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Multi-Poly-Bernoulli Numbers and Finite Multiple Zeta Values

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Abstract

We define the multi-poly-Bernoulli numbers slightly differently from the similar numbers given in earlier papers by Bayad, Hamahata, and Masubuchi, and study their basic properties. Our motivation for the new definition is the connection to "finite multiple zeta values", which have been studied by Hoffman and Zhao, among others, and are recast in a recent work by Zagier and the second author. We write the finite multiple zeta value in terms of our new multi-poly-Bernoulli numbers.

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1 Introduction

For any index set $(k_1, \ldots, k_r)$ with $k_i \in \mathbb{Z}$ and $r \geq 1$, we define two kinds of *multi-poly-Bernoulli numbers* $B_n^{(k_1,\ldots,k_r)}$ and $C_n^{(k_1,\ldots,k_r)}$ by the following generating series:

$$\frac{\mathrm{Li}_{k_1,\ldots,k_r}(1-e^{-t})}{1-e^{-t}} = \sum_{n=0}^{\infty} B_n^{(k_1,\ldots,k_r)} \frac{t^n}{n!}, \tag{1}$$

$$\frac{\mathrm{Li}_{k_1,\ldots,k_r}(1-e^{-t})}{e^{t}-1} = \sum_{n=0}^{\infty} C_n^{(k_1,\ldots,k_r)} \frac{t^n}{n!}, \tag{2}$$

where $\mathrm{Li}_{k_1,\ldots,k_r}(z)$ is the multiple polylogarithm series given by

$$\mathrm{Li}_{k_1,\ldots,k_r}(z) = \sum_{m_1>\cdots>m_r>0} \frac{z^{m_1}}{m_1^{k_1}\cdots m_r^{k_r}}. \tag{3}$$

When $r=1$, the numbers $B_n^{(k)}$ and $C_n^{(k)}$ are "poly-Bernoulli numbers" defined and studied in our previous work [7, 1]. When $r=1$ and $k_1=1$, the multiple polylogarithm is just the usual logarithm $-\log(1-z)$, and thus both $B_n^{(1)}$ and $C_n^{(1)}$ are the classical Bernoulli numbers (with $B_1^{(1)} = 1/2$ and $C_1^{(1)} = -1/2$).

The *finite multiple zeta value* is the collection of truncated sums

$$\sum_{p>m_1>\cdots>m_r>0} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}}$$

modulo p for all primes p, considered in the quotient ring $\prod_p \mathbb{Z}/p\mathbb{Z}/\bigoplus_p \mathbb{Z}/p\mathbb{Z}$. (For more details on finite multiple zeta values, we refer the reader to papers by Hoffman [6], Zhao [10], and Kaneko-Zagier [8], among others.) In §4 we show the congruence

$$\sum_{p>m_1>\cdots>m_r>0} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}} \equiv -C_{p-2}^{(k_1-1,k_2,\ldots,k_r)} \pmod{p},$$

which reveals that each component of a finite multiple zeta value is a multi-poly-Bernoulli number.

2 Recurrences and explicit formulas for the multi-poly-Bernoulli numbers

In this section, we first prove a simple relation between $B_n^{(k_1,\ldots,k_r)}$ and $C_n^{(k_1,\ldots,k_r)}$ and proceed to give recursion relations and explicit formulas for the multi-poly-Bernoulli numbers. Since the two generating series (1) and (2) differ only by the factor e^t, two obvious relations between $B_n^{(k_1,\dots,k_r)}$ and $C_n^{(k_1,\dots,k_r)}$ are

$$B_n^{(k_1,\dots,k_r)} = \sum_{i=0}^{n} \binom{n}{i} C_i^{(k_1,\dots,k_r)} \quad \text{and} \quad C_n^{(k_1,\dots,k_r)} = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} B_i^{(k_1,\dots,k_r)}.$$

A simpler relation expressing $B_n^{(k_1,\dots,k_r)}$ in terms of $C_n^{(k_1,\dots,k_r)}$ is the following.

