Journal of Integer Sequences, Vol. 17 (2014),
Article 14.11.2

Exact Divisibility by Powers of the Fibonacci and Lucas Numbers

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Abstract

We give new results on exact divisibility by powers of the Fibonacci and Lucas numbers. For example, we prove that if F_n^k exactly divides m and n is not congruent to 3 modulo 6, then F_n^{k+1} exactly divides F_{nm}. We also provide some examples and open questions.

1 Introduction

Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$, and let $(L_n)_{n\geq 0}$ be the Lucas sequence given by $L_0 = 2$, $L_1 = 1$ with the same recursive pattern as the Fibonacci sequence.

The divisibility property of Fibonacci and Lucas numbers has always been a popular area of research. In particular, the divisibility by powers of the Fibonacci numbers became a popular topic when Matijasevič [10, 11, 12] proved in 1970 that

$$F_n^2 \mid F_{nm} \text{ if and only if } F_n \mid m \tag{1}$$

which led to the solution to Hilbert's 10th Problem. Since (1) appeared, there have been several results [2, 5, 9, 13, 14, 17] on divisibility by powers of the Fibonacci numbers in the literature. However, exact divisibility by powers of the Fibonacci and Lucas numbers seems to be new. (Recall that for integers a, $d \geq 2$ and $k \geq 1$, we say that d^k exactly divides a and write $d^k \parallel a$ if $d^k \mid a$ and $d^{k+1} \nmid a$). Now let us consider the following divisibility:

$$F_n^{k+1} \mid F_{nF_n^k}. \tag{2}$$

Marques [9, p. 241] mentioned that to the best of his knowledge, (2) was first proved by Benjamin and Rouse [2], using a combinatorial approach, and a second proof of (2) is due to Seibert and Trojovsky [16] by using mathematical induction together with an identity for $\frac{F_{nm}}{F_m}$. Marques [9] himself also gave another proof of (2) by applying Lengyel's theorem [8] on the p-adic order of Fibonacci and Lucas numbers. However, there was an older result which implied (2); it was proved in 1977 by Hoggatt and Bicknell-Johnson [5], as follows:

Theorem 1. *[2, 5] Let k, m, and n be positive integers. If $F_n^k \mid m$, then $F_n^{k+1} \mid F_{nm}$.*

Note that, from Theorem 1, we can easily obtain (2) by substituting $m = F_n^k$, and obtain one direction of (1) by substituting $k = 1$. Now it is natural to ask if $k+1$ in Theorem 1 is the largest exponent of F_n such that F_n^{k+1} divides F_{nm}. Our purpose is to give another proof of Theorem 1, and generalize the result to include the divisibility and exact divisibility by powers of the Fibonacci and Lucas numbers. Our results are as follows:

Theorem 2. *Let k, m, n be positive integers and $n \geq 3$. Then*

(i) if $F_n^k \parallel m$ and $n \not\equiv 3 \pmod 6$, then $F_n^{k+1} \parallel F_{nm}$;

(ii) if $F_n^k \parallel m$, $n \equiv 3 \pmod 6$, and $\frac{F_n^{k+1}}{2} \nmid m$, then $F_n^{k+1} \parallel F_{nm}$;

(iii) if $F_n^k \parallel m$, $n \equiv 3 \pmod 6$, and $\frac{F_n^{k+1}}{2} \mid m$, then $F_n^{k+2} \parallel F_{nm}$.

Theorem 3. *Let k, m, n be positive integers and m is odd. Then*

(i) if $L_n^k \mid m$, then $L_n^{k+1} \mid L_{nm}$;

(ii) if $n \geq 2$ and $L_n^k \parallel m$, then $L_n^{k+1} \parallel L_{nm}$.

