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Asymptotic Series for Hofstadter's Figure-Figure Sequences

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Abstract

We compute asymptotic series for Hofstadter's figure-figure sequences.

1 Introduction

We consider disjoint partitions of the set of strictly positive integers into two subsets such that one set, B, consists of the differences of consecutive elements of the other set, A, and a given difference appears at most once. There are many such partitions. We call a the (strictly increasing) sequence enumerating A, and b the (injective) sequence of its first differences, both with offset 1. Hofstadter's figure-figure sequences are the sequences a and b corresponding to the partition with the set A lexicographically minimal. This is equivalent to b being increasing. The sequences read

$$a_n = 1, 3, 7, 12, 18, 26, 35, 45, 56, 69, \dots \qquad \text{(OEIS A005228)},$$
$$b_n = 2, 4, 5, 6, 8, 9, 10, 11, 13, 14, \dots \qquad \text{(OEIS A030124)}.$$

These sequences were introduced by Hofstadter in [2, p. 73]. They appear as an example of complementary sequences in [3]. Their asymptotic behavior does not seem to be given

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anywhere in the literature except for the asymptotic equivalents mentioned by Hasler and Wilson in the related OEIS entries [1]. In this article, we compute asymptotic series for these sequences.

We have by definition $b_n = a_{n+1} - a_n$, so $a_n = 1 + \sum_{k=1}^{n-1} b_k$. Since the sequence a is strictly increasing, given any $n \geq 1$, there is a unique $k \geq 1$ such that $a_k - k < n \leq a_{k+1} - (k+1)$. This defines a sequence u by letting u_n be this k. Therefore,

$$a(u_n) - u_n < n \leq a(u_n + 1) - (u_n + 1). \tag{1}$$

The sequence u is non-decreasing (actually, $u_{n+1} - u_n \in \{0, 1\}$) and $u_1 = 1$. It reads

$$u_n = 1, 2, 2, 2, 3, 3, 3, 3, 4, 4, \ldots \qquad \text{(OEIS A225687)}.$$

The partition condition implies

$$b_n = n + u_n.$$

As a consequence,

$$a_n = 1 + \frac{(n-1)n}{2} + \sum_{k=1}^{n-1} u_k. \tag{2}$$

2 Bounds and asymptotic equivalents

Since $u_n \geq 1$, we have $a_n \geq \frac{1}{2}n(n+1)$. Therefore, the left inequality of (1) implies $\frac{1}{2}u_n(u_n + 1) - u_n \leq n - 1$, or $u_n^2 - u_n - 2(n-1) \leq 0$, so $u_n \leq \frac{1}{2} + \sqrt{\frac{1}{4} + 2(n-1)}$, and finally

$$1 \leq u_n < \sqrt{2n} + \frac{1}{2}.$$

This implies $n + 1 \leq b_n < n + \sqrt{2n} + \frac{1}{2}$, so

$$b_n \sim n.$$

The upper bound on u implies in turn $a_n < 1 + \frac{1}{2}(n-1)n + \sum_{k=1}^{n-1}(\sqrt{2k} + \frac{1}{2})$. Since the function $\sqrt{x}$ is strictly increasing, we have $\sum_{k=1}^{n-1}\sqrt{k} < \int_1^n \sqrt{x}\,dx = \frac{2}{3}(n^{3/2} - 1)$. Therefore

$$\frac{n^2}{2} + \frac{n}{2} \leq a_n < \frac{n^2}{2} + \frac{2^{3/2}}{3}n^{3/2} - \frac{1}{3}$$

and in particular

$$a_n \sim \frac{n^2}{2}.$$

The relation $a_n < \frac{n^2}{2} + \frac{2^3}{3}\left(\frac{n}{2}\right)^{3/2} - \frac{1}{3}$ and the right inequality of (1) imply $n < \frac{(u_n+1)^2}{2} + \frac{2^{3/2}}{3}(u_n + 1)^{3/2} - u_n - \frac{4}{3}$, which implies $u_n \to +\infty$. Therefore $2n \leq u_n^2 + O(u_n^{3/2})$, but we saw that $u_n = O(\sqrt{n})$, so $O({u_n}^{3/2}) \subseteq O(n^{3/4}) \subseteq o(n)$, so $u_n^2 \geq 2n + o(n)$, so $u_n \geq \sqrt{2n} + o(\sqrt{n})$. Combining this with the above upper bound, we obtain

$$u_n \sim \sqrt{2n}$$

and in particular $O(u_n) = O(\sqrt{n})$.