Proposition 1. *For any $r \geq 1, k_i \in \mathbb{Z}$ and $n \geq 1$, we have*

$$B_n^{(k_1,\dots,k_r)} = C_n^{(k_1,\dots,k_r)} + C_{n-1}^{(k_1-1,k_2,\cdots,k_r)}.$$

Proof. From the obvious identity

$$\frac{d}{dz}\operatorname{Li}_{k_1,\dots,k_r}(z) = \frac{1}{z}\operatorname{Li}_{k_1-1,k_2,\dots,k_r}(z)$$

(note that we allow the first index to be 0 or negative), we have

$$\frac{d}{dt}\operatorname{Li}_{k_1,\dots,k_r}(1-e^{-t}) = \frac{\operatorname{Li}_{k_1-1,k_2,\dots,k_r}(1-e^{-t})}{1-e^{-t}} \cdot e^{-t} = \frac{\operatorname{Li}_{k_1-1,k_2,\dots,k_r}(1-e^{-t})}{e^t-1},$$

and hence

$$\begin{aligned}
\sum_{n=1}^{\infty} C_{n-1}^{(k_1-1,k_2,\dots,k_r)} \frac{t^n}{n!} &= \int_0^t \sum_{n=0}^{\infty} C_n^{(k_1-1,k_2,\dots,k_r)} \frac{t^n}{n!}\, dt = \int_0^t \frac{\operatorname{Li}_{k_1-1,k_2,\dots,k_r}(1-e^{-t})}{e^t-1}\, dt \\
&= \operatorname{Li}_{k_1,\dots,k_r}(1-e^{-t}) = \operatorname{Li}_{k_1,\dots,k_r}(1-e^{-t}) \left(\frac{e^t}{e^t-1} - \frac{1}{e^t-1} \right) \\
&= \frac{\operatorname{Li}_{k_1,\dots,k_r}(1-e^{-t})}{1-e^{-t}} - \frac{\operatorname{Li}_{k_1,\dots,k_r}(1-e^{-t})}{e^t-1} \\
&= \sum_{n=0}^{\infty} B_n^{(k_1,\dots,k_r)} \frac{t^n}{n!} - \sum_{n=0}^{\infty} C_n^{(k_1,\dots,k_r)} \frac{t^n}{n!}.
\end{aligned}$$

This proves the proposition. $\square$

Proposition 2. *For any $k_1, \dots, k_r \in \mathbb{Z}$ and $n \geq 0$, we have*

$$B_n^{(k_1,\dots,k_r)} = \frac{1}{n+1} \left(B_n^{(k_1-1,k_2,\dots k_r)} - \sum_{m=1}^{n-1} \binom{n}{m-1} B_m^{(k_1,\dots,k_r)} \right)$$

and

$$C_n^{(k_1,\dots,k_r)} = \frac{(-1)^n}{n+1} \left(\sum_{m=0}^{n} (-1)^m \binom{n}{m} C_m^{(k_1-1,k_2,\dots,k_r)} - \sum_{m=1}^{n-1} (-1)^m \binom{n}{m-1} C_m^{(k_1,\dots,k_r)} \right),$$

where an empty sum is understood to be 0.

Proof. For $B_n^{(k_1,\dots,k_r)}$, we multiply the defining equation (1) by $1-e^{-t}$ and differentiate with respect to t to obtain

$$\frac{e^{-t}}{1-e^{-t}}\,\mathrm{Li}_{k_1-1,k_2,\dots,k_r}(1-e^{-t}) = e^{-t}\sum_{n=0}^{\infty} B_n^{(k_1,\dots,k_r)}\frac{t^n}{n!} + (1-e^{-t})\sum_{n=1}^{\infty} B_n^{(k_1,\dots,k_r)}\frac{t^{n-1}}{(n-1)!},$$

and from this we have

$$\sum_{n=0}^{\infty} B_n^{(k_1,\dots,k_r)}\frac{t^n}{n!} = \sum_{n=0}^{\infty} B_n^{(k_1-1,k_2,\dots,k_r)}\frac{t^n}{n!} - (e^t-1)\sum_{n=1}^{\infty} B_n^{(k_1,\dots,k_r)}\frac{t^{n-1}}{(n-1)!}.$$

