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On a Congruence Modulo n^3 Involving Two Consecutive Sums of Powers

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Abstract

For various positive integers k, the sums of kth powers of the first n positive integers, $S_k(n) := 1^k + 2^k + \cdots + n^k$, are some of the most popular sums in all of mathematics. In this note we prove a congruence modulo n^3 involving two consecutive sums $S_{2k}(n)$ and $S_{2k+1}(n)$. This congruence allows us to establish an equivalent formulation of Giuga's conjecture. Moreover, if k is even and $n \geq 5$ is a prime such that $n - 1 \nmid 2k - 2$, then this congruence is satisfied modulo n^4. This suggests a conjecture about when a prime can be a Wolstenholme prime. We also propose several Giuga-Agoh-like conjectures. Further, we establish two congruences modulo n^3 for two binomial-type sums involving sums of powers $S_{2i}(n)$ with $i = 0, 1, \ldots, k$. Finally, we obtain an extension of a result of Carlitz-von Staudt for odd power sums.

1 Introduction and basic results

The sum of powers of integers $\sum_{i=1}^{n} i^k$ is a well-studied problem in mathematics (see, e.g., [9, 40]). Finding formulas for these sums has interested mathematicians for more than 300 years since the time of Jakob Bernoulli (1654–1705). Several methods were used to find the sum $S_k(n)$ (see, for example, Vakil [49]). These lead to numerous recurrence relations. For a nice account of sums of powers, see Edwards [15]. For simplicity, here as always in the sequel, for all integers $k \geq 1$ and $n \geq 2$ we write

$$S_k(n) := \sum_{i=1}^{n-1} i^k = 1^k + 2^k + 3^k + \cdots + (n-1)^k.$$

The study of these sums led Jakob Bernoulli to develop numbers later named in his honor. Namely, the celebrated *Bernoulli formula* (sometimes called *Faulhaber's formula*) gives the sum $S_k(n)$ explicitly as (see, e.g., Beardon [4])

$$S_k(n) = \frac{1}{k+1} \sum_{i=0}^{k} \binom{k+1}{i} n^{k+1-i} B_i, \tag{1}$$

where B_i ($i = 0, 1, 2, \ldots$) are the *Bernoulli numbers* defined by the generating function

$$\sum_{i=0}^{\infty} B_i \frac{x^i}{i!} = \frac{x}{e^x - 1}.$$

It is easy to find the values $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, and $B_i = 0$ for odd $i \geq 3$. Furthermore, $(-1)^{i-1} B_{2i} > 0$ for all $i \geq 1$. These and many other properties can be found, for instance, in [23]. Several generalizations of the formula (1) were established by Z.-H. Sun ([46, Thm. 2.1] and [47]) and Z.-W. Sun [48].

By the well-known *Pascal's identity* proven by Pascal in 1654 (see, e.g., [29]), we have

$$\sum_{i=0}^{k-1} \binom{k}{i} S_i(n+1) = (n+1)^k - 1. \tag{2}$$

Recall also that the formula (2) is also presented in Bernoulli's *Ars Conjectandi* [6], (also see [19, pp. 269–270]) published posthumously in 1713.

On the other hand, divisibility properties of the sums $S_k(n)$ were investigated by many authors [13, 27, 30, 42]. For example, in 2003 Damianou and Schumer [13, Thm. 1, p. 221 and Thm. 2, p. 222] proved, respectively:

(1) if k is odd, then n divides $S_k(n)$ if and only if n is incogruent to 2 modulo 4;

(2) if k is even, then n divides $S_k(n)$ if and only if n is not divisible by any prime p such that $p \mid D_k$, where D_k is the denominator of the kth Bernoulli number B_k.

Denominators of Bernoulli numbers B_k ($k = 0, 1, 2, \ldots$) are given as the sequence A027642 in [41] (cf. also its subsequence A002445 consisting of the terms with even indices k).

Motivated by the recurrence formula for $S_k(n)$ recently obtained in [34, Corollary 1.7], in this note we prove the following basic result.

Theorem 1. *Let k and n be positive integers. Then for each $k \geq 2$*

$$2S_{2k+1}(n) - (2k+1)nS_{2k}(n) \equiv \begin{cases} 0 \pmod{n^3}, & \text{if } k \text{ is even or } n \text{ is odd} \\ & \text{or } n \equiv 0 \text{ (mod 4)}; \\ \frac{n^3}{2} \pmod{n^3}, & \text{if } k \text{ is odd and } n \equiv 2 \text{ (mod 4)}. \end{cases} \tag{3}$$

Furthermore,

$$2S_3(n) - 3nS_2(n) \equiv \begin{cases} 0 \pmod{n^3}, & \text{if } n \text{ is odd;} \\ \frac{n^3}{2} \pmod{n^3}, & \text{if } n \text{ is even.} \end{cases} \tag{4}$$

In particular, for all $k \geq 1$ and $n \geq 1$, we have

$$2S_{2k+1}(n) \equiv (2k+1)nS_{2k}(n) \pmod{n^2}, \tag{5}$$

and for all $k \geq 1$ and $n \not\equiv 2 \pmod 4$

$$2S_{2k+1}(n) \equiv (2k+1)nS_{2k}(n) \pmod{n^3}. \tag{6}$$

Combining the congruence (5) and the "even case" (2) of a result of Damianou and Schumer [13, Thm. 1, p. 221 and Thm. 2, p. 222] mentioned above, we obtain the following "odd" extension of their result.

