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# A Note on Extended Binomial Coefficients

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**Abstract**

We study the distribution of the extended binomial coefficients by deriving a complete asymptotic expansion with uniform error terms. We obtain the expansion from a local central limit theorem and we state all coefficients explicitly as sums of Hermite polynomials and Bernoulli numbers.

## 1 Introduction

The extended binomial coefficients, occasionally called polynomial coefficients [5, p. 77], are defined as the coefficients in the expansion

$$\sum_{k=0}^{\infty} \binom{n}{k}^{(q)} x^k = \left(1 + x + x^2 + \cdots + x^q\right)^n, \quad n, q \in \mathbb{N} = \{1, 2, \ldots\}. \tag{1}$$

In written form, they presumably appeared for the first time in the work by De Moivre [6, p. 41] and later they also were addressed by Euler [9]. Since then, the extended binomial coefficients played a role mainly in the theory of compositions of integers, as the number $c(k, n, q)$ of compositions of $k$ with $n$ parts not exceeding $q$ is given by [10, p. 45]

$$c(k, n, q) = \binom{n}{k-n}^{(q-1)}.$$

Thus, the extended binomial coefficients and their modifications have been studied in various papers and from different perspectives [1, 2, 3, 4, 7, 8, 12, 13, 15], and among the properties their distribution is of particular interest. Recently, Eger [8] showed (using slightly different notation) that

$$\binom{n}{nq/2}^{(q)} \sim \frac{(q+1)^n}{\sqrt{2\pi n \frac{q(q+2)}{12}}},$$

as $n \to \infty$, meaning that the quotient of both sides tends to unity. Moreover, based upon numerical simulations [8] the question arises how well those coefficients can be approximated by "normal approximations" in general. It is the aim of this note to give a precise and comprehensive answer to this question by establishing a complete asymptotic expansion for the extended binomial coefficients with error terms holding uniformly with respect to all integer $k$. More precisely, we show the following.

**Theorem 1.** *For all integers $N \geq 2$ we have*

$$\sqrt{\frac{q(q+2)n}{12}} \frac{1}{(1+q)^n} \binom{n}{k}^{(q)} = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} + \sum_{\nu=1}^{\lfloor (N-2)/2 \rfloor} \frac{q_{2\nu}(x)}{n^\nu} + o\left(\frac{1}{n^{(N-2)/2}}\right),$$

*as $n \to \infty$, uniformly with respect to all $k \in \mathbb{Z}$, with*

$$x = \frac{\sqrt{12}}{\sqrt{q(q+2)n}} \left(k - \frac{q}{2} n\right),$$

*where the functions $q_{2\nu}(x)$ are given explicitly as sums of Hermite polynomials and Bernoulli numbers (see Theorem 5 below for the exact formulas).*

Although we only deal with the very basic situation of the extended binomial coefficients in (1) here, the presented approach is a general one, which admits the derivation of (complete) asymptotic expansions in many applications. However, usually it is not possible to obtain the involved quantities in a very explicit form, which is an instance making the case of extended binomial coefficients especially interesting.

A general overview of the analytic theory of compositions can be found in Flajolet and Sedgewick's standard book [10], where essentially two asymptotic results on restricted compositions are given [10, pp. 43–44]. The first one states that the numbers $C_k^{\{1,\dots,q\}}$ of compositions of $k$ with parts restricted to $\{1, \dots, q\}$ asymptotically behave like

$$C_k^{\{1,\dots,q\}} \sim c_q \rho_q^{-k},$$

as $k \to \infty$, for fixed $q \geq 2$. Here, $c_q > 0$ is some constant and $\rho_q$ is the singularity of the associated generating function located in the interval $\left(\frac{1}{2}, 1\right)$. The second result deals with the number $C_k^{(n)}$ of compositions of $k$ having $n$ parts. As these numbers are given explicitly by

$$C_k^{(n)} = \binom{k-1}{n-1},$$

we immediately obtain the asymptotic formula

$$C_k^{(n)} \sim \frac{k^{n-1}}{(n-1)!},$$

as $k \to \infty$, for fixed $n$. Interpreting the extended binomial coefficients as numbers of restricted compositions, in the present paper we are concerned with the numbers $c(k, n, q)$ counting compositions of $k$ with $n$ parts restricted to $\{1, \ldots, q\}$, which can be considered as a mixed type of restricted compositions in the above sense. For fixed integer $q$, the result in Theorem 5 gives a complete and explicit description of the behavior of $c(k, n, q)$ for large values of $n$, valid uniformly in $k$, meaning that with $n$ growing to infinity it is not necessary to specify the way $k$ tends to infinity. This feature usually is not available by methods in the context of singularity analysis.