3 Asymptotic series

Since $a_n \sim \frac{n^2}{2}$, we have $a_{n+1} - a_n = O(n)$. Now (1) gives $a(u_n) = n + O(u_n)$. On the other hand, (2) gives $a_n = \frac{n^2}{2} + \sum_{k=1}^{n-1} u_k + O(n)$, therefore $\frac{u_n^2}{2} + \sum_{k=1}^{u_n-1} u_k = n + O(u_n)$. Since $u_n = O(\sqrt{n})$, we can increment the upper limit of the summation index by 1, and since $O(u_n) = O(\sqrt{n})$, we obtain the main relation

$$\frac{u_n^2}{2} + \sum_{k=1}^{u_n} u_k = n + O(\sqrt{n}).$$

We are now ready to prove by induction that for all $K \geq 1$, we have the asymptotic expansion

$$u_n = \sum_{k=1}^{K} (-1)^{k+1} \frac{2^{1+(k-1)k/2}}{\prod_{j=1}^{k-1}(2^j+1)} \left(\frac{n}{2}\right)^{1/2^k} + o\left(n^{1/2^K}\right). \tag{3}$$

Indeed, the case $K = 1$ reduces to $u_n \sim \sqrt{2n}$, which we already proved. We also prove the case $K = 2$ separately since it is slightly different from the general case. We write $u_n = \sqrt{2n} + v_n$ with $v_n = o(\sqrt{n})$. We have

$$\frac{u_n^2}{2} - n = \sqrt{2n}\, v_n + \frac{v_n^2}{2}.$$

We do not know *a priori* that $v_n^2 = O(\sqrt{n})$, and that is why we have to prove this case separately. We also have

$$\sum_{k=1}^{u_n} u_k = \sqrt{2} \sum_{k=1}^{u_n} \sqrt{k} + \sum_{k=1}^{u_n} v_k = \frac{2^{3/2}}{3} {u_n}^{3/2} + o\big(O(u_n)^{3/2}\big) + \sum_{k=1}^{u_n} v_k.$$

We have $\sum_{k=1}^{u_n} v_k = o\big(O(\sqrt{n})^{3/2}\big) \subseteq o(n^{3/4})$ and $o\big(O(u_n)^{3/2}\big) \subseteq o(n^{3/4})$. We also have ${u_n}^{3/2} \sim (\sqrt{2n})^{3/2} = (2n)^{3/4}$. Therefore,

$$\frac{u_n^2}{2} + \sum_{k=1}^{u_n} u_k - n = \sqrt{2n}\, v_n + \frac{v_n^2}{2} + \frac{2^{9/4}}{3} n^{3/4} + o(n^{3/4}).$$

This has to be $O(\sqrt{n})$ by the main relation. Dividing the right-hand side by $\sqrt{2n}$, we obtain

$$v_n + \frac{{v_n}^2}{2\sqrt{2n}} + \frac{2^{7/4}}{3} n^{1/4} = o(n^{1/4}).$$

Since $v_n = o(\sqrt{n})$, we have $\frac{{v_n}^2}{2\sqrt{2n}} = o(v_n)$, so

$$v_n + \frac{2^{7/4}}{3} n^{1/4} = o(n^{1/4}) + o(v_n),$$

so $v_n \sim -\frac{2^2}{3}\left(\frac{n}{2}\right)^{1/4}$, as desired.