Corollary 2. *If k is an odd positive integer and n a positive integer such that n is not divisible by any prime p such that $p \mid D_{k-1}$, where D_{k-1} is the denominator of the $(k-1)$th Bernoulli number B_{k-1}, then n^2 divides $2S_k(n)$.*

Conversely, if k is an odd positive integer and n a positive integer relatively prime to k such that n^2 divides $2S_k(n)$, then n is not divisible by any prime p such that $p \mid D_{k-1}$, where D_{k-1} is the denominator of the $(k-1)$th Bernoulli number B_{k-1}.

The paper is organized as follows. Some applications of Theorem 1 are presented in the following section. In Subsection 2.1 we give three particular cases of the congruence (3) of Theorem 1 (Corollary 3). One of these congruences immediately yields a reformulation of Giuga's conjecture in terms of the divisibility of $2S_n(n) + n^2$ by n^3 (Proposition 4).

In the next subsection we establish the fact that the congruence (6) holds modulo n^4 whenever $n \geq 5$ is a prime such that $n - 1 \nmid 2k - 2$ ((14) of Proposition 7). Motivated by some particular cases of this congruence and related computations in `Mathematica 8`, we propose several Giuga-Agoh-like conjectures. In particular, Conjecture 12 characterizes Wolstenholme primes as positive integers n such that $S_{n-2}(n) \equiv 0 \pmod{n^3}$.

In Subsection 2.3 we establish two congruences modulo n^3 for two binomial sums involving sums of powers $S_{2i}(n)$ with $i = 0, 1, \ldots, k$ (Proposition 17).

Combining the congruence (5) of Theorem 1 with the Carlitz-von Staudt result for determining $S_{2k}(n) \pmod n$ (Theorem 20), in the last subsection of Section 2, we extend this result modulo n^2 for power sums $S_{2k+1}(n)$ (Theorem 23).

Recall that *Erdős-Moser Diophantine equation* is the equation of the form

$$1^k + 2^k + \cdots + (m-2)^k + (m-1)^k = m^k \tag{7}$$

where $m \geq 2$ and $k \geq 2$ are positive integers. Notice that $(m, k) = (3, 1)$ is the only solution for $k = 1$. In letter to Leo Moser around 1950, Paul Erdős conjectured that such solutions of the above equation do not exist (see [39]). Using remarkably elementary methods, Moser [39] showed in 1953 that if (m, k) is a solution of (7) with $m \geq 2$ and $k \geq 2$, then $m > 10^{10^6}$ and k is even. We believe that Theorem 23 can be useful for study some Erdős-Moser-like Diophantine equations with odd k (see Remark 26).

Proofs of all our results are given in Section 3.

2 Applications of Theorem 1

2.1 Variations of Giuga-Agoh's conjecture

Taking $k = (n-1)/2$ if n is odd, $k = n/2$ if n is even and $k = (n-2)/2$ if n is even into (3) of Theorem 1 we obtain respectively the following three congruences.

Corollary 3. *If n is an odd positive integer, then*

$$2S_n(n) \equiv n^2 S_{n-1}(n) \pmod{n^3}. \tag{8}$$

If n is even, then

$$2S_{n+1}(n) - n(n+1)S_n(n) \equiv \begin{cases} 0 \pmod{n^3}, & \text{if } n \equiv 0 \ (\text{mod } 4); \\ \frac{n^3}{2} \pmod{n^3}, & \text{if } n \equiv 2 \ (\text{mod } 4) \end{cases} \tag{9}$$

and

$$2S_{n-1}(n) \equiv n(n-1)S_{n-2}(n) \pmod{n^3}. \tag{10}$$

In particular, for each even n we have

$$S_{n-1}(n) \equiv \begin{cases} 0 \pmod{n}, & \text{if } n \equiv 0 \ (\text{mod } 4); \\ \frac{n}{2} \pmod{n}, & \text{if } n \equiv 2 \ (\text{mod } 4). \end{cases} \tag{11}$$

Notice that if n is any prime, then by Fermat's little theorem we have $S_{n-1}(n) \equiv -1 \pmod{n}$. In 1950 Giuga [18] conjectured that a positive integer $n \geq 2$ is a prime if and only if $S_{n-1}(n) \equiv -1 \pmod{n}$. This conjecture is related to the sequences A029875, A007850, A198391, A199767, A226365 and A029876 in [41]. The following proposition provides an equivalent formulation of Giuga's conjecture.

Proposition 4. *The following conjectures are equivalent:*

(i) *A positive integer $n \geq 3$ is a prime if and only if*

$$S_{n-1}(n) \equiv -1 \pmod{n}. \tag{12}$$

(ii) *A positive integer $n \geq 3$ is a prime if and only if*

$$2S_n(n) \equiv -n^2 \pmod{n^3}. \tag{13}$$

The above conjecture (ii) is related to the sequence A219540 in [41]. Since by the congruence (11), $S_{n-1}(n) \not\equiv -1 \pmod{n}$ for each even $n \geq 4$, without loss of generality Giuga's conjecture may be restricted to the set of odd positive integers. In view of this fact and the fact that by (8), $n^2 \mid S_n(n)$ for each odd n, Proposition 4 yields the following equivalent formulation of Giuga's conjecture.