Theorem 4. *Let k, m, and n be positive integers, m is even, and $n \geq 2$. Then the following statements hold.*

(i) If $L_n^k \mid m$, then $L_n^{k+1} \mid F_{nm}$;

(ii) If $L_n^k \parallel m$ and $n \not\equiv 0 \pmod 3$, then $L_n^{k+1} \parallel F_{nm}$;

(iii) If $L_n^k \parallel m$, $n \equiv 0 \pmod 6$, and $\frac{L_n^{k+1}}{2} \nmid m$, then $L_n^{k+1} \parallel F_{nm}$;

(iv) If $L_n^k \parallel m$, $n \equiv 0 \pmod 6$, and $\frac{L_n^{k+1}}{2} \mid m$, then $L_n^{k+2} \mid F_{nm}$;

(v) If $L_n^k \parallel m$, $n \equiv 3 \pmod 6$, and $\frac{L_n^{k+1}}{4} \nmid m$, then $L_n^{k+1} \parallel F_{nm}$;

(vi) If $L_n^k \parallel m$, $n \equiv 3 \pmod 6$, and $\frac{L_n^{k+1}}{4} \mid m$, then $L_n^{k+2} \parallel 4F_{nm}$.

To prove the above theorems, we will need a number of lemmas given in the next section.

2 Preliminaries and lemmas

In this section, we will give some auxiliary results that will be used in this article. Let m, n, and r be positive integers. Then the following statements hold.

$$\gcd(F_m, F_n) = F_{\gcd(m,n)}. \tag{3}$$

$$F_{nm} = \sum_{j=1}^{m} \binom{m}{j} F_n^j F_{n-1}^{m-j} F_j. \tag{4}$$

$$\text{If } m \geq 3 \text{, then } F_m \mid F_n \text{ if and only if } m \mid n. \tag{5}$$

$$L_n = F_{n-1} + F_{n+1}. \tag{6}$$

$$\gcd(L_n, F_n) = \begin{cases} 1, & \text{if } 3 \nmid n; \\ 2, & \text{if } 3 \mid n. \end{cases} \tag{7}$$

For the reader's convenience, let us give references for the above statements. Properties (3) and (5) can be found in [7, pp. 196–198]. Property (4) can be found in several articles such as [2, 4, 5]. Property (6) appears in [7, p. 80], and (7) can be proved by using (6), (3), and (5). We refer the reader to [1, 3, 6, 7, 15, 18] for more details and additional references.

Next we give the following results, similar to (4).

$$L_{nm} \cdot 2^{m-1} = \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{m-2j} L_n^{m-2j} F_n^{2j} 5^j. \tag{8}$$

$$F_{nm} \cdot 2^{m-1} = \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m}{m-2j-1} L_n^{m-(2j+1)} F_n^{2j+1} 5^j. \tag{9}$$

These are probably less well known, so we will give a proof for completeness.

Proof. Recall Binet's formula that $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ and $L_n = \alpha^n + \beta^n$, where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Then $\alpha^n = \frac{1}{2}(L_n + (\alpha - \beta)F_n) = \frac{1}{2}\left(L_n + \sqrt{5}F_n\right)$, and $\beta^n = \frac{1}{2}(L_n - (\alpha - \beta)F_n) = \frac{1}{2}\left(L_n - \sqrt{5}F_n\right)$. Then

$$\begin{aligned} L_{nm} = \alpha^{nm} + \beta^{nm} &= \frac{1}{2^m}\left(L_n + \sqrt{5}F_n\right)^m + \frac{1}{2^m}\left(L_n - \sqrt{5}F_n\right)^m \\ &= \frac{1}{2^m} \sum_{k=0}^{m} \binom{m}{m-k} L_n^{m-k} (\sqrt{5}F_n)^k \left(1 + (-1)^k\right). \end{aligned}$$

When k is odd, $1+(-1)^k = 0$, so we replace k by $2j$ and obtain

$$L_{nm} = \frac{1}{2^{m-1}} \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{m-2j} L_n^{m-2j} F_n^{2j} 5^j,$$

which gives (8). For (9), we write

$$\begin{aligned}
\sqrt{5}F_{nm} = (\alpha-\beta)F_{nm} = \alpha^{nm} - \beta^{nm} &= \frac{1}{2^m}(L_n + \sqrt{5}F_n)^m - \frac{1}{2^m}(L_n - \sqrt{5}F_n)^m \\
&= \frac{1}{2^m} \sum_{k=0}^{m} \binom{m}{m-k} L_n^{m-k} (\sqrt{5}F_n)^k \left(1-(-1)^k\right).
\end{aligned}$$

As in the previous case, we replace k by $2j+1$ to obtain (9). $\square$

The next lemma is about the p-adic orders of Fibonacci and Lucas numbers given by Lengyel [8]. Recall that the order of appearance of m and the period modulo m of the Fibonacci sequence, respectively, is the smallest positive integer k such that $m \mid F_k$ and the smallest positive integer k such that $F_{n+k} \equiv F_n \pmod{m}$ for all $n \geq 1$.