# 2 Proof of the main result

Our approach is based on an application of a local central limit theorem. To this end, we choose a sequence of independent random variables with common uniform distribution on the integers $\{0, \ldots, q\}$. This way, the extended binomial coefficients can be represented (up to a normalization) as certain probabilities for the sums of the random variables. Before stating the details, we will fix some notation following Petrov [14]. For a (real) random variable $X$ we denote its characteristic function by

$$\varphi_X(t) = Ee^{itX}, \quad t \in \mathbb{R},$$

where, as usual, $E$ means the mathematical expectation with respect to the underlying probability distribution. If $X$ has finite moments up to $k$-th order, then $\varphi_X$ is $k$ times continuously differentiable on $\mathbb{R}$ and we have

$$\frac{d^k}{dt^k}\varphi_X(t)\Big|_{t=0} = \frac{1}{i^k} EX^k.$$

Moreover, in this case we define the cumulants of order $k$ by

$$\gamma_k = \frac{1}{i^k}\frac{d^k}{dt^k} \log \varphi_X(t)\Big|_{t=0},$$

where the logarithm takes its principal branch. Now, let $(X_n)$ be a sequence of independent integer-valued random variables having a common distribution and suppose that for all positive integer values of $k$ we have

$$E|X_1|^k < \infty$$

and

$$EX_1 = \mu, \quad VarX_1 = \sigma^2 > 0.$$

Thus, for the sum given by

$$S_n = \sum_{\nu=1}^{n} X_\nu$$

we obtain

$$ES_n = n\mu, \quad VarS_n = n\sigma^2,$$

and for integer $k$ we define the probabilities

$$p_n(k) = P\left(S_n = k\right).$$

Furthermore, we introduce the Hermite polynomials (in the probabilist's version)

$$H_m(x) = (-1)^m e^{x^2/2} \frac{d^m}{dx^m} e^{-x^2/2},$$

and for positive integers $\nu$ we define the functions

$$q_\nu(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \sum_{\substack{k_1,\ldots,k_\nu \geq 0 \\ k_1+2k_2+\cdots+\nu k_\nu=\nu}} H_{\nu+2s}(x) \prod_{m=1}^{\nu} \frac{1}{k_m!} \left( \frac{\gamma_{m+2}}{(m+2)!\sigma^{m+2}} \right)^{k_m}, \tag{2}$$

where $s = k_1 + \cdots + k_\nu$ and $\gamma_{m+2}$ denotes the cumulant of order $m+2$ of $X_1$.

Finally, we demand (for convenience) that the maximal span of the distribution of $X_1$ is equal to one. This means that there are no numbers $a$ and $h > 1$ such that the values taken on by $X_1$ with probability one can be expressed in the form $a + hk$ ($k \in \mathbb{Z}$). Under all these assumptions we have the following complete asymptotic expansion in the sense of a local central limit theorem [14, p. 205].

**Theorem 2.** *For all integers $N \geq 2$ we have*

$$\sigma\sqrt{n} p_n(k) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} + \sum_{\nu=1}^{N-2} \frac{q_\nu(x)}{n^{\nu/2}} + o\left(\frac{1}{n^{(N-2)/2}}\right), \tag{3}$$

*as $n \to \infty$, uniformly with respect to all $k \in \mathbb{Z}$, where we have*

$$x = \frac{k - n\mu}{\sigma\sqrt{n}}.$$

In the following we choose $X_1$ to take the integer values $\{0, \ldots, q\}$ with

$$P(X_1 = k) = \frac{1}{q+1}, \quad k \in \{0, \ldots, q\}.$$

Hence, we obtain

$$p_n(k) = P(S_n = k) = \frac{1}{(1+q)^n} \binom{n}{k}^{(q)}, \quad k \in \mathbb{Z}. \tag{4}$$

It is our aim to apply Theorem 2 in full generality and we want to compute all cumulants as explicitly as possible.