Conjecture 5 (Giuga's conjecture)**.** An odd integer $n \geq 3$ is a prime if and only if

$$\frac{2S_n(n)}{n^2} \equiv -1 \pmod{n}.$$

Remark 6. It is known that $S_{n-1}(n) \equiv -1 \pmod{n}$ if and only if for each prime divisor p of n, $(p-1) \mid (n/p-1)$ and $p \mid (n/p-1)$ (see [18], [7, Thm. 1]). Therefore, any counterexample to Giuga's conjecture must be squarefree. Giuga [18] showed that there are no exceptions to the conjecture up to 10^{1000}. In 1985 Bedocchi [5] improved this bound to $n > 10^{1700}$. Finally, in 1996 Borwein et. al. raised the bound to $n > 10^{13887}$. Recently, Luca, Pomerance, and Shparlinski [28] proved that for any real number x, the number of counterexamples to Giuga's conjecture $G(x) := \#\{n < x : n \text{ is composite and } S_{n-1}(n) \equiv -1 \pmod{n}\}$ satisfies the estimate $G(x) = o(\sqrt{x})$ as $x \to \infty$.

Independently, in 1990 Agoh [1] (published in 1995; see also [8] and the sequence A046094 in [41]) conjectured that a positive integer $n \geq 2$ is a prime if and only if $nB_{n-1} \equiv -1$ (mod n). Note that the denominator of the number nB_{n-1} can be greater than 1, but since by the von Staudt-Clausen theorem (1840) (see, e.g., [22, Thm. 118]; cf. the equality (25) given below), the denominator of any Bernoulli number B_{2k} is squarefree, it follows that the denominator of nB_{n-1} is invertible modulo n. In 1996 it was reported by Agoh [7] that his conjecture is equivalent to Giuga's conjecture, and hence the name "Giuga-Agoh conjecture" found in the literature. It was pointed out in [7] that this can be seen from the Bernoulli formula (1) after some analysis involving the von Staudt-Clausen theorem. The equivalence of both conjectures is proved in detail in 2002 by Kellner [24, Satz 3.1.3, Section 3.1, p. 97] (also see [25, Thm. 2.3]).

Quite recently, Grau and Oller-Marcén [20, Corollary 4] proved that an integer n is a counterexample to Giuga's conjecture if and only if it is both a Carmichael and a Giuga number (for definitions and more information on Carmichael numbers see Alford et al. [2] and Banks and Pomerance [3], and for Giuga numbers see Borwein et al. [7], Borwein and Wong [8], and Wong [50, Chapter 2]; also see the sequences A007850 and A002997 in [41]). Furthermore, several open problems concerning Giuga's conjecture can be found in [8, E Open Problems].

2.2 The congruence (3) holds modulo n^4 for a prime $n \geq 5$

The following result shows that for each prime $n \geq 5$ the first congruence of (3) also holds modulo n^4.

Proposition 7. *Let $p \geq 5$ be a prime and let $k \geq 2$ be a positive integer such that $p-1 \nmid 2k-2$. Then*

$$2S_{2k+1}(p) \equiv (2k+1)pS_{2k}(p) \pmod{p^4}. \tag{14}$$

Furthermore, if $p-1 \nmid 2k$, then

$$S_{2k-1}(p) \equiv 0 \pmod{p^2}. \tag{15}$$

As a consequence of Proposition 7, we obtain the following "supercongruence" which generalizes Lemma 2.4 in [32].

Corollary 8. *Let $p \geq 5$ be a prime and let k be a positive integer such that $k \leq (p^4-p^3-4)/2$ and $p-1 \nmid 2k+2$. Then*

$$2R_{2k-1}(p) \equiv (1-2k)pR_{2k}(p) \pmod{p^4} \tag{16}$$

where

$$R_s(p) := \sum_{i=1}^{p-1} \frac{1}{i^s}, \quad s = 1, 2, \ldots.$$

Remark 9. Z.-H. Sun [45, Section 5, Thm. 5.1] in terms of Bernoulli numbers explicitly determined $\sum_{i=1}^{p-1}(1/i^k) \pmod{p^3}$ for each prime $p \geq 5$ and $k = 1, 2, \ldots, p-1$. In particular, substituting the second congruence of Theorem 5.1(a) in [45] (with $2k$ instead of even k) into (14), we immediately obtain the following "supercongruence":

$$R_{2k-1}(p) \equiv \frac{k(1-2k)}{2}\left(\frac{B_{2p-2-2k}}{p-1-k} - 4\frac{B_{p-1-2k}}{p-1-2k}\right)p^2 \pmod{p^4}$$

for all primes $p \geq 7$ and $k = 1, \ldots, (p-5)/2$.

By [45, (5.1) on p. 206],

$$S_{2k}(p) \equiv \frac{p}{3}(3B_{2k} + k(2k-1)p^2B_{2k-2}) \pmod{p^3}, \tag{17}$$

which, inserting into (14), gives

$$S_{2k+1}(p) \equiv \frac{2k+1}{2}p^2B_{2k} \pmod{p^4} \tag{18}$$

for all primes $p \geq 5$ and positive integers $k \geq 2$ such that $p-1 \nmid 2k-2$. Moreover, (14) with $2k = p-1 \geq 4$ (i.e., $p \geq 5$) directly gives

$$S_p(p) \equiv \frac{p^2}{2}S_{p-1}(p) \pmod{p^4}.$$

Taking $2k+1=p$ into (18), we find that

$$S_p(p) \equiv \frac{p^3}{2}B_{p-1} \pmod{p^4},$$

which reducing modulo p^3, and using the congruence $pB_{p-1} \equiv -1 \pmod{p}$, yields $2S_p(p) \equiv -p^2 \pmod{p^3}$. This is actually the congruence (13) of Proposition 4 with a prime $n=p \geq 5$.

