k}$. Hence, $(n-k+1)\binom{n}{n-k}$ equals the number of parts in all compositions of $n+1$ with $(n-k+1)$ parts. Since k ranges from 0 to n, the assertion follows. ☐

The following very short (non-bijective) proof is suggested by the referee.

Proof. The number of parts in all composition of $n+1$ is simply

$$\sum_{k\geq} kc(n+1,k) = \sum_{k\geq 1} k\binom{n}{k-1} = \sum_{k\geq 0}(k+1)\binom{n}{k} = 2^{n-1}(n+2).$$

$\square$

We now present two configurations counted by the number $\binom{1,n-1}{1,\,3}$, $(n>1)$ A027471.

Proposition 42. *Given n points on a straight line, the number $\binom{1,n-1}{1,\,3}$ equals the number of colorings of $n-1$ points with three colors.*

Proof. Let $X_i = \{x_{i1}, x_{i2}, x_{i3}\}$, $(i = 1, \ldots, n-1)$ be the main blocks of X, and $Y = \{y\}$ be the additional block. We define a correspondence between n-insets of X and the above-defined colorings in the following way:

1. If U is an n-inset such that $y \in U$, then U contains exactly one element from each of the main blocks. If $x_{ij} \in U$, then we color the point i by the color j. In this way, the point n remains uncolored.

2. If $y \notin U$, then there is exactly one main block X_k, two elements of which are in U. In this case, the kth point remains uncolored. If $x_{km} \notin U$, then the point n is colored by the color m. If $x_{ij} \in U$, $(i \neq k)$, then we color the point i by the color j.

   The correspondence is clearly bijective.

$\square$

Proposition 43. *Let $n>1$ is an integer. Then*

$$\binom{1,n-1}{1,\ 3} = \sum_{X\subseteq Y\subseteq[n]} (|Y|-|X|). \tag{21}$$

Proof. Take k such that $1 \leq k \leq n$. We count all pairs $X \subseteq Y \subseteq [n]$, such that $|Y|-|X| = k$. If $X_k = \{x_1, x_2, \ldots, x_k\}$ is a given k-subset of $[n]$, and if Z_k is an arbitrary subset of $[n] \setminus \{x_1, x_2, \ldots, x_k\}$, ($\emptyset$ included), then $|X_k \cup Z_k| - |Z_k| = k$. Hence, for a fixed X_k there are 2^{n-k} mutually different Z_k's. On the other hand, there are $\binom{n}{k}$ mutually different X_k's. We conclude that there are $\binom{n}{k}2^{n-k}$ pairs (U,V) of subsets, where $U-V$ has k elements. The sum on the right side of (21) thus equals $\sum_{k=1}^{n} k\binom{n}{k}2^{n-k}$. It is easy to see that

$$\sum_{k=1}^{n} k\binom{n}{k}2^{n-k} = n3^{n-1}.$$

On the other hand, if $X_i = \{x_{i1}, x_{i2}, x_{i3}\}$, $(i = 1, 2, \ldots, n-1)$ are the main blocks, and $Y = \{y\}$ the additional block of X, then there are obviously 3^{n-1} n-insets of X containing y. The n-insets of X, not containing y, must contain two elements from one main block, and one element from the remaining main blocks. For this, we have $3(n-1)3^{n-2}$ possibilities. Hence, there are $3^{n-1} + 3(n-1)3^{n-2} = n3^{n-1}$ n-insets of X. $\square$

Next, we prove that our function, in one particular case, counts the number of the so-called weak compositions. We let $c(n)$ denote the number of the compositions of n. It is well-known that $c(n) = 2^{n-1},\ (n > 0)$. Additionally, we put $c(0) = 1$. Compositions in which some parts may be zero are called weak compositions. We let $cw(r, s)$ denote the number of the weak compositions of r in which s parts equal zero.

Proposition 44. *Let r be a positive integer, and let s be a nonnegative integer. Then*

$$cw(r,s) = \sum_{j_1+j_2+\cdots+j_{s+1}=r} c(j_1)c(j_2)\cdots c(j_{s+1}), \tag{22}$$

where the sum is taken over $j_t \geq 0,\ (t = 1, 2, \ldots, s + 1)$.