Comparing the coefficients of $t^n/n!$ on both sides, we obtain the desired relation for $B_n^{(k_1,\dots,k_r)}$. The relation for $C_n^{(k_1,\dots,k_r)}$ is obtained similarly. $\square$

We make a remark on the following point: since we allow k_1 to be positive or negative, the recursions above need never 'stop'. However, by the identity

$$\mathrm{Li}_{0,k_2,\dots,k_r}(z) = \frac{z}{1-z}\,\mathrm{Li}_{k_2,\dots,k_r}(z),$$

the $B_n^{(0,k_2,\dots,k_r)}$ or $C_n^{(0,k_2,\dots,k_r)}$ can be written as simple linear combinations of $B_m^{(k_2,\dots,k_r)}$ or $C_m^{(k_2,\dots,k_r)}$ $(0\le m<n)$. The proposition therefore gives a way to express $B_n^{(k_1,\dots,k_r)}$ or $C_n^{(k_1,\dots,k_r)}$ in terms of numbers with smaller n and lower "depth" (length r of the index set).

Next, we write our multi-poly-Bernoulli numbers as finite sums involving Stirling numbers of the second kind. Recall that the Stirling numbers of the first kind $\begin{bmatrix} n \\ m \end{bmatrix}$ and the second kind $\begin{Bmatrix} n \\ m \end{Bmatrix}$ are the integers uniquely determined by the following recursions for all integers m,n (see Knuth [9]):

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1,\ \begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ m \end{bmatrix} = 0 \quad (n,m\neq 0), \qquad \begin{bmatrix} n+1 \\ m \end{bmatrix} = \begin{bmatrix} n \\ m-1 \end{bmatrix} + n\begin{bmatrix} n \\ m \end{bmatrix} \quad (\forall n,m\in\mathbb{Z}),$$

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = 1,\ \begin{Bmatrix} n \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ m \end{Bmatrix} = 0, \quad (n,m\neq 0), \qquad \begin{Bmatrix} n+1 \\ m \end{Bmatrix} = \begin{Bmatrix} n \\ m-1 \end{Bmatrix} + m\begin{Bmatrix} n \\ m \end{Bmatrix} \quad (\forall n,m\in\mathbb{Z}).$$

Theorem 3. *For any $r\ge 1$, $k_1,k_2,\cdots,k_r\in\mathbb{Z}$, and $n\ge 0$, we have*

$$B_n^{(k_1,\dots,k_r)} \ = \ (-1)^n \sum_{n+1\ge m_1>m_2>\cdots>m_r>0} \frac{(-1)^{m_1-1}(m_1-1)!\left\{ {n \atop m_1-1} \right\}}{{m_1}^{k_1}\cdots {m_r}^{k_r}}$$

and

$$C_n^{(k_1,\dots,k_r)} \ = \ (-1)^n \sum_{n+1\ge m_1>m_2>\cdots>m_r>0} \frac{(-1)^{m_1-1}(m_1-1)!\left\{ {n+1 \atop m_1} \right\}}{m_1^{k_1}\cdots m_r^{k_r}}.$$

Proof. Using the well-known generating series (cf. Graham et al. [3, §7.4])

$$\frac{(e^t-1)^m}{m!} = \sum_{n=m}^{\infty} \left\{ {n \atop m} \right\} \frac{t^n}{n!} \quad (m \geq 0), \tag{4}$$

we have

$$\begin{aligned}
\sum_{n=0}^{\infty} B_n^{(k_1,\dots,k_r)} \frac{t^n}{n!} &= \frac{\mathrm{Li}_{k_1,\cdots,k_r}(1-e^{-t})}{1-e^{-t}} \\
&= \sum_{m_1>m_2>\cdots>m_r>0} \frac{(1-e^{-t})^{m_1-1}}{{m_1}^{k_1}\cdots{m_r}^{k_r}} \\
&= \sum_{m_1>m_2>\cdots>m_r>0} \frac{(-1)^{m_1-1}(m_1-1)!}{{m_1}^{k_1}\cdots{m_r}^{k_r}} \sum_{n=m_1-1}^{\infty} \left\{ {n \atop m_1-1} \right\} \frac{(-t)^n}{n!} \\
&= \sum_{n=r-1}^{\infty} (-1)^n \sum_{n+1\geq m_1>m_2>\cdots>m_r>0} \frac{(-1)^{m_1-1}(m_1-1)!\left\{ {n \atop m_1-1} \right\}}{{m_1}^{k_1}\cdots{m_r}^{k_r}} \frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients of t^n, we obtain the formula for $B_n^{(k_1,\dots,k_r)}$.