Comparing the above two congruences gives $S_{p-1}(p) \equiv pB_{p-1} \pmod{p^2}$ for each prime $p \geq 5$. However, the congruence (17) with $2k=p-1$ implies that for all primes $p \geq 5$

$$S_{p-1}(p) \equiv pB_{p-1} \pmod{p^3}.$$

Remark 10. A computation shows that each of the congruences

$$S_n(n) \equiv \frac{n^3}{2}B_{n-1} \pmod{n^4}$$

and

$$S_{n-1}(n) \equiv nB_{n-1} \pmod{n^3}$$

is also satisfied for numerous odd composite positive integers n. However, we propose the following conjecture.

Conjecture 11. Each of the congruences

$$S_n(n) \equiv \frac{n^3}{2}B_{n-1} \pmod{n^5},$$

$$S_{n-1}(n) \equiv nB_{n-1} \pmod{n^4}$$

is satisfied for none integer $n \geq 2$.

Similarly, taking $k=(p-3)/2$ into (18) for each prime $p \geq 5$ we get

$$S_{p-2}(p) \equiv \frac{(p-2)p^2}{2}B_{p-3} \pmod{p^4}. \tag{19}$$

Therefore, $p^3 \mid S_{p-2}(p)$ if and only if the numerator of the Bernoulli number B_{p-3} is divisible by p, and such a prime is said to be *Wolstenholme prime* (see, e.g., [33, Section 7]). Numerators of Bernoulli numbers B_k $(k=0,1,2,\ldots)$ are given as the sequence A027641 in [41] (cf. also its subsequence A000367 consisting of the terms with even indices k). The only two known such primes are 16843 and 2124679, and by a result of McIntosh and Roettger from [31], these primes are the only two Wolstenholme primes less than 10^9. Wolstenholme primes are given as the sequence A088164 which is a subsequence of irregular primes A000928 in [41] (cf. the sequence A177783). In view of the above congruence, and our computation via `Mathematica 8` up to $n=20000$ we have the following two conjectures.

Conjecture 12. A positive integer $n \geq 2$ is a Wolstenholme prime if and only if

$$S_{n-2}(n) \equiv 0 \pmod{n^3}.$$

Conjecture 13. The congruence

$$S_{n-2}(n) \equiv 0 \pmod{n^4}$$

is satisfied for none integer $n \geq 2$.

Remark 14. Quite recently, inspired by Giuga's conjecture, Grau, Luca, and Oller-Marcén [21] studied the odd positive integers n satisfying the congruence

$$S_{(n-1)/2}(n) \equiv 0 \pmod{n}.$$

Grau et al. [21, Section 2, Proposition 2.1] observed that this congruence is satisfied for each odd prime n and for each odd positive integer $n \equiv 3 \pmod 4$. Notice that if $n = 4k+3$ with $k \geq 0$, then the first part of the congruence (3) yields

$$2S_{(n-1)/2}(n) \equiv \frac{(n-1)n}{2} S_{(n-3)/2}(n) \pmod{n^3}$$

which by the congruence (14) holds modulo n^4 for each prime $n \geq 7$ such that $n \equiv 3 \pmod 4$. Multiplying the above congruence by 2 and reducing the modulus, immediately gives

$$4S_{(n-1)/2}(n) \equiv -nS_{(n-3)/2}(n) \pmod{n^2}.$$

The above congruence shows that $S_{(n-1)/2}(n) \equiv 0 \pmod{n^2}$ for some $n \equiv 3 \pmod 4$ if and only if $S_{(n-3)/2}(n) \equiv 0 \pmod{n}$. Furthermore, reducing the congruence (18) with $k = (p-3)/4$ where $p \geq 7$ is a prime such that $p \equiv 3 \pmod 4$ gives

$$S_{(p-1)/2}(p) \equiv -\frac{p^2}{4} B_{(p-3)/2} \pmod{p^3}, \tag{20}$$

whence it follows that for such a prime p, $S_{(p-1)/2}(p) \equiv 0 \pmod{p^2}$.

On the other hand, if $n \equiv 1 \pmod 4$, that is $n = 4k+1$ with $k \geq 1$, the first part of the congruence (3) yields

$$2S_{(n+1)/2}(n) \equiv \frac{(n+1)n}{2} S_{(n-1)/2}(n) \pmod{n^3}$$

which by the congruence (14) holds modulo n^4 for each prime $n \equiv 1 \pmod 4$. Multiplying the above congruence by 2 and reducing the modulus, immediately gives

$$4S_{(n+1)/2}(n) \equiv nS_{(n-1)/2}(n) \pmod{n^2}.$$

The above congruence shows that $S_{(n-1)/2}(n) \equiv 0 \pmod{n}$ for some $n \equiv 1 \pmod{4}$ if and only if $S_{(n+1)/2}(n) \equiv 0 \pmod{n^2}$. For example, by [21, Proposition 2.3] (cf. Corollary 24 given below) both previous congruences are satisfied for every odd prime power $n = p^{2s+1}$ with any prime $p \equiv 1 \pmod{4}$ and a positive integer s. Moreover, reducing the congruence (17) with $k = (p-1)/4$ where $p \geq 5$ is a prime such that $p \equiv 1 \pmod{4}$ gives

