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The Number of Relatively Prime Subsets of a Finite Union of Sets of Consecutive Integers

Mohamed Ayad
Laboratoire de Mathématiques Pures et Appliquées
Université du Littoral
F-62228 Calais
France
Ayad@lmpa.univ-littoral.fr

Vincenzo Coia
Department of Statistics
University of British Columbia
Vancouver, BC V6T 1Z4
Canada
vincen.coia@stat.ubc.ca

Omar Kihel
Department of Mathematics
Brock University
St. Catharines, ON L2S 3A1
Canada
okihel@brocku.ca

Abstract

Let A be a finite union of disjoint sets of consecutive integers and let n be a positive integer. We give a formula for the number of relatively prime subsets (resp.,

relatively prime subsets of cardinality k) of A, which generalizes results of Nathanson, El Bachraoui and others. We give as well similar formulas for the number of subsets with gcd coprime to n.

1 Introduction

A nonempty set S of integers is said to be *relatively prime* if $\gcd(S) = 1$, where $\gcd(S)$ denotes the greatest common divisor of the elements of S. Nathanson [10] defines $f(n)$ to be the number of relatively prime subsets of $\{1, 2, \ldots, n\}$ and, for $k \geq 1$, $f_k(n)$ to be the number of relatively prime subsets of $\{1, 2, \ldots, n\}$ of cardinality k. By analogy with Euler's phi function $\phi(n)$ that counts the number of positive integers a in the set $\{1, 2, \ldots, n\}$ such that $\gcd(a, n) = 1$, Nathanson [10] defines $\Phi(n)$ to be the number of nonempty subsets S of the set $\{1, 2, \ldots, n\}$ such that $\gcd(S)$ is relatively prime to n and, for integer $k \geq 1$, $\Phi_k(n)$ to be the number of subsets S of the set $\{1, 2, \ldots, n\}$ such that $\gcd(S)$ is relatively prime to n and $|S| = k$. He obtains explicit formulas for these four functions and deduces asymptotic estimates.

For simplicity, we use a more general notation than Nathanson [10]. For a nonempty set of integers S, we define

- $f(S) = |\{H \subseteq S : \gcd(H) = 1, H \neq \emptyset\}|$ as the number of nonempty relatively prime subsets of S;
- $f_k(S) = |\{H \subseteq S : \gcd(H) = 1, |H| = k\}|$ as the number of relatively prime subsets of S of cardinality k;
- $\Phi(S, n) = |\{H \subseteq S : \gcd(H \cup \{n\}) = 1, H \neq \emptyset\}|$ as the number of nonempty subsets of S with gcd relatively prime to integer n;
- $\Phi_k(S, n) = |\{H \subseteq S : \gcd(H \cup \{n\}) = 1, |H| = k\}|$ as the number of subsets of S of cardinality k and with gcd relatively prime to integer n.

Further, we define $[a, b]_{\mathbb{Z}} = [a, b] \cap \mathbb{Z} = \{a, a+1, \ldots, b\}$ for integers $a < b$ as the set of consecutive integers from a to b, inclusive.

El Bachraoui [4] and Nathanson and Orosz [11] generalize the results of Nathanson [10] to subsets of $[\ell, m]_{\mathbb{Z}}$ for integers $0 \leq \ell < m$, and prove Theorem 1.

Theorem 1. *For non-negative integers $\ell < m$ and $k \geq 1$, using the notation $f(\ell, m) = f([\ell, m]_{\mathbb{Z}})$ and $f_k(\ell, m) = f_k([\ell, m]_{\mathbb{Z}})$ of El Bachraoui [4] and Nathanson and Orosz [11] we have*

$$f(\ell, m) = \sum_{d=1}^{m} \mu(d) \left(2^{\lfloor \frac{m}{d} \rfloor - \lfloor \frac{\ell}{d} \rfloor} - 1\right) \tag{1}$$

and

$$f_k(\ell, m) = \sum_{d=1}^{m} \mu(d) \binom{\lfloor m/d \rfloor - \lfloor \ell/d \rfloor}{k}, \tag{2}$$

where μ is the Möbius function.