To obtain the formula for $C_n^{(k_1,\dots,k_r)}$, we proceed similarly by using the equation

$$\frac{e^t(e^t-1)^{m-1}}{(m-1)!} = \sum_{n=m-1}^{\infty} \left\{ {n+1 \atop m} \right\} \frac{t^n}{n!} \quad (m \geq 1),$$

which follows from equation (4) by differentiation:

$$\begin{aligned}
\sum_{n=0}^{\infty} C_n^{(k_1,\dots,k_r)} \frac{t^n}{n!} &= \frac{\mathrm{Li}_{k_1,\dots,k_r}(1-e^{-t})}{e^t-1} \\
&= \sum_{m_1>m_2>\cdots>m_r>0} \frac{e^{-t}(1-e^{-t})^{m_1-1}}{{m_1}^{k_1}\cdots{m_r}^{k_r}} \\
&= \sum_{m_1>m_2>\cdots>m_r>0} \frac{(-1)^{m_1-1}(m_1-1)!}{{m_1}^{k_1}\cdots{m_r}^{k_r}} \sum_{n=m_1-1}^{\infty} \left\{ {n+1 \atop m_1} \right\} \frac{(-t)^n}{n!} \\
&= \sum_{n=r-1}^{\infty} (-1)^n \sum_{n+1\geq m_1>m_2>\cdots>m_r>0} \frac{(-1)^{m_1-1}(m_1-1)!\left\{ {n+1 \atop m_1} \right\}}{{m_1}^{k_1}\cdots{m_r}^{k_r}} \frac{t^n}{n!}.
\end{aligned}$$

□

We end this section by giving formulas for the special case $k_1 = \cdots = k_r = 1$.

Proposition 4. *For any $r \geq 1$ and $n \geq r-1$, we have*

$$B_n^{(\overbrace{1,1,\dots,1}^{r})} = \frac{1}{n+1}\binom{n+1}{r} B_{n-r+1}^{(1)}, \quad C_n^{(\overbrace{1,1,\dots,1}^{r})} = \frac{1}{n+1}\binom{n+1}{r} C_{n-r+1}^{(1)}.$$

Proof. Use the identity $\mathrm{Li}_{\underbrace{1,\ldots,1}_{r}}(z) = (-\log(1-z))^r/r!$ to obtain

$$\sum_{n=0}^{\infty} B_n^{(\overbrace{1,\ldots,1}^{r})} \frac{t^n}{n!} = \frac{t^{r-1}}{r!} \frac{t}{1-e^{-t}} = \frac{t^{r-1}}{r!} \sum_{n=0}^{\infty} B_n^{(1)} \frac{t^n}{n!}$$

and

$$\sum_{n=0}^{\infty} C_n^{(\overbrace{1,\ldots,1}^{r})} \frac{t^n}{n!} = \frac{t^{r-1}}{r!} \frac{t}{e^{t}-1} = \frac{t^{r-1}}{r!} \sum_{n=0}^{\infty} C_n^{(1)} \frac{t^n}{n!}.$$

The proposition follows. $\square$

3 Multi-poly-Bernoulli numbers with negative indices

Proposition 5. [2] *We have the following generating series identities for multi-poly-Bernoulli numbers with non-positive upper indices:*

1)

$$\begin{aligned}
&\sum_{k_1,\ldots,k_r=0}^{\infty} \sum_{n=0}^{\infty} B_n^{(-k_1,\ldots,-k_r)} \frac{t^n}{n!} \frac{x_1^{k_1}}{k_1!} \cdots \frac{x_r^{k_r}}{k_r!} \\
= \; & \frac{(1-e^{-t})^{r-1}}{(e^{-x_1}+e^{-t}-1)(e^{-x_1-x_2}+e^{-t}-1)\cdots(e^{-x_1-\cdots-x_r}+e^{-t}-1)},
\end{aligned}$$