$$S_{(p-1)/2}(p) \equiv pB_{(p-1)/2} \pmod{p^2}. \tag{21}$$

The congruence (21) shows that $S_{(p-1)/2}(p) \equiv 0 \pmod{p^2}$ whenever $p \equiv 1 \pmod{4}$ is an irregular prime for which $B_{(p-1)/2} \equiv 0 \pmod{p}$. To see that the converse is not true, consider the composite number $n = 3737 = 37 \cdot 101$ satisfying $S_{(n-1)/2}(n) \equiv 0 \pmod{n^2}$ (this is the only such a composite number less than 16000).

Nevertheless, in view of the congruences (20) and using arguments similar to those preceding Conjecture 12 (including a computation up to $n = 20000$), we have the following conjecture.

Conjecture 15. An odd positive integer $n \geq 3$ such that $n \equiv 3 \pmod{4}$ satisfies the congruence

$$S_{(n-1)/2}(n) \equiv 0 \pmod{n^3}$$

if and only if n is an irregular prime for which $B_{(n-3)/2} \equiv 0 \pmod{n}$.

We also propose the following conjecture.

Conjecture 16. The congruence

$$S_{(n-1)/2}(n) \equiv 0 \pmod{n^3}$$

is satisfied for none odd positive integer $n \geq 5$ such that $n \equiv 1 \pmod{4}$.

2.3 Two congruences modulo n^3 involving power sums $S_k(n)$

Combining the congruences of Theorem 1 with Pascal's identity, we can arrive to the following congruences.

Proposition 17. *Let k and n be positive integers. Then*

$$2\sum_{i=0}^{k}(1+n(k+1-i))\binom{2k+2}{2i}S_{2i}(n) \equiv -2 \pmod{n^3} \tag{22}$$

and

$$2\sum_{i=0}^{k}\left(\binom{2k+2}{2i}+n(k+1)\binom{2k+1}{2i}\right)S_{2i}(n) \equiv -2 \pmod{n^3}. \tag{23}$$

Remark 18. Clearly, if n is odd, then both congruences (22) and (23) remain also valid after dividing by 2. However, a computation in `Mathematica 8` for small values k and even n suggests that this would be true for each k and even n, and so we have

Conjecture 19. The congruence

$$\sum_{i=0}^{k}(1+n(k+1-i))\binom{2k+2}{2i}S_{2i}(n) \equiv -1 \pmod{n^3}$$

is satisfied for all $k \geq 1$ and even n.

2.4 An extension of Carlitz-von Staudt result for odd power sums

The following congruence is known as a *Carlitz-von Staudt's result* [10] in 1961 (for an easier proof see [37, Thm. 3]).

Theorem 20. ([10], [37, Thm. 3]) *Let k and $n > 1$ be positive integers. Then*

$$S_k(n) \equiv \begin{cases} 0 \pmod{\frac{(n-1)n}{2}}, & \text{if } k \text{ is odd;} \\ -\sum_{(p-1)|k, p|n} \frac{n}{p} \pmod{n}, & \text{if } k \text{ is even,} \end{cases} \tag{24}$$

where the summation is taken over all primes p such that $(p-1) \mid k$ and $p \mid n$.

Remark 21. It is easy to show the first ("odd") part of Theorem 20 (see, e.g., [37, Proof of Theorem 3] or [30, Proposition 1]) whose proof is a modification of Lengyel's arguments in [27]. Recall also that the classical *theorem of Faulhaber* ([16]; also see [4, 14, 26]) states that every sum $S_{2k-1}(n)$ (of odd power) can be expressed as a polynomial in the triangular number $T_{n-1} := (n-1)n/2$ A000217 (cf. the sequences A079618 and A064538 in [41]). For even powers, it has been shown that the sum $S_{2k}(n)$ is a polynomial in the triangular number T_{n-1} multiplied by a linear factor in n (see, e.g., [26]). Quite recently, Dzhumadil'daev and Yeliussizov [14] established an analog of Faulhaber's theorem for a power sum of binomial coefficients.

Remark 22. The second part of the congruence (24) in Theorem 20 can be proved using the famous von Staudt-Clausen theorem (given below) discovered without proof by Clausen [12] in 1840, and independently by von Staudt in 1840 [43]; for alternative proofs, see, e.g., Carlitz [10], Moree [35] or Moree [37, Thm. 3]. This also follows from Chowla's proof of the von Staudt-Clausen theorem given in [11]. Namely, Chowla proved that the difference

$$\frac{S_{2k}(n+1)}{n} - B_{2k}$$

is an integer for all positive integers k and n. This together with the facts that $S_{2k}(n+1) \equiv S_{2k}(n) \pmod{n}$ and that by the *von Staudt-Clausen theorem*,

$$B_{2k} = A_{2k} - \sum_{\substack{(p-1)|2k \\ p \text{ prime}}} \frac{1}{p}, \tag{25}$$

where A_{2k} is an integer, immediately gives the second part of the congruence (24). This theorem is related to the sequences A000146 in [41] (cf. A165908 and A027762 in [41]). Recall also that in many places, the von Staudt-Clausen theorem is stated in the following equivalent statement (see, e.g., [44, page 153]):

$$pB_{2k} \equiv \begin{cases} 0 \pmod{p}, & \text{if } p-1 \nmid 2k; \\ -1 \pmod{p}, & \text{if } p-1 \mid 2k, \end{cases}$$

where p is a prime and k a positive integer.