For brevity, define the arithmetic sequence $\mathcal{A}_n^{(a,b)} = \{a, a+b, \ldots, a+(n-1)b\}$ for positive integers n, a, and b. Ayad and Kihel [1] generalize Theorem 1 to obtain Theorem 2.

Theorem 2. *For all positive integers n, a, and b, with $\gcd(a,b) = 1$, using the notation $f^{(a,b)}(n) = f\left(\mathcal{A}_n^{(a,b)}\right)$ and $f_k^{(a,b)}(n) = f_k\left(\mathcal{A}_n^{(a,b)}\right)$ of Ayad and Kihel [1], we have*

$$f^{(a,b)}(n) = \sum_{\substack{d=1 \\ \gcd(b,d)=1}}^{a+(n-1)b} \mu(d)\left(2^{\left\lfloor \frac{n}{d} \right\rfloor + \varepsilon_d} - 1\right) \tag{3}$$

and

$$f_k^{(a,b)}(n) = \sum_{\substack{d=1 \\ \gcd(b,d)=1}}^{a+(n-1)b} \mu(d)\binom{\left\lfloor \frac{n}{d} \right\rfloor + \varepsilon_d}{k}, \tag{4}$$

where

$$\varepsilon_d = \begin{cases} 0, & \text{if } d \mid n; \\ 1, & \text{if } d \nmid n \text{ and } \left(-ab^{-1}\right) \bmod d \in \left\{\left\lfloor \frac{n}{d} \right\rfloor d, \ldots, n-1\right\}; \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

El Bachraoui [6] extends Theorem 1 to the union of two sets of consecutive integers, to obtain Theorem 3.

Theorem 3. *For nonnegative integers $\ell_1 < m_1 < \ell_2 < m_2$ and for $k \geq 1$,*

$$f\left([\ell_1, m_1]_{\mathbb{Z}} \cup [\ell_2, m_2]_{\mathbb{Z}}\right) = \sum_{d=1}^{m_2} \mu(d)\left(2^{\left\lfloor \frac{m_1}{d} \right\rfloor + \left\lfloor \frac{m_2}{d} \right\rfloor - \left\lfloor \frac{\ell_1 - 1}{d} \right\rfloor - \left\lfloor \frac{\ell_2 - 1}{d} \right\rfloor} - 1\right) \tag{6}$$

and

$$f_k\left([\ell_1, m_1]_{\mathbb{Z}} \cup [\ell_2, m_2]_{\mathbb{Z}}\right) = \sum_{d=1}^{m_2} \mu(d)\binom{\left\lfloor \frac{m_1}{d} \right\rfloor + \left\lfloor \frac{m_2}{d} \right\rfloor - \left\lfloor \frac{\ell_1 - 1}{d} \right\rfloor - \left\lfloor \frac{\ell_2 - 1}{d} \right\rfloor}{k}. \tag{7}$$

We now switch our attention to analogous results for functions Φ and Φ_k. For the consecutive integers case, El Bachraoui [4] and Nathanson and Orosz [11] prove Theorem 4.

Theorem 4. *For non-negative integers $\ell < m$ and $k \geq 1$, using the notation $\Phi(\ell, m) = \Phi\left([\ell, m]_{\mathbb{Z}}, m\right)$ and $\Phi_k(\ell, m) = \Phi_k\left([\ell, m]_{\mathbb{Z}}, m\right)$ of El Bachraoui [4] and Nathanson and Orosz [11] we have*

$$\Phi(\ell, m) = \sum_{d \mid m} \mu(d)\, 2^{\left(\frac{m}{d} - \left\lfloor \frac{\ell}{d} \right\rfloor\right)} \tag{8}$$

and

$$\Phi_k(\ell, m) = \sum_{d \mid m} \mu(d)\binom{\frac{m}{d} - \left\lfloor \frac{\ell}{d} \right\rfloor}{k}. \tag{9}$$

Ayad and Kihel [1] generalize Theorem 4 to obtain Theorem 5.