Proof. We use induction with respect to s. For $s = 0$, the assertion is obvious. Assume that the assertion is true for $s - 1$. Using the induction hypothesis, we may write equation (22) in the following form:

$$cw(r,s) = \sum_{j=0}^{r} c(j)cw(r-j, s-1). \tag{23}$$

Let $(i_1, i_2, \ldots,)$ be a weak composition of r, in which exactly s parts equal 0. Assume that i_p is the first part equal to zero. Then $(i_1, \ldots, i_{p-1})$ is a composition of $i_1 + \cdots + i_{p-1} = j$ without zeroes. Note that j can be zero. Furthermore, $(i_{p+1}, \cdots)$ is a weak composition of $r - j$ with $s - 1$ zeroes. For a fixed j, there are $c(j)cw(r - j, s - 1)$ such compositions. Changing j, we conclude that the right side of (23) counts all weak compositions. □

Proposition 45. *Let r be a positive integer, and let s be a nonnegative integer. Then*

$$cw(r,s) = \binom{s+1, r-1}{s,\ 2}. \tag{24}$$

Proof. Collecting terms in (22), in which the indices j_t equal zero, we obtain

$$cw(r,s) = \sum_{i=0}^{s} \binom{s+1}{i} \sum_{j_1+j_2+\cdots+j_{s-i+1}=r} 2^{j_1-1}2^{j_2-1}\cdots 2^{j_{s-i+1}-1},$$

where the sum is taken over $j_t \geq 1$. Hence,

$$cw(r,s) = 2^{r-s-1} \sum_{i=0}^{s} 2^i \binom{s+1}{i} \sum_{j_1+j_2+\cdots+j_{s-i+1}=r} 1.$$

Since the last sum is taken over all compositions of r with $s - i + 1$ parts, we finally have

$$cw(r,s) = 2^{r-s-1} \sum_{i=0}^{s+1} 2^i \binom{s+1}{i} \binom{r-1}{s-i},$$

and the proof follows from (11). □

Remark 46. The referee found that this result connects our function with the notion of the weighted composition of integers, which are considered in papers [1], [3], [4], and [7].

Remark 47. As a consequence of (11), and Propositions 44 and 45, the referee found the following identity:

$$\sum_{j=0}^{r-1}\binom{r-1}{j}\binom{s+j+1}{s}=\sum_{j=0}^{r-1}2^j\binom{r-1}{j}\binom{s+1}{r-j},$$

which is similar to identities (19) and(20).

Remark 48. The formula (24) produces the following sequences in OEIS [8] A000297, A058396, A062109, A169792, A169793, A169794, A169795, A169796, A169797.

We conclude the paper with three chessboard combinatorial problems.

Proposition 49. *Let n be a positive integer. Then the number*

$$\binom{n-1,\ 2}{1,n}$$

equals the number of possible rook moves on an $n\times n$ chessboard A035006.

Proof. Let $X_i=\{x_{i1},\dots,x_{i,n}\},\ (i=1,2)$ be the main blocks of X, and $Y=\{y_1,\dots,y_{n-1}\}$ be the additional block. We need a bijection of 3-insets of the set X, and all the possible rook moves on an $n\times n$ chessboard. The correspondence goes as follows:

$$\begin{array}{rr}
1. & \{x_{1i},x_{1j},x_{2k}\}\leftrightarrow[(i,k)\to(j,k)],\\
2. & \{x_{1k},x_{2i},x_{2j}\}\leftrightarrow[(j,k)\to(i,k)],\\
3. & \{x_{1i},x_{2j},y_k\}\leftrightarrow[(i,k)\to(i,j)],\ (j\neq k),\\
4. & \{x_{1i},x_{2j},y_j\}\leftrightarrow[(i,n)\to(i,j)],\ (j=k).
\end{array}$$

According to (7), the number of possible moves equals $2(n-1)n^2$. $\square$

Proposition 50. *If $n\geq 2$, then the number*

$$\binom{1,n}{n-2,2}$$

equals the total number of possible bishop moves on an $n\times n$ chessboard A002492.

Proof. We give two proofs.

1. This proof is bijective.

   It is enough to count the number of moves from the field (i,j) to the field $(i+k,j+k)$, for a positive k, such that $i+k\leq n,\ j+k\leq n$. If N is the number of such moves, then $4N$ is the number of all possible moves.

Let set X consists of n main blocks $X_i = \{x_{i,1}, x_{i,2}\},\ (i = 1, 2, \ldots, n)$, and the additional block $Y = \{y\}$. We define a bijective correspondence between the set of moves described above and one fourth of all $(2n-2)$-insets of X. In fact, we define a bijection between the moves and the complements of $(2n-2)$-insets of X. The complements are 3-sets $\{a, b, c\}$ of X, such that no two of its elements can be in the same main block. The correspondence goes as follows:

(a) $\{x_{i,1}, x_{j,1}, x_{k,1}\} \leftrightarrow [(i, j) \to (i + k - j, k)],\ (1 \le i < j < k)$. In this correspondence we have $\binom{n}{3}$ elements.