Combining the congruence (5) in Theorem 1 with the second ("even") part of the congruence (24), we immediately obtain an improvement of its first ("odd") part as follows.

Theorem 23. *Let Let k and n be positive integers. Then*

$$2S_{2k+1}(n) \equiv -(2k+1)n \sum_{(p-1)|2k, p|n} \frac{n}{p} \pmod{n^2}, \tag{26}$$

where the summation is taken over all primes p such that $p-1 \mid k$ and $p \mid n$.

In particular, taking $n = p^s$ and $k = (p^s - 1)/4$ into (26) where p is an odd prime p and $s \geq 1$ such that $p^s \equiv 1 \pmod 4$, we immediately obtain an analogue of Proposition 2.3 in a recent paper [21].

Corollary 24. *Let p be an odd prime. Then*

$$S_{(p^s+1)/2}(p^s) \equiv \begin{cases} 0 \pmod{p^{2s}}, & \text{if } p \equiv 1 \ (\text{mod } 4) \text{ and } s \geq 1 \text{ is odd}; \\ -\frac{p^{2s-1}}{4} \pmod{p^{2s}}, & \text{if } s \geq 2 \text{ is even.} \end{cases}$$

Finally, comparing (24), (25) and (26), we immediately obtain an "odd" extension of a result due to Kellner [25, Thm. 1.2] in 2004 (the congruence (27) given below).

Corollary 25. *Let k and n be positive integers. Then*

$$S_{2k}(n) \equiv nB_{2k} \pmod{n} \quad \text{(Kellner [20])} \tag{27}$$

and

$$2S_{2k+1}(n) \equiv (2k+1)n^2 B_{2k} \pmod{n^2}. \tag{28}$$

Remark 26. Notice also that Theorem 20 plays a key role in a recent study ([17, 35, 37, 38]) of the Erdős-Moser Diophantine equation given by (7). As noticed in Introduction, in 1953 Moser [39] showed that if (m, k) is a solution of the equation (7) with $m \geq 2$ and $k \geq 2$, then $m > 10^{10^6}$ and k is even. Recently, using Theorem 20, Moree [37, Thm. 4] improved the bound on m to $1.485 \cdot 10^{9321155}$. That Theorem 20 can be used to reprove Moser's result was first observed in 1996 by Moree [36], where it played a key role in the study of the more general equation

$$1^k + 2^k + \cdots + (m-2)^k + (m-1)^k = am^k \tag{29}$$

where a is a given positive integer. Moree [36] generalized Erdős-Moser conjecture in the sense that the only solution of the "generalized" Erdős-Moser Diophantine equation (29) is the trivial solution $1 + 2 + \cdots + 2a = a(2a+1)$. Notice also that Moree [36, Proposition 9] proved that in any solution of the equation (29), m is odd. Nevertheless, motivated by the Moser's technique [37, proof of Theorem 3] previously mentioned, to study (7), we believe that Theorem 23 would be applicable in investigations of some other Erdős-Moser type Diophantine equations with odd k.

3 Proofs of Theorem 1, Corollaries 2, 8, Propositions 4, 7 and 17

Proof of Theorem 1. If $k \geq 1$ then by the binomial formula, for each $i = 1, 2, \ldots, n-1$ we have

$$\begin{aligned}
&2(i^{2k+1} + (n-i)^{2k+1}) - (2k+1)n(i^{2k} + (n-i)^{2k})\\
\equiv&2\left(i^{2k+1} - i^{2k+1}) + \binom{2k+1}{1}ni^{2k} - \binom{2k+1}{2}n^2 i^{2k-1}\right)\\
&- (2k+1)n\left(i^{2k} + i^{2k} - \binom{2k}{1}ni^{2k-1}\right) \pmod{n^3}\\
=&2(2k+1)ni^{2k} - 2(2k+1)kn^2 i^{2k-1} - 2(2k+1)ni^{2k} + 2(2k+1)kn^2 i^{2k-1}\\
=&0 \pmod{n^3}.
\end{aligned} \tag{30}$$

If $k \geq 3$ and n is odd then after summation of (30) over $i = 1, 2, \ldots, (n-1)/2$ we obtain

$$2\sum_{i=1}^{n-1} i^{2k+1} - (2k+1)n\sum_{i=1}^{n-1} i^{2k} \equiv 0 \pmod{n^3}. \tag{31}$$

If $k \geq 2$ and n is even then after summation of (30) over $i = 1, 2, \ldots, n/2$ we get

$$2\sum_{i=1}^{n-1} i^{2k+1} + 2\left(\frac{n}{2}\right)^{2k+1} - (2k+1)n\sum_{i=1}^{n-1} i^{2k} - (2k+1)n\left(\frac{n}{2}\right)^{2k} \equiv 0 \pmod{n^3},$$

or equivalently,

$$2S_{2k+1} - (2k+1)nS_{2k} \equiv \frac{kn^{2k+1}}{2^{2k-1}} = \frac{n^3}{2}\cdot k \cdot \left(\frac{n}{2}\right)^{2k-2} \pmod{n^3}. \tag{32}$$

Since for even n

$$\frac{n^3}{2} \cdot k \cdot \left(\frac{n}{2}\right)^{2k-2} \equiv \begin{cases} 0 \pmod{n^3}, & \text{if } k \text{ is even or } n \equiv 0 \pmod 4; \\ \frac{n^3}{2} \pmod{n^3}, & \text{if } k \text{ is odd and } n \equiv 2 \pmod 4), \end{cases}$$

this together with (32) and (31) yields both congruences of (3) in Theorem 1.