Theorem 5. *For nonnegative integers a, b, and n, with $\gcd(a,b)=1$, using the notation $\Phi^{(a,b)}(n)=\Phi\left(\mathcal{A}_n^{(a,b)},n\right)$ and $\Phi_k^{(a,b)}(n)=\Phi_k\left(\mathcal{A}_n^{(a,b)},n\right)$ of Ayad and Kihel [1] we have*

$$\Phi^{(a,b)}(n)=\sum_{\substack{d\mid n\\ \gcd(b,d)=1}}\mu(d)\left(2^{\frac{n}{d}}-1\right) \tag{10}$$

and

$$\Phi_k^{(a,b)}(n)=\sum_{\substack{d\mid n\\ \gcd(b,d)=1}}\mu(d)\binom{\frac{n}{d}}{k}. \tag{11}$$

El Bachraoui and Salim [9] extend Theorem 4 to the union of two sets of consecutive integers, to obtain Theorem 6.

Theorem 6. *For nonnegative integers $\ell_1<m_1<\ell_2<m_2$ and for $k\geq 1$,*

$$\Phi\left([\ell_1,m_1]_{\mathbb{Z}}\cup[\ell_2,m_2]_{\mathbb{Z}},n\right)=\sum_{d\mid n}\mu(d)\,2^{\left\lfloor\frac{m_1}{d}\right\rfloor+\left\lfloor\frac{m_2}{d}\right\rfloor-\left\lfloor\frac{\ell_1-1}{d}\right\rfloor-\left\lfloor\frac{\ell_2-1}{d}\right\rfloor} \tag{12}$$

and

$$\Phi_k\left([\ell_1,m_1]_{\mathbb{Z}}\cup[\ell_2,m_2]_{\mathbb{Z}},n\right)=\sum_{d\mid n}\mu(d)\binom{\left\lfloor\frac{m_1}{d}\right\rfloor+\left\lfloor\frac{m_2}{d}\right\rfloor-\left\lfloor\frac{\ell_1-1}{d}\right\rfloor-\left\lfloor\frac{\ell_2-1}{d}\right\rfloor}{k}. \tag{13}$$

In Section 2, we extend Theorems 3 and 6 to the union of any finite number of disjoint sets of consecutive integers. The approach we take is simple and much different from the approach of El Bachraoui [6] and El Bachraoui and Salim [9] for the union of two sets. Several authors [2, 3, 12, 7, 13, 14, 5] discuss other properties and generalizations.

2 Finite union of disjoint sets of consecutive integers

For positive integers $\ell_i\leq m_i$ for $i=1,\ldots,r$, denote $A^{(i)}=[\ell_i,m_i]_{\mathbb{Z}}$ for brevity and assume $A^{(i)}\cap A^{(j)}=\emptyset$ for $i\neq j$. Consider the union

$$A=\bigcup_{i=1}^{r}A^{(i)}. \tag{14}$$

El Bachraoui [6] derives equations for $f(A)$ and $f_k(A)$ for $r=2$, as in Theorem 3. We extend this to any $r\in\mathbb{N}$ in Theorem 8, but first we need Lemma 7. Also, throughout this section, for a set of integers S we denote $\mathcal{P}(S)=\{H\subseteq S:H\neq\emptyset\}$ and $\mathcal{P}_k(S)=\{H\subseteq S:|H|=k\}$.

Lemma 7. *Let $A_d = \{x \in A : d \mid x\}$ be all the multiples of d found in A, where A is defined in equation (14). Then,*

$$|A_d| = \sum_{i=1}^{r} \left(\left\lfloor \frac{m_i}{d} \right\rfloor - \left\lfloor \frac{\ell_i - 1}{d} \right\rfloor \right).$$

Proof. For $i = 1, \ldots, r$, let $A_d^{(i)} = \left\{x \in A^{(i)} : d \mid x\right\}$, $M_d^{(i)} = \{x \in [0, m_i]_{\mathbb{Z}} : d \mid x\}$, and $L_d^{(i)} = \{x \in [0, \ell_i - 1]_{\mathbb{Z}} : d \mid x\}$. Clearly, we have $\left|A_d^{(i)}\right| = \left|M_d^{(i)}\right| - \left|L_d^{(i)}\right|$. But, we simply have $\left|M_d^{(i)}\right| = \left\lfloor \frac{m_i}{d} \right\rfloor$ and $\left|L_d^{(i)}\right| = \left\lfloor \frac{\ell_i - 1}{d} \right\rfloor$. So,

$$\left|A_d^{(i)}\right| = \left\lfloor \frac{m_i}{d} \right\rfloor - \left\lfloor \frac{\ell_i - 1}{d} \right\rfloor.$$