2)

$$\begin{aligned}
&\sum_{k_1,\ldots,k_r=0}^{\infty} \sum_{n=0}^{\infty} C_n^{(-k_1,\ldots,-k_r)} \frac{t^n}{n!} \frac{x_1^{k_1}}{k_1!} \cdots \frac{x_r^{k_r}}{k_r!} \\
= \; & \frac{e^{-t}(1-e^{-t})^{r-1}}{(e^{-x_1}+e^{-t}-1)(e^{-x_1-x_2}+e^{-t}-1)\cdots(e^{-x_1-\cdots-x_r}+e^{-t}-1)}.
\end{aligned}$$

Proof. Use the defining identity

$$\sum_{n=0}^{\infty} B_n^{(-k_1,\ldots,-k_r)} \frac{t^n}{n!} = (1-e^{-t})^{-1} \sum_{m_1>\cdots>m_r>0} m_1^{k_1} \cdots m_r^{k_r} (1-e^{-t})^{m_1}$$

[2]This proposition and the corollary below are also pointed out by Genki Shibukawa.

to obtain

$$\sum_{k_1,\dots,k_r=0}^{\infty}\sum_{n=0}^{\infty} B_n^{(-k_1,\dots,-k_r)}\frac{t^n}{n!}\frac{x_1^{k_1}}{k_1!}\cdots\frac{x_r^{k_r}}{k_r!} = (1-e^{-t})^{-1}\sum_{m_1>\cdots>m_r>0}(1-e^{-t})^{m_1}e^{m_1x_1+\cdots+m_rx_r}$$

$$= (1-e^{-t})^{-1}\sum_{n_1,\dots,n_r=1}^{\infty}(1-e^{-t})^{n_1+\cdots+n_r}e^{(n_1+\cdots+n_r)x_1+(n_2+\cdots+n_r)x_2+\cdots+n_rx_r}$$

$$= (1-e^{-t})^{-1}\sum_{n_1,\dots,n_r=1}^{\infty}\left((1-e^{-t})e^{x_1}\right)^{n_1}\left((1-e^{-t})e^{x_1+x_2}\right)^{n_2}\cdots\left((1-e^{-t})e^{x_1+\cdots+x_r}\right)^{n_r}$$

$$= (1-e^{-t})^{r-1}\frac{e^{x_1}}{1-\left((1-e^{-t})e^{x_1}\right)}\frac{e^{x_1+x_2}}{1-\left((1-e^{-t})e^{x_1+x_2}\right)}\cdots\frac{e^{x_1+\cdots+x_r}}{1-\left((1-e^{-t})e^{x_1+\cdots+x_r}\right)}$$

$$= \frac{(1-e^{-t})^{r-1}}{(e^{-x_1}+e^{-t}-1)(e^{-x_1-x_2}+e^{-t}-1)\cdots(e^{-x_1-\cdots-x_r}+e^{-t}-1)}.$$

The identity for $C_n^{(-k_1,\dots,-k_r)}$ readily follows from this because the defining generating series differ only by the factor e^{-t}. $\square$

Corollary 6. *For any integers $k_1,\dots,k_r \geq 0$ and $n \geq 0$, the multi-poly-Bernoulli numbers $B_n^{(-k_1,\dots,-k_r)}$ and $C_n^{(-k_1,\dots,-k_r)}$ are positive integers.*

Proof. The right-hand sides of 1) and 2) of the proposition can be rewritten as

$$e^t(e^t-1)^{r-1}\cdot\frac{e^{x_1}}{1-(e^t-1)(e^{x_1}-1)}\cdot\frac{e^{x_1+x_2}}{1-(e^t-1)(e^{x_1+x_2}-1)}\cdots\frac{e^{x_1+\cdots+x_r}}{1-(e^t-1)(e^{x_1+\cdots+x_r}-1)}$$

and

$$(e^t-1)^{r-1}\cdot\frac{e^{x_1}}{1-(e^t-1)(e^{x_1}-1)}\cdot\frac{e^{x_1+x_2}}{1-(e^t-1)(e^{x_1+x_2}-1)}\cdots\frac{e^{x_1+\cdots+x_r}}{1-(e^t-1)(e^{x_1+\cdots+x_r}-1)}$$

respectively, from which the positivity of $B_n^{(-k_1,\dots,-k_r)}$ and $C_n^{(-k_1,\dots,-k_r)}$ is obvious. That both are integers follows from the explicit formulas in Theorem 3. $\square$

We can give some explicit formulas different from Theorem 3 for special indices.