Finally, for $k = 1$ we have

$$2S_3(n) - 3nS_2(n) = \frac{n^3}{2} \cdot (1-n) \equiv \begin{cases} 0 \pmod{n^3}, & \text{if } n \text{ is odd;} \\ \frac{n^3}{2} \pmod{n^3}, & \text{if } n \text{ is even.} \end{cases}$$

This completes the proof. □

Proof of Corollary 2. Both assertions follow immediately from the congruence (5) and a result of Damianou and Schumer [13, Thm. 2, p. 222] which asserts that if k is even, then n divides $S_k(n)$ if and only if n is not divisible by any prime p such that $p \mid D_k$, where D_k is the denominator of the kth Bernoulli number B_k. □

Proof of Proposition 4. Proof of (*i*) $\Rightarrow$ (*ii*). Suppose that Giuga's conjecture is true. Then if n is an odd positive integer satisfying the congruence (13) of Proposition 4, using this and (8) of Corollary 3, we find that

$$n^2 S_{n-1}(n) \equiv 2S_n(n) \equiv -n^2 \pmod{n^3},$$

whence we have

$$S_{n-1}(n) \equiv -1 \pmod{n}.$$

By Giuga's conjecture, the above congruence implies that n is a prime.

If $n \geq 4$ is an even positive integer, then the congruence (11) shows that $S_{n-1}(n) \not\equiv -1 \pmod{n}$. We will show that for such a n, $2S_n(n) \not\equiv -n^2 \pmod{n^3}$. Take $n = 2^s(2l-1)$, where s and l are positive integers. Since for $i = 1, 2, \ldots$ we have $(2i)^n \equiv 0 \pmod{2^n}$, this together with the inequality $2^{2^s} \geq 2^{s+1}$ yields $(2i)^n \equiv 0 \pmod{2^{s+1}}$. Therefore, we obtain

$$2S_n(n) \equiv 2 \sum_{\substack{1 \leq j \leq n-1 \\ j \text{ odd}}} j^n \pmod{2^{s+1}}.$$

By Euler's theorem, for each odd j we have

$$j^n = j^{2^s(2l-1)} = \left(j^{2^s}\right)^{2l-1} = \left(j^{\varphi(2^{s+1})}\right)^{2l-1} \equiv 1 \pmod{2^{s+1}} \equiv 1 \pmod{2^s},$$

where $\varphi(m)$ is Euler's totient function. Substituting this into the above congruence, we get

$$2S_n(n) \equiv n = 2^s(2l-1) \not\equiv 0 \pmod{2^{s+1}}.$$

Now, if we suppose that $2S_n(n) \equiv -n^2 \pmod{n^3}$, then must be $2S_n(n) \equiv 0 \pmod{n^2}$, and so, $2S_n(n) \equiv 0 \pmod{2^{2s}} \equiv 0 \pmod{2^{s+1}}$. This contradicts the above congruence, and the implication $(i) \Rightarrow (ii)$ is proved.

Proof of $(ii) \Rightarrow (i)$. Now suppose that Conjecture (ii) of Proposition 4 is true. Then if n is an odd positive integer satisfying the congruence (12), multiplying this by n^2 and using (8) of Corollary 3, we find that

$$2S_n(n) \equiv n^2 S_{n-1}(n) \equiv -n^2 \pmod{n^3},$$

which implies that

$$2S_n(n) \equiv -n^2 \pmod{n^3}.$$

By our Conjecture (ii), the above congruence implies that n is a prime.

If $n \geq 4$ is an even positive integer, then we have previously shown that for such a n, $2S_n(n) \not\equiv -n^2 \pmod{n^3}$ and $S_{n-1}(n) \not\equiv -1 \pmod{n}$. This completes the proof of implication $(ii) \Rightarrow (i)$. □

Proof of Proposition 7. If we extend the congruence (30) modulo n^4, then in the same manner we obtain

$$\begin{aligned} &2(i^{2k+1} + (n-i)^{2k+1}) - (2k+1)n(i^{2k} + (n-i)^{2k}) \\ &\equiv 2\binom{2k+1}{3} n^3 i^{2k-2} - (2k+1)\binom{2k}{2} n^3 i^{2k-2} \pmod{n^4}, \end{aligned}$$

whence it follows that

$$2S_{2k+1}(n) - (2k+1)nS_{2k}(n) \equiv \frac{k(1-4k^2)}{3} n^3 S_{2k-2}(n) \pmod{n^4}. \tag{33}$$

If $n = p$ is a prime such that $p-1 \nmid 2k-2$, then the well known congruence $S_{2k-2}(p) \equiv 0 \pmod{p}$ (see, e.g., [46, the congruence (6.3)] or [29, Thm. 1]) and (33) yield the congruence (14). Finally, (15) immediately follows reducing (14) modulo p^2 and using the previous fact that $S_{2k}(p) \equiv 0 \pmod{p}$ whenever $p-1 \nmid 2k$. □

Remark 27. Applying a result of Damianou and Schumer [13, Thm. 2, p. 222] used in the proof of Corollary 2 to the congruence (33), it follows that

$$2S_{2k+1}(n) \equiv (2k+1)nS_{2k}(n) \pmod{n^4}$$

whenever n is not divisible by any prime p such that $p \mid D_{2k-2}$, where D_{2k-2} is the denominator of the $(2k-2)$th Bernoulli number B_{2k-2}. The converse assertion is true if n is relatively prime to the integer $k(1-4k^2)/3$.