Now, since $A_d = \bigcup\limits_{i=1}^{r} A_d^{(i)}$ and since $A_d^{(i)} \cap A_d^{(j)} = \emptyset$ for $i \neq j$, we have $|A_d| = \sum\limits_{i=1}^{r} \left|A_d^{(i)}\right|$ which completes the proof. ∎

Theorem 8. *For A defined in equation (14), we have*

$$f(A) = \sum_{d=1}^{\max\{m_1,\ldots,m_r\}} \mu(d) \left(2^{\sum\limits_{i=1}^{r} \left(\left\lfloor \frac{m_i}{d} \right\rfloor - \left\lfloor \frac{\ell_i - 1}{d} \right\rfloor \right)} - 1 \right); \tag{15}$$

$$f_k(A) = \sum_{d=1}^{\max\{m_1,\ldots,m_r\}} \mu(d) \binom{\sum\limits_{i=1}^{r} \left(\left\lfloor \frac{m_i}{d} \right\rfloor - \left\lfloor \frac{\ell_i - 1}{d} \right\rfloor \right)}{k}. \tag{16}$$

Proof. We begin by proving equation (15). From the total amount of nonempty subsets of A, remove those subsets that are *not* relatively prime:

$$f(A) = |\mathcal{P}(A)| - \left| \bigcup_{p \text{ prime}} \mathcal{P}(A_p) \right|.$$

Using inclusion-exclusion and the same principle as in the proof of Ayad and Kihel [1, Theorem 5], we obtain

$$f(A) = \sum_{d=1}^{\max\{m_1,\ldots,m_r\}} \mu(d) \left(2^{|A_d|} - 1\right).$$

Applying Lemma (7), we obtain equation (15).

To prove equation (16), from the total amount of subsets of A with cardinality k, remove those subsets that are *not* relatively prime:

$$f_k(A) = |\mathcal{P}_k(A)| - \left| \bigcup_{p \text{ prime}} \mathcal{P}_k(A_p) \right|.$$

Using inclusion-exclusion and the same principle as in the proof of Ayad and Kihel [1, Theorem 5], we obtain

$$f_k(A) = \sum_{d=1}^{\max\{m_1,\ldots,m_r\}} \mu(d) \binom{|A_d|}{k}.$$

Applying Lemma (7), we obtain equation (16). ☐

Similarly, we now extend Theorem 6.

Theorem 9. *Define A as in equation (14). Then for any integer $k \geq 1$,*

$$\Phi(A,n) = \sum_{d|n} \mu(d) \left(2^{\sum_{i=1}^{r} \left(\left\lfloor \frac{m_i}{d} \right\rfloor - \left\lfloor \frac{\ell_i - 1}{d} \right\rfloor \right)} - 1 \right); \tag{17}$$

$$\Phi_k(A,n) = \sum_{d|n} \mu(d) \binom{\sum_{i=1}^{r} \left(\left\lfloor \frac{m_i}{d} \right\rfloor - \left\lfloor \frac{\ell_i - 1}{d} \right\rfloor \right)}{k}. \tag{18}$$

Proof. We begin by proving equation (17). Notice that

$$\Phi(A,n) = |\mathcal{P}(A)| - \left| \bigcup_{\substack{p \text{ prime} \\ p|n}} \mathcal{P}(A_p) \right|.$$

As in the proof of Theorem 8, we have

$$\Phi(A,n) = \sum_{d|n} \mu(d) \left(2^{|A_d|} - 1 \right).$$

Applying Lemma 7 proves equation (17).

To prove equation (18), notice that

$$\Phi_k(A,n) = |\mathcal{P}_k(A)| - \left| \bigcup_{\substack{p \text{ prime} \\ p|n}} \mathcal{P}_k(A_p) \right|.$$

As in the proof of Theorem 8, we have

$$\Phi_k(A,n) = \sum_{d|n} \mu(d) \binom{|A_d|}{k}.$$

Applying Lemma 7 proves equation (18). ☐

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