Proposition 7. *For $r \geq 1$, $k \geq 0$ and $n \geq 0$, we have*

1)

$$B_n^{(-k,\overbrace{0,\dots,0}^{r-1})} = (-1)^n\sum_{j=0}^{n}(-1)^j(j+1)^k j!\binom{j}{r-1}\left\{{n \atop j}\right\},$$

$$C_n^{(-k,\overbrace{0,\dots,0}^{r-1})} = (-1)^n\sum_{j=0}^{n}(-1)^j(j+1)^k j!\binom{j}{r-1}\left\{{n+1 \atop j+1}\right\}.$$

2)

$$
\begin{aligned}
B_n^{(\overbrace{0,\dots,0}^{r-1},-k)} &= \sum_{j=0}^{\min\{n+1-r,k\}} (r+j-1)!j!\left\{ {n+1 \atop r+j} \right\}\left\{ {k+1 \atop j+1} \right\},\\
C_n^{(\overbrace{0,\dots,0}^{r-1},-k)} &= \sum_{j=0}^{\min\{n+1-r,k\}} (r+j-1)!j!\left\{ {n \atop r+j-1} \right\}\left\{ {k+1 \atop j+1} \right\}.
\end{aligned}
$$

Proof. Set $x_2 = \cdots = x_r = 0$ in 1) of Proposition 5 to obtain

$$
\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} B_n^{(-k,\overbrace{0,\dots,0}^{r-1})}\frac{t^n}{n!}\frac{x^k}{k!} = \frac{(1-e^{-t})^{r-1}}{(e^{-x}+e^{-t}-1)^r}.
$$

From this we have

$$
\begin{aligned}
\sum_{r=1}^{\infty}\sum_{k,n=0}^{\infty} B_n^{(-k,\overbrace{0,\dots,0}^{r-1})}\frac{t^n}{n!}\frac{x^k}{k!}y^{r-1} &= \frac{1}{(e^{-x}+e^{-t}-1)}\sum_{r=1}^{\infty}\left(\frac{1-e^{-t}}{e^{-x}+e^{-t}-1}\right)^{r-1} y^{r-1}\\
&= \frac{1}{(e^{-x}+e^{-t}-1)}\frac{1}{1-\frac{1-e^{-t}}{e^{-x}+e^{-t}-1}y}\\
&= \frac{1}{e^{-x}+e^{-t}-1-(1-e^{-t})y}\\
&= \frac{e^x}{1-e^x(1+y)(1-e^{-t})}\\
&= \sum_{j=0}^{\infty} e^{(j+1)x}(1+y)^j(1-e^{-t})^j\\
&= \sum_{j=0}^{\infty} e^{(j+1)x}(1+y)^j(-1)^j j!\sum_{n=j}^{\infty}(-1)^n\left\{ {n \atop j} \right\}\frac{t^n}{n!}\\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{n} e^{(j+1)x}(1+y)^j(-1)^{n+j}j!\left\{ {n \atop j} \right\}\frac{t^n}{n!}.
\end{aligned}
$$

Comparing the coefficients of $\frac{t^n}{n!}\frac{x^k}{k!}y^{r-1}$ on both sides, we obtain 1) for $B_n^{(-k,0,\dots,0)}$.