Proof of Corollary 8. By Euler's theorem [22], for all positive integers m and i such that $1 \leq m < p^4 - p^3$ and $1 \leq i \leq p-1$ we have $1/i^m \equiv i^{\varphi(p^4)-m} \pmod{p^4}$, where $\varphi(p^4) = p^4 - p^3$ is the Euler's totient function. Therefore, $R_m \equiv S_{p^4-p^3-m} \pmod{p^4}$. Applying the last congruence for $m = 2k-1$ and $m = 2k$, and substituting this into (14) of Proposition 7 with $p^4 - p^3 - 2k \geq 4$ instead of $2k$, we immediately obtain

$$2R_{2k-1}(p) \equiv (p^4 - p^3 - 2k + 1)pR_{2k}(p) \equiv (1-2k)pR_{2k}(p) \pmod{p^4},$$

as desired. $\square$

Proof of Proposition 17. As $S_0(n) = n-1$ and $S_1(n) = (n-1)n/2$, Pascal's identity (2) yields

$$\begin{aligned} 2(n^{2k+2} - 1) &= 2\sum_{i=0}^{2k+1} \binom{2k+2}{i} S_i(n) \\ &= 2(n-1)(1+(k+1)n) + \sum_{i=1}^{k} \left(2\binom{2k+2}{2i} S_{2i}(n) + 2\binom{2k+2}{2i+1} S_{2i+1}(n) \right). \end{aligned} \tag{34}$$

If n is odd, then multiplying the congruence (6) of Theorem 1 by $\binom{2k+2}{2i+1}$ and using the identity $\binom{2k+2}{2i+1} = \frac{2k+2-2i}{2i+1}\binom{2k+2}{2i}$, we find that

$$\begin{aligned} \binom{2k+2}{2i+1} 2S_{2i+1}(n) &\equiv \frac{2k+2-2i}{2i+1}\binom{2k+2}{2i}(2i+1)nS_{2i}(n) \pmod{n^3} \\ &= (2k+2-2i)\binom{2k+2}{2i} nS_{2i}(n) \pmod{n^3} \end{aligned} \tag{35}$$

for each $i = 1, \ldots, k$. Now substituting (35) into (34), we obtain

$$2(n-1)(1+(k+1)n) + 2\sum_{i=1}^{k}(1+n(k+1-i))\binom{2k+2}{2i} S_{2i}(n) \equiv -2 \pmod{n^3}, \tag{36}$$

which is obviously the same as (22).

If n is even, then since $\binom{2k+2}{2i+1}$ is even (this is true by the identity $\binom{2k+2}{2i+1} = \frac{2(k+1)}{2i+1}\binom{2k+1}{2i}$), we have that $\binom{2k+2}{2i+1}\frac{n^3}{2} \equiv 0 \pmod{n^3}$. This shows that (22) is satisfied for even n and each $i = 1, \ldots, k$, and hence, proceeding in the same manner as in the previous case, we obtain (22).

Further, applying the identities $2i\binom{2k+2}{2i} = (2k+2)\binom{2k+1}{2i-1}$ and $\binom{2k+2}{2i} - \binom{2k+1}{2i-1} = \binom{2k+1}{2i}$, the left hand side of (23) is equal to

$$
\begin{aligned}
&2(1+n(k+1))\sum_{i=0}^{k}\binom{2k+2}{2i}S_{2i}(n)-n\sum_{i=0}^{k}2i\binom{2k+2}{2i}S_{2i}(n)\\
=&2\sum_{i=0}^{k}\binom{2k+2}{2i}S_{2i}(n)+2n(k+1)(n-1)+2n(k+1)\sum_{i=1}^{k}\binom{2k+2}{2i}S_{2i}(n)\\
&-2n(k+1)\sum_{i=1}^{k}\binom{2k+1}{2i-1}S_{2i}(n)\\
=&2\sum_{i=0}^{k}\binom{2k+2}{2i}S_{2i}(n)+2n(k+1)(n-1)\\
&+2n(k+1)\sum_{i=1}^{k}\left(\binom{2k+2}{2i}-\binom{2k+1}{2i-1}\right)S_{2i}(n)\\
=&2\sum_{i=0}^{k}\binom{2k+2}{2i}S_{2i}(n)+2n(k+1)(n-1)+2n(k+1)\sum_{i=1}^{k}\binom{2k+1}{2i}S_{2i}(n)\\
=&2\sum_{i=0}^{k}\binom{2k+2}{2i}S_{2i}(n)+2n(k+1)\sum_{i=0}^{k}\binom{2k+1}{2i}S_{2i}(n)\\
=&2\sum_{i=0}^{k}\left(\binom{2k+2}{2i}+n(k+1)\binom{2k+1}{2i}\right)S_{2i}(n).
\end{aligned}
$$

Comparing the above equality with (22) immediately gives (23). □

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