**Lemma 3.** *For the $k$-th order cumulant $\gamma_k$ of $X_1$ we have*

$$\gamma_k = \begin{cases} \frac{q}{2}, & \text{if } k = 1; \\ 0, & \text{if } k \text{ odd and } k > 1; \\ \frac{\mathcal{B}_{2l}}{2l}\left((q+1)^{2l} - 1\right), & \text{if } k = 2l, l \geq 1, \end{cases} \tag{5}$$

*where $\mathcal{B}_\nu$, $\nu \geq 0$, denotes the Bernoulli numbers [11, p. 22].*

*Proof.* First, we observe that the characteristic function of $X_1$ is given by

$$\varphi_{X_1}(t) = \frac{1 + e^{it} + \cdots + e^{qit}}{1+q}.$$

According to the definition of the cumulants we obtain for a positive integer $k$

$$\begin{aligned} \gamma_k &= \frac{1}{i^k}\frac{d^k}{dt^k}\log \varphi_{X_1}(t)\Big|_{t=0} \\ &= \frac{1}{i^k}\frac{d^k}{dt^k}\left\{\log\left(1 + e^{it} + \cdots + e^{qit}\right) - \log(1+q)\right\}\Big|_{t=0} \\ &= \frac{1}{i^k}\frac{d^k}{dt^k}\log\left(\frac{e^{(q+1)it} - 1}{e^{it} - 1}\right)\Big|_{t=0} \\ &= \frac{1}{i^k}\frac{d^k}{dt^k}\left\{\frac{q}{2}it + \log\left(\frac{\sin\frac{q+1}{2}t}{\sin\frac{t}{2}}\right)\right\}\Big|_{t=0} \\ &= \frac{q}{2}\delta_{k,1} + \frac{1}{i^k}\frac{d^k}{dt^k}\left\{\log\left(\frac{\sin\frac{q+1}{2}t}{\frac{q+1}{2}t}\right) - \log\left(\frac{\sin\frac{t}{2}}{\frac{t}{2}}\right)\right\}\Big|_{t=0}, \end{aligned}$$

where $\delta_{k,1}$ denotes the Kronecker delta. Using

$$\frac{d}{dz}\log\left(\frac{\sin z}{z}\right) = \operatorname{cotan} z - \frac{1}{z}$$

yields

$$\gamma_k = \frac{q}{2}\delta_{k,1} + \frac{1}{i^k}\frac{d^{k-1}}{dt^{k-1}}\left\{\frac{q+1}{2}\left(\operatorname{cotan}\frac{q+1}{2}t - \frac{2}{(q+1)t}\right) - \frac{1}{2}\left(\operatorname{cotan}\frac{t}{2} - \frac{2}{t}\right)\right\}\Big|_{t=0}.$$

Now, making use of the following expansion [11, p. 35]

$$\operatorname{cotan} z - \frac{1}{z} = \sum_{m=1}^{\infty}(-1)^m\frac{4^m}{(2m)!}\mathcal{B}_{2m}z^{2m-1} \quad , \quad 0 < |z| < \pi,$$

after some algebra we obtain

$$\gamma_k = \frac{q}{2}\delta_{k,1} + \frac{1}{i^k}\frac{d^{k-1}}{dt^{k-1}}\sum_{m=1}^{\infty}(-1)^m\frac{\mathcal{B}_{2m}}{(2m)!}\left((q+1)^{2m} - 1\right)t^{2m-1}\Big|_{t=0}.$$

Carrying out the differentiation under the summation sign immediately gives us (5). ∎

*Remark* 4. As an immediate consequence of Lemma 3 we obtain

$$EX_1 = \mu = \gamma_1 = \frac{q}{2}$$

and, as we know $\mathcal{B}_2 = \frac{1}{6}$,

$$VarX_1 = \sigma^2 = \gamma_2 = \frac{\mathcal{B}_2}{2}\left((q+1)^2 - 1\right) = \frac{q(q+2)}{12}.$$

We now are ready to state the main theorem in form of a complete asymptotic expansion with explicit coefficients for the extended binomial coefficients $\binom{n}{k}^{(q)}$.