(b) $\{x_{i,2}, x_{j,2}, x_{k,2}\} \leftrightarrow [(j, i) \to (k, i + k - j)],\ (i < j < k)$. In the correspondence we also have $\binom{n}{3}$ elements.

(c) $\{x_{i,1}, x_{j,2}, y\} \leftrightarrow [(i, i) \to (j, j)],\ (i < j)$. Now, we have $\binom{n}{2}$ moves.

It is clear that all moves are counted. For this we need

$$2\binom{n}{3} + \binom{n}{2} = \frac{n(2n-1)(n-1)}{6}$$

insets. On the other hand, according to (11), we have

$$\binom{1,\ n}{n-2, 2} = \frac{2n(2n-1)(n-1)}{3},$$

which proves the assertion.

2. We let T_n be an $n \times n$ chessboard, and let a_n denote the total number of possible bishop moves. We may consider that T_{n+1} is obtained by adding to T_n one row at the top, and one column at the right. We calculate $a_{n+1} - a_n$, which is the number of moves on T_{n+1} that are not possible on T_n.

   (a) Firstly, if the bishop is on the main diagonal of T_n or below, then only one additional move is produced. We have thus obtained $\frac{n(n+1)}{2}$ new moves.

   (b) For the bishop on T_n, and above the main diagonal, there are 3 additional moves, or $3\frac{n(n-1)}{2}$ additional moves in total.

   (c) For each bishop on T_{n+1} which is not on T_n, we have n additional moves. Hence, we have $n(2n+1)$ additional moves in total.

   We thus have $\frac{n(n+1)}{2} + 3\frac{n(n-1)}{2} + n(2n+1) = 4n^2$ additional moves in total. Hence, the following recurrence is obtained:

$$a_{n+1} - a_n = 4n^2.$$

   It is easy to see that $\binom{1,\ n}{n-2, 2}$ satisfies the recurrence.

□

Since the queen can move both as a rook and as a bishop, we have

Proposition 51. *Let $n \geq 2$ be an integer. The number*

$$\binom{1, n}{n-2, 2} + \binom{n-1,\ 2}{1,\ n}, \ (n \geq 2)$$

equals the total number of possible queen moves on an $n \times n$ chessboard. This number is $\frac{2n(5n-1)(n-1)}{3}$ A035005.

Finally, we give a number of additional configurations, counted by our function, and described in sequences in OEIS [8].

Function	Numbers of sequences
$\binom{0,n}{k,2}$	A000918, A001787, A001788, A001789, A003472, A054849, A002409, A054851, A140325, A140354, A172242
$\binom{1,n}{1,Q}$	A059270, A094952, A069072, A007531, A000466, A019583, A076301
$\binom{m,1}{k,q}$	A015237, A160378, A027620, A028347, A028560, A034428, A000567, A045944, A123865, A034828
$\binom{m,2}{k,Q}$	A080838, A015237, A091361, A017593, A063488 A039623, A116882, A081266, A202804, A194715
$\binom{m,n}{k,2}$	A002002, A049600, A142978, A099776, A014820, A069039, A099195, A006325, A061927, A191596 A001792, A045623, A045891, A034007, A111297 A159694, A001788, A049611, A058396, A158920

References

[1] S. Eger, Restricted weighted integer compositions and extended binomial coefficients, *J. Integer Seq.* **16** (2013), Article 13.1.3.

[2] J. H. Conway and N. J. A. Sloane, Low-dimensional lattices VII: coordination sequences, *Proc. Royal Soc. London*, **A453** (1997), 2369–2389.

[3] N. E. Fahssi, Polynomial triangles revisited, preprint, http://arxiv.org/abs/1202.0228.

[4] J. H. Guo, Some n-color compositions, *J. Integer Seq.* **15** (2012), Article 12.1.2.

[5] M. Janjić, An enumerative function, preprint, `http://arxiv.org/abs/0801.1976`.

[6] M. Janjić, On a class of polynomials with integer coefficients, *J. Integer Seq.* **11** (2008), Article 08.5.2.

[7] S. Shapcott, C-color compositions and palindromes, *Fib. Quart.* 50 (2012), 297–303.

[8] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, `http://oeis.org`.

[9] R. A. Sulanke, Objects counted by the central Delannoy numbers, *J. Integer Seq.* **6** (2003), Article 03.1.5.

[10] C. de J. Pita Ruiz Velasco, Convolution and Sulanke numbers, *J. Integer Seq.* **13** (2010), Article 10.1.8.

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