Lemma 5. *[8] For each $m \in \mathbb{N}$, let $\nu_p(m)$ be the p-adic order of m, $z(m)$ the order of appearance of m, and $s(m)$ the period modulo m of the Fibonacci sequence. For $n \geq 1$, we have*

$$\nu_2(F_n) = \begin{cases} 0, & \text{if } n \equiv 1,2 \pmod{3}; \\ 1, & \text{if } n \equiv 3 \pmod{6}; \\ \nu_2(n)+2, & \text{if } n \equiv 0 \pmod{6}, \end{cases}$$

$$\nu_2(L_n) = \begin{cases} 0, & \text{if } n \equiv 1,2 \pmod{3}; \\ 2, & \text{if } n \equiv 3 \pmod{6}; \\ 1, & \text{if } n \equiv 0 \pmod{6}, \end{cases}$$

$\nu_5(F_n) = \nu_5(n)$, $\nu_5(L_n) = 0$, and if p is a prime, $p \neq 2$ and $p \neq 5$, then

$$\nu_p(F_n) = \begin{cases} \nu_p(n) + \nu_p(F_{z(p)}), & \text{if } n \equiv 0 \pmod{z(p)}; \\ 0, & \text{otherwise.} \end{cases}$$

$$\nu_p(L_n) = \begin{cases} \nu_p(n) + \nu_p(F_{z(p)}), & \text{if } s(p) \neq 4z(p) \text{ and } n \equiv \frac{z(p)}{2} \pmod{z(p)}; \\ 0, & \text{otherwise.} \end{cases}$$

The next result given by Onphaeng and Pongsriiam [13] is an important tool for obtaining the main results of this article.

Lemma 6. *[13] Let k, ℓ, m, s be positive integers and $s^k \mid m$. Then $s^{k+\ell} \mid \binom{m}{j} s^j$ for all $1 \leq j \leq m$ satisfying $2^{j-\ell+1} > j$. In particular, $s^{k+1} \mid \binom{m}{j} s^j$ for all $1 \leq j \leq m$, and $s^{k+2} \mid \binom{m}{j} s^j$ for all $3 \leq j \leq m$.*

3 Proof of the main results

In this section, we will give the proof of Theorems 1, 2, 3, and 4.

Another proof of Theorem 1. Assume that $F_n^k \mid m$. Then by Lemma 6, $F_n^{k+1} \mid \binom{m}{j} F_n^j$ for all $1 \le j \le m$. Therefore $F_n^{k+1} \mid F_{nm}$ by (4). $\square$

Proof of Theorem 2. Assume that $F_n^k \parallel m$. By Theorem 1, $F_n^{k+1} \mid F_{nm}$. So to show that $F_n^{k+1} \parallel F_{nm}$, it is enough to show that $F_n^{k+2} \nmid F_{nm}$. First we write

$$m = F_n^k c \quad \text{where } c \ge 1 \text{ and } F_n \nmid c. \tag{10}$$

By Lemma 6, $F_n^{k+2} \mid \binom{m}{j} F_n^j$ for all $3 \le j \le m$. Then we obtain by (4) that

$$F_{nm} \equiv m F_n F_{n-1}^{m-1} + \frac{m(m-1)}{2} F_n^2 F_{n-1}^{m-2} \pmod{F_n^{k+2}}. \tag{11}$$

Proof of (i):