**Theorem 5.** *For all integers $N \geq 2$ we have*

$$\sqrt{\frac{q(q+2)n}{12}}\frac{1}{(1+q)^n}\binom{n}{k}^{(q)} = \frac{1}{\sqrt{2\pi}}e^{-x^2/2} + \sum_{\nu=1}^{\lfloor (N-2)/2 \rfloor} \frac{q_{2\nu}(x)}{n^\nu} + o\left(\frac{1}{n^{(N-2)/2}}\right),$$

*as $n \to \infty$, uniformly with respect to all $k \in \mathbb{Z}$, with*

$$x = \frac{\sqrt{12}}{\sqrt{q(q+2)n}}\left(k - \frac{q}{2}n\right),$$

*and*

$$\begin{aligned} q_{2\nu}(x) &= \frac{1}{\sqrt{2\pi}}\left(\frac{12}{q(q+2)}\right)^\nu e^{-x^2/2} \\ &\times \sum_{\substack{k_2,k_4,\ldots,k_{2\nu}\geq 0 \\ k_2+2k_4+\cdots+\nu k_{2\nu}=\nu}} H_{2(\nu+s)}(x)\left(\frac{6}{q(q+2)}\right)^s \prod_{m=1}^{\nu}\frac{1}{k_{2m}!}\left(\frac{\mathcal{B}_{2(m+1)}\left((q+1)^{2m+2}-1\right)}{(2m+2)!(m+1)}\right)^{k_{2m}}, \end{aligned} \tag{6}$$

*where $s = k_2 + k_4 + \cdots + k_{2\nu}$.*

*Proof.* The proof is based on an application of Theorem 2 to the probabilities defined in (4). First we observe that in our situation the functions given in (2) vanish identically for odd indices, which turns out to be a consequence of (5). Indeed, if $\nu = 2l+1$ for an integer $l \geq 0$, then in every solution $k_1, \ldots, k_{2l+1} \geq 0$ of the equation

$$k_1 + 2k_2 + \cdots + (2l+1)k_{2l+1} = 2l+1$$

there is at least one odd index $i$ with $k_i > 0$. Consequently, using (5) we have

$$\prod_{m=1}^{2l+1}\frac{1}{k_m!}\left(\frac{\gamma_{m+2}}{(m+2)!\sigma^{m+2}}\right)^{k_m} = 0,$$

from which follows that $q_{2l+1}(x)$ vanishes identically. Thus, only the functions $q_{2\nu}(x)$ appear in (3) and here we have

$$q_{2\nu}(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \sum_{\substack{k_1,\ldots,k_{2\nu}\geq 0 \\ k_1+2k_2+\cdots+2\nu k_{2\nu}=2\nu}} H_{2(\nu+s)}(x) \prod_{m=1}^{2\nu} \frac{1}{k_m!} \left( \frac{\gamma_{m+2}}{(m+2)!\sigma^{m+2}} \right)^{k_m},$$

where $s = k_1 + \cdots + k_{2\nu}$. An analogous argument as in the odd case above shows that a solution $k_1, \ldots, k_{2\nu}$ of the equation

$$k_1 + 2k_2 + \cdots + 2\nu k_{2\nu} = 2\nu$$

with a positive entry at an odd index does not give any contribution to the whole sum, so that we can write

$$q_{2\nu}(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \sum_{\substack{k_2,k_4,\ldots,k_{2\nu}\geq 0 \\ k_2+2k_4+\cdots+\nu k_{2\nu}=\nu}} H_{2(\nu+s)}(x) \prod_{m=1}^{\nu} \frac{1}{k_{2m}!} \left( \frac{\gamma_{2m+2}}{(2m+2)!\sigma^{2m+2}} \right)^{k_{2m}},$$

where $s = k_2 + k_4 + \cdots + k_{2\nu}$. Now, taking the explicit form of the cumulants in (5) into account, after some elementary computation we obtain (6). □

For the purpose of illustration we state Theorem 5 for $N = 5$ explicitly.

**Example 6.** Using the known facts

$$H_4(x) = x^4 - 6x^2 + 3, \quad \mathcal{B}_4 = -\frac{1}{30},$$

we obtain

$$\sqrt{\frac{q(q+2)n}{12}} \frac{1}{(1+q)^n} \binom{n}{k}^{(q)} = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \left\{ 1 - \frac{\left((q+1)^4 - 1\right)\left(x^4 - 6x^2 + 3\right)}{20nq^2(q+2)^2} \right\} + o\left( \frac{1}{n^{3/2}} \right),$$

as $n \to \infty$, uniformly with respect to all $k \in \mathbb{Z}$, where we have

$$x = \frac{\sqrt{12}}{\sqrt{q(q+2)n}} \left( k - \frac{q}{2} n \right).$$

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