Similarly, we compute

$$
\begin{aligned}
\sum_{r=1}^{\infty}\sum_{k,n=0}^{\infty} C_n^{(-k,\overbrace{0,\dots,0}^{r-1})}\frac{t^n}{n!}\frac{x^k}{k!}y^{r-1} &= e^{-t}\cdot\frac{e^x}{1-e^x(1+y)(1-e^{-t})}\\
&= \sum_{j=0}^{\infty} e^{(j+1)x}(1+y)^j e^{-t}(1-e^{-t})^j\\
&= \sum_{j=0}^{\infty} e^{(j+1)x}(1+y)^j(-1)^j j!\sum_{n=j}^{\infty}(-1)^n\left\{{n+1 \atop j+1}\right\}\frac{t^n}{n!}\\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{n} e^{(j+1)x}(1+y)^j(-1)^{n+j}j!\left\{{n+1 \atop j+1}\right\}\frac{t^n}{n!}
\end{aligned}
$$

and obtain the formula for $C_n^{(-k,0,\dots,0)}$.

For 2), we set $x_1=\cdots=x_{r-1}=0$ in the formulas in Proposition 5 and have

$$
\begin{aligned}
\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} B_n^{(\overbrace{0,\dots,0}^{r-1},-k)}\frac{t^n}{n!}\frac{x^k}{k!} &= \frac{(e^t-1)^{r-1}}{e^{-x}+e^{-t}-1}\\
&= (e^t-1)^{r-1}\cdot\frac{e^{t+x}}{1-(e^t-1)(e^x-1)}\\
&= \sum_{j=0}^{\infty} e^t(e^t-1)^{r+j-1}e^x(e^x-1)^j\\
&= \sum_{j=0}^{\infty}(r+j-1)!\sum_{n=r+j-1}^{\infty}\left\{{n+1 \atop r+j}\right\}\frac{t^n}{n!}\cdot j!\sum_{k=j}^{\infty}\left\{{k+1 \atop j+1}\right\}\frac{x^k}{k!}
\end{aligned}
$$

and

$$
\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} C_n^{(\overbrace{0,\dots,0}^{r-1},-k)}\frac{t^n}{n!}\frac{x^k}{k!} = \sum_{j=0}^{\infty}(r+j-1)!\sum_{n=r+j-1}^{\infty}\left\{{n \atop r+j-1}\right\}\frac{t^n}{n!}\cdot j!\sum_{k=j}^{\infty}\left\{{k+1 \atop j+1}\right\}\frac{x^k}{k!}.
$$

Comparing the coefficients of $\frac{t^n}{n!}\frac{x^k}{k!}$ on both sides, we obtain 2). $\square$

4 Connection to the finite multiple zeta values

As recalled in the introduction, the finite multiple zeta value $\zeta_{\mathcal{A}}(k_1,\dots,k_r)$ is an element in the ring $\mathcal{A} := \prod_p \mathbb{Z}/p\mathbb{Z}/\bigoplus_p \mathbb{Z}/p\mathbb{Z}$ represented by $\left(\zeta_{\mathcal{A}}(k_1,\dots,k_r)_{(p)}\right)_p \in \prod_p \mathbb{Z}/p\mathbb{Z}$ where

$$
\zeta_{\mathcal{A}}(k_1,\dots,k_r)_{(p)} := \sum_{p>m_1>\cdots>m_r>0}\frac{1}{m_1^{k_1}\cdots m_r^{k_r}} \mod p.
$$

Theorem 8. *For any $r \geq 1$ and $k_i \in \mathbb{Z}$, we have*

$$\zeta_{\mathcal{A}}(k_1, \ldots, k_r)_{(p)} \;=\; -C_{p-2}^{(k_1-1,k_2,\ldots,k_r)} \bmod p.$$

More generally, we have for any $r \geq 1$, $j \geq 0$, and $k_i \in \mathbb{Z}$

$$\zeta_{\mathcal{A}}(\underbrace{1,\ldots,1}_{j}, k_1, \ldots, k_r)_{(p)} \;=\; -C_{p-j-2}^{(k_1-1,k_2,\ldots,k_r)} \bmod p.$$

Proof. By the explicit formula in Theorem 3, we have

$$\begin{aligned} C_{p-2}^{(k_1-1,k_2,\ldots,k_r)} &= -\sum_{p-1\geq m_1>m_2>\cdots>m_r>0} \frac{(-1)^{m_1-1}(m_1-1)!\left\{ {p-1 \atop m_1} \right\}}{m_1^{k_1-1}m_2^{k_2}\cdots m_r^{k_r}} \\ &= \sum_{p-1\geq m_1>m_2>\cdots>m_r>0} \frac{(-1)^{m_1}m_1!\left\{ {p-1 \atop m_1} \right\}}{m_1^{k_1}m_2^{k_2}\cdots m_r^{k_r}}. \end{aligned}$$