Now, suppose that the expansion holds for some $K \geq 2$. We prove it for $K+1$. It will be convenient to denote the coefficients of the expansion by

$$\alpha_k = (-1)^{k+1}\frac{2^{1+(k-1)k/2}}{\prod_{j=1}^{k-1}(2^j+1)},$$

so $\alpha_1 = 2$. We write $v_n = o\big(n^{1/2^K}\big)$ for the remainder in (3). Then (3) gives

$$\begin{aligned} u_n^2 &= \left(\sqrt{2n} + \sum_{k=2}^{K} \alpha_k \left(\frac{n}{2}\right)^{1/2^k} + v_n\right)^2 = 2n\left(1 + \frac{1}{\sqrt{2n}}\sum_{k=2}^{K}\alpha_k\left(\frac{n}{2}\right)^{1/2^k} + \frac{v_n}{\sqrt{2n}}\right)^2 \\ &= 2n\left(1 + \frac{2}{\sqrt{2n}}\sum_{k=2}^{K}\alpha_k\left(\frac{n}{2}\right)^{1/2^k} + 2\frac{v_n}{\sqrt{2n}} + O\left(n^{1/4+1/4-1}\right) + O\left(\frac{v_n^2}{n}\right) + O\left(n^{1/4-1}v_n\right)\right). \end{aligned}$$

Since $K \geq 2$, we have $v_n = o\left(n^{1/2^K}\right) \subseteq o\left(n^{1/4}\right)$. Therefore

$$\frac{u_n^2}{2} - n = 2\sum_{k=2}^{K}\alpha_k\left(\frac{n}{2}\right)^{1/2+1/2^k} + \sqrt{2n}\,v_n + O(\sqrt{n}).$$

On the other hand,

$$\begin{aligned} \sum_{k=1}^{u_n} u_k &= \sum_{k=1}^{K} 2\frac{2^k}{2^k+1}\alpha_k\left(\frac{u_n}{2}\right)^{1+1/2^k} + o\left({u_n}^{1+1/2^K}\right) \\ &= \sum_{k=1}^{K}\frac{2^{k+1}}{2^k+1}\alpha_k\left(\frac{n}{2}\right)^{1/2+1/2^{k+1}} + o\left(n^{1/2+1/2^{K+1}}\right). \end{aligned}$$

Therefore

$$\begin{aligned} \frac{u_n^2}{2} + \sum_{k=1}^{u_n} u_k - n &= 2\sum_{k=2}^{K}\alpha_k\left(\frac{n}{2}\right)^{1/2+1/2^k} + \sum_{k=1}^{K}\alpha_k\frac{2^{k+1}}{2^k+1}\left(\frac{n}{2}\right)^{1/2+1/2^{k+1}} \\ &\quad + \sqrt{2n}\,v_n + o\left(n^{1/2+1/2^{K+1}}\right) \\ &= \alpha_K\frac{2^{K+1}}{2^K+1}\left(\frac{n}{2}\right)^{1/2+1/2^{K+1}} + 2\left(\frac{n}{2}\right)^{1/2} v_n + o\left(n^{1/2+1/2^{K+1}}\right) \end{aligned}$$

since the terms in the sums cancel out except for the last in the second sum. This expression has to be $O(\sqrt{n})$ by the main relation, so $v_n \sim -\frac{2^K}{2^K+1}\alpha_K\left(\frac{n}{2}\right)^{1/2^{K+1}}$, as desired.

From the expansion of u_n, we find that of $b_n = n + u_n$, and that of a_n by term-by-term integration. We obtain

$$b_n = n + \sum_{k=1}^{K}(-1)^{k+1}\frac{2^{1+(k-1)k/2}}{\prod_{j=1}^{k-1}(2^j+1)}\left(\frac{n}{2}\right)^{1/2^k} + o\left(n^{1/2^K}\right)$$

and

$$a_n = \frac{n^2}{2} + \sum_{k=1}^{K} (-1)^{k+1} \frac{2^{k(k+1)/2}}{\prod_{j=1}^{k}(2^j+1)} \left(\frac{n}{2}\right)^{1+1/2^k} + o\left(n^{1+1/2^K}\right).$$

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References

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