Case 1 of (i): $n \equiv 1, 2, 4, 5 \pmod 6$. Then by Lemma 5, F_n is odd. Since $F_n^k \mid 2\left(\frac{m(m-1)}{2}\right)$ and F_n is odd, we obtain that $F_n^k \mid \frac{m(m-1)}{2}$. Therefore

$$F_n^{k+2} \left| \frac{m(m-1)}{2} F_n^2 F_{n-1}^{m-2} \right. . \tag{12}$$

By (11) and (12), we obtain $F_{nm} \equiv m F_n F_{n-1}^{m-1} \pmod{F_n^{k+2}}$. Since $\gcd(F_n, F_{n-1}) = 1$ with respect to (3), we see that

$$F_n^{k+2} \mid F_{nm} \;\Leftrightarrow\; F_n^{k+2} \mid m F_n F_{n-1}^{m-1} \;\Leftrightarrow\; F_n^{k+1} \mid m. \tag{13}$$

Since $F_n^k \parallel m$, $F_n^{k+1} \nmid m$. So we obtain by (13) that $F_n^{k+2} \nmid F_{nm}$.

Case 2 of (i): $n \equiv 0 \pmod 6$. Then by Lemma 5 we have $F_n \equiv 0 \pmod 4$. Let

$$d = F_{n-1} + (m-1)\frac{F_n}{2}.$$

Since F_n is even,

$$d \text{ is an integer and } \left(\frac{F_n}{2}, d\right) = 1 \tag{14}$$

By (11) and (10), we obtain

$$F_{nm} \equiv m F_n F_{n-1}^{m-2}\left(F_{n-1} + \frac{m-1}{2} F_n\right) \equiv c F_n^{k+1} F_{n-1}^{m-2} d \pmod{F_n^{k+2}}.$$

Suppose, in order to get a contradiction, that $F_n^{k+2} \mid F_{nm}$. Then $F_n^{k+2} \mid cF_n^{k+1}F_{n-1}^{m-2}d$ which implies that $F_n \mid cd$. By $F_n \equiv 0 \pmod 4$, $F_n \mid cd$, and (14) we obtain that d is odd, c is even, and $\frac{F_n}{2} \mid \frac{c}{2}d$. This and (14) imply that $F_n \mid c$, which contradicts (10). Hence $F_n^{k+2} \nmid F_{nm}$.

Proof of (ii):

Assume that $n \equiv 3 \pmod 6$ and $\frac{F_n^{k+1}}{2} \nmid m$. Then by Lemma 5 and (10), respectively, we obtain $\nu_2(F_n) = 1$ and $\frac{F_n}{2} \nmid c$. We can follow the argument in Case 2 of (i) to see that $F_{nm} \equiv cF_n^{k+1}F_{n-1}^{m-2}d \pmod{F_n^{k+2}}$, and that $F_n^{k+2} \mid F_{nm}$ if and only if $F_n \mid cd$. But if $F_n \mid cd$, then $\frac{F_n}{2} \mid cd$ which implies by (14) that $\frac{F_n}{2} \mid c$, a contradiction. So $F_n \nmid cd$ and therefore $F_n^{k+2} \nmid F_{nm}$. Hence $F_n^{k+1} \parallel F_{nm}$.

Proof of (iii):

Assume that $n \equiv 3 \pmod 6$ and $\frac{F_n^{k+1}}{2} \mid m$. In this case, we will show that $F_n^{k+2} \parallel F_{nm}$. We can still follow the argument in Case 2 of (i) and the proof of (ii) to obtain $F_{nm} \equiv cF_n^{k+1}F_{n-1}^{m-2}d \pmod{F_n^{k+2}}$ and $F_n^{k+2} \mid F_{nm}$ if and only if $F_n \mid cd$. Since $n \equiv 3 \pmod 6$, F_n is even and $\frac{F_n}{2}$ is odd. Therefore F_{n-1} and $m-1$ are odd and d is even. Since $m = F_n^k c$ and $\frac{F_n^{k+1}}{2} \mid m$, we obtain $F_n \mid 2c$. Since $2c \mid cd$, we obtain $F_n \mid cd$ and therefore $F_n^{k+2} \mid F_{nm}$. It remains to show that $F_n^{k+3} \nmid F_{nm}$. Suppose for a contradiction that $F_n^{k+3} \mid F_{nm}$. So for every prime p dividing F_n, $\nu_p(F_n^{k+3}) \leq \nu_p(F_{nm})$. By Lemma 5, we obtain the following inequalities:

$$\begin{aligned}
2 + \nu_2(F_n^{k+1}) = \nu_2(F_n^{k+3}) &\leq \nu_2(F_{nm}) = \nu_2(nm) + 2 = \nu_2(m) + 2,\\
2\nu_5(n) + \nu_5(F_n^{k+1}) = \nu_5(F_n^{k+3}) &\leq \nu_5(F_{nm}) = \nu_5(n) + \nu_5(m),
\end{aligned}$$

and for every prime $p \notin \{2, 5\}$ and $p \mid F_n$, we have

$$\begin{aligned}
2\nu_p(F_n) + \nu_p(F_n^{k+1}) = \nu_p(F_n^{k+3}) &\leq \nu_p(F_{nm})\\
&= \nu_p(nm) + \nu_p(F_{z(p)}) = \nu_p(m) + \nu_p(n) + \nu_p(F_{z(p)})\\
&= \nu_p(m) + \nu_p(F_n).
\end{aligned}$$

From the above inequalities, we obtain that $\nu_p(m) \geq \nu_p(F_n^{k+1})$ for every prime p dividing F_n. Therefore $F_n^{k+1} \mid m$ which contradicts $F_n^k \parallel m$. This gives (iii). Hence the proof is complete. ∎

Proof of Theorem 3. Proof of (i):

Assume that $L_n^k \mid m$. Since m is odd and $L_n \mid m$, L_n is odd too. By Lemma 6, $L_n^{k+1} \mid \binom{m}{j} L_n^j$ for each $1 \leq j \leq m$. Then $L_n^{k+1} \mid \binom{m}{m-2j} L_n^{m-2j}$ for all $0 \leq j \leq \frac{m-1}{2}$. Then by (8), we see that $L_n^{k+1} \mid L_{nm} \cdot 2^{m-1}$. Hence $L_n^{k+1} \mid L_{nm}$. This proves (i).

Proof of (ii):

Next assume that $n \geq 2$ and $L_n^k \parallel m$. Then $m \geq 3$ and by Lemma 6, $L_n^{k+2} \mid \binom{m}{j} L_n^j$ for all $3 \leq j \leq m$. This implies that $L_n^{k+2} \mid \binom{m}{m-2j} L_n^{m-2j}$ for all $0 \leq j \leq \frac{m-3}{2}$. Then by (8), we see that

$$L_{nm} \cdot 2^{m-1} \equiv mL_nF_n^{m-1}5^{\frac{m-1}{2}} \pmod{L_n^{k+2}} \tag{15}$$

Since L_n is odd, we obtain

$$\begin{aligned} L_n^{k+2} \mid L_{nm} &\Leftrightarrow L_n^{k+2} \mid L_{nm} \cdot 2^{m-1} \\ &\Leftrightarrow L_n^{k+1} \mid mF_n^{m-1} 5^{\frac{m-1}{2}}, \quad \text{by (15)} \\ &\Leftrightarrow L_n^{k+1} \mid m, \quad \text{by (7) and Lemma 5.} \end{aligned}$$

Now $L_n^k \parallel m$, so $L_n^{k+1} \nmid m$ and thus $L_n^{k+2} \nmid L_{nm}$. Hence $L_n^{k+1} \parallel L_{nm}$ as desired. $\square$

Proof of Theorem 4. Assume that $L_n^k \mid m$. Since $n \geq 2$, we have $m \geq 4$. Then by Lemma 6, $L_n^{k+1} \mid \binom{m}{j} L_n^j$ for every $1 \leq j \leq m$. Therefore $L_n^{k+1} \mid \binom{m}{m-2j-1} L_n^{m-(2j+1)}$ for every $0 \leq j \leq \frac{m-2}{2}$. Then by (9), we obtain $F_{nm} \cdot 2^{m-1} \equiv 0 \pmod{L_n^{k+1}}$. From this point on, p-adic orders will be used and we sometimes apply Lemma 5 without referring to it explicitly.

Proof of (i):

Case 1 of (i): $n \equiv 1, 2 \pmod 3$. Then $\gcd(2, L_n) = 1$. Since $L_n^{k+1} \mid F_{nm} \cdot 2^{m-1}$, $L_n^{k+1} \mid F_{nm}$.

Case 2 of (i): $n \equiv 0 \pmod 6$. Then $\nu_2(L_n) = 1$. Since $L_n^k \mid m$, $\nu_2(m) \geq \nu_2(L_n^k) = k$. Since $n \equiv 0 \pmod 6$, $nm \equiv 0 \pmod 6$ and therefore

$$\nu_2(F_{nm}) = \nu_2(nm) + 2 = \nu_2(n) + \nu_2(m) + 2 \geq k + 2.$$

Since $L_n^{k+1} \mid F_{nm} \cdot 2^{m-1}$ and $\nu_2(L_n) = 1$, we obtain $\left(\frac{L_n}{2}\right)^{k+1} \mid \frac{F_{nm}}{2^{k+1}} \cdot 2^{m-1}$, which implies $\left(\frac{L_n}{2}\right)^{k+1} \mid \frac{F_{nm}}{2^{k+1}}$. Therefore $L_n^{k+1} \mid F_{nm}$.

Case 3 of (i): $n \equiv 3 \pmod 6$. Then $\nu_2(L_n) = 2$ and $\nu_2(F_n) = 1$. Since $L_n^k \mid m$, $\nu_2(m) \geq \nu_2(L_n^k) = 2k$. Therefore $nm \equiv 0 \pmod 6$ and $\nu_2(F_{nm}) = \nu_2(nm) + 2 = \nu_2(n) + \nu_2(m) + 2 \geq 2k + 2$. Since $L_n^{k+1} \mid F_{nm} \cdot 2^{m-1}$, $\nu_2(L_n) = 2$, and $\nu_2(F_{nm}) \geq 2k + 2$, we obtain

$$\left.\frac{L_n^{k+1}}{(2^2)^{k+1}}\right| \frac{F_{nm}}{2^{2k+2}} \cdot 2^{m-1},$$

which implies that $\frac{L_n^{k+1}}{(2^2)^{k+1}} \mid \frac{F_{nm}}{2^{2k+2}}$. Therefore $L_n^{k+1} \mid F_{nm}$.

In any case, we obtain that $L_n^{k+1} \mid F_{nm}$. This proves (i).

Proof of (ii):

Next assume that $L_n^k \parallel m$. Recall that $m \geq 4$. By Lemma 6, $L_n^{k+2} \mid \binom{m}{j} L_n^j$ for every $3 \leq j \leq m$. Therefore $L_n^{k+2} \mid \binom{m}{m-2j-1} L_n^{m-(2j+1)}$ for every $0 \leq j \leq \frac{m-4}{2}$. Then by (9),

$$F_{nm} \cdot 2^{m-1} \equiv mL_n F_n^{m-1} 5^{\frac{m-2}{2}} \pmod{L_n^{k+2}}. \tag{16}$$

Assume that $n \equiv 1, 2 \pmod 3$. Then by Lemma 5, (16), and (7), we obtain that

$$L_n^{k+2} \mid F_{nm} \Leftrightarrow L_n^{k+2} \mid F_{nm} \cdot 2^{m-1} \Leftrightarrow L_n^{k+2} \mid mL_n F_n^{m-1} 5^{\frac{m-2}{2}} \Leftrightarrow L_n^{k+1} \mid m.$$

But $L_n^k \parallel m$, so $L_n^{k+1} \nmid m$ and thus $L_n^{k+2} \nmid F_{nm}$. Then $L_n^{k+1} \parallel F_{nm}$. This proves (ii).

Proof of (iii) and (iv):

Assume that $L_n^k \parallel m$ and $n \equiv 0 \pmod 6$. In analogy with the proof of Case 2 of (i), we see that

$$L_n^{k+2} \mid F_{nm} \cdot 2^{m-1} \Leftrightarrow \frac{L_n^{k+2}}{2^{k+2}} \mid \frac{F_{nm}}{2^{k+2}} \cdot 2^{m-1} \Leftrightarrow \frac{L_n^{k+2}}{2^{k+2}} \mid \frac{F_{nm}}{2^{k+2}} \Leftrightarrow L_n^{k+2} \mid F_{nm}. \tag{17}$$

Let $m = L_n^k c$ where $c \geq 1$ and $L_n \nmid c$. Then by (17), (16), and Lemma 5, we obtain that

$$L_n^{k+2} \mid F_{nm} \Leftrightarrow L_n^{k+2} \mid mL_nF_n^{m-1}5^{\frac{m-2}{2}} \Leftrightarrow L_n \mid cF_n^{m-1}. \tag{18}$$

Now $\gcd(L_n, F_n) = 2$ and $\nu_2(L_n) = 1$, so $\gcd\left(\frac{L_n}{2}, F_n\right) = 1$, and therefore

$$L_n \mid cF_n^{m-1} \Leftrightarrow \frac{L_n}{2} \mid c\frac{F_n^{m-1}}{2} \Leftrightarrow \frac{L_n}{2} \mid c \Leftrightarrow \frac{L_n^{k+1}}{2} \mid m. \tag{19}$$

By (18) and (19), we see that $L_n^{k+2} \mid F_{nm} \Leftrightarrow \frac{L_n^{k+1}}{2} \mid m$. This proves (iii) and (iv).

Proof of (v) and (vi):

Assume that $L_n^k \parallel m$ and $n \equiv 3 \pmod 6$. In analogy with the proof of Case 3 of (i), we obtain that

$$\begin{aligned} L_n^{k+2} \mid F_{nm} \cdot 2^{m-1} &\Leftrightarrow \frac{L_n^{k+2}}{2^{2(k+2)}} \mid \frac{F_{nm}}{2^{2k+2}} \cdot \frac{2^{m-1}}{2^2} \\ &\Leftrightarrow \frac{L_n^{k+2}}{2^{2(k+2)}} \mid \frac{F_{nm}}{2^{2k+2}} \\ &\Leftrightarrow L_n^{k+2} \mid 4F_{nm}. \end{aligned} \tag{20}$$

Next, we write $m = L_n^k c$, use (16), and use the similar idea in the proof of (iii) and (iv) to obtain that

$$\begin{aligned} L_n^{k+2} \mid F_{nm} \cdot 2^{m-1} \Leftrightarrow L_n^{k+2} \mid mL_nF_n^{m-1}5^{\frac{m-2}{2}} \Leftrightarrow L_n \mid cF_n^{m-1} \Leftrightarrow \frac{L_n}{4} \mid c\frac{F_n^{m-1}}{4} \\ \Leftrightarrow \frac{L_n}{4} \mid c \Leftrightarrow \frac{L_n^{k+1}}{4} \mid m. \end{aligned} \tag{21}$$

From (20) and (21), we conclude that $L_n^{k+2} \mid 4F_{nm} \Leftrightarrow \frac{L_n^{k+1}}{4} \mid m$. This gives (v) and to prove (vi), it suffices to show that $L_n^{k+3} \nmid 4F_{nm}$. Suppose for a contradiction that $L_n^{k+3} \mid 4F_{nm}$. Then for every prime p dividing L_n, $\nu_p(L_n^{k+3}) \leq \nu_p(4F_{nm})$. Consider the following inequalities:

$$4 + \nu_2(L_n^{k+1}) = \nu_2(L_n^{k+3}) \leq \nu_2(4F_{nm}) = 2 + \nu_2(nm) + 2 = 4 + \nu_2(m),$$

where we use that $mn \equiv 0 \pmod 6$, and for every prime $p \notin \{2, 5\}$ and $p \mid L_n$, we have

$$\begin{aligned}\nu_p(L_n^{k+1}) + 2\nu_p(L_n) = \nu_p(L_n^{k+3}) &\le \nu_p(4F_{nm}) \le \nu_p(nm) + \nu_p