By the well-known formula (cf. Graham et al. [3, §6.1])

$$(-1)^{m_1}m_1!\left\{ {p-1 \atop m_1} \right\} = \sum_{l=0}^{m_1}(-1)^l\binom{m_1}{l}l^{p-1},$$

we see

$$(-1)^{m_1}m_1!\left\{ {p-1 \atop m_1} \right\} \equiv \sum_{l=1}^{m_1}(-1)^l\binom{m_1}{l} \pmod p,$$

the sum on the right being equal to $(1-1)^{m_1}-1=-1$. This proves that

$$C_{p-2}^{(k_1-1,k_2,\ldots,k_r)} \bmod p = -\zeta_{\mathcal{A}}(k_1,\ldots,k_r)_{(p)}.$$

For the second identity, we proceed as follows. First note

$$\begin{aligned} &\sum_{p-1\geq i_1>\cdots>i_j>m_1>\cdots>m_r\geq 1} \frac{1}{i_1\cdots i_j m_1^{k_1}\cdots m_r^{k_r}} \\ &= \sum_{p-j>m_1>\cdots>m_r\geq 1} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}} \sum_{p-1\geq i_1>\cdots>i_j>m_1} \frac{1}{i_1\cdots i_j}. \end{aligned}$$

By changing $i_l \to p-i_l$, we see that

$$\sum_{p-1\geq i_1>\cdots>i_j>m_1} \frac{1}{i_1\cdots i_j} \equiv (-1)^j \sum_{p-m_1>i_j>\cdots>i_1\geq 1} \frac{1}{i_1\cdots i_j} \pmod p.$$

Using the formula

$$\sum_{p-m_1>i_j>\cdots>i_1\geq 1} \frac{1}{i_1\cdots i_j} = \frac{1}{(p-m_1-1)!}\left[{p-m_1 \atop j+1} \right],$$

and the congruences

$$\begin{bmatrix} n \\ m \end{bmatrix} \equiv \begin{Bmatrix} p-m \\ p-n \end{Bmatrix} \pmod p \quad (1 \le m \le n \le p-1)$$

(see Hoffman [6, §5]) and

$$\frac{1}{(p-m_1-1)!} \equiv (-1)^{m_1+1} m_1! \pmod p,$$

we obtain (for odd p)

$$\begin{aligned} \zeta_{\mathcal{A}}(\underbrace{1,\dots,1}_{j}, k_1, \dots, k_r)_{(p)} &= \sum_{p-j>m_1>\cdots>m_r\ge 1} \frac{(-1)^{j+m_1+1} m_1! \left\{ {p-j-1 \atop m_1} \right\}}{m_1^{k_1}\cdots m_r^{k_r}} \bmod p \\ &= -C_{p-j-2}^{(k_1-1,k_2,\dots,k_r)} \bmod p. \end{aligned}$$

This concludes the proof of the theorem. □

Combining the theorem with Proposition 4, we see that

$$\begin{aligned} \zeta_{\mathcal{A}}(2, \overbrace{1,\dots,1}^{k-2}) &= \left(-C_{p-2}^{(\overbrace{1,\dots,1}^{k-1})} \bmod p\right)_p = \left(-\frac{1}{p-1}\binom{p-1}{k-1} B_{p-k} \bmod p\right)_p \\ &= (B_{p-k} \bmod p)_p. \end{aligned}$$

(We used $\binom{p-1}{k-1} \equiv (-1)^{k-1} \pmod p$ and $(-1)^{p-k}B_{p-k} = B_{p-k}$ for large enough p.) As is discussed in Kaneko-Zagier [8], the element $(B_{p-k} \bmod p)_p$ on the right is regarded as an analogue of $k\zeta(k)$. (Note that the obvious analogue $\zeta_{\mathcal{A}}(k)$ of $\zeta(k)$ is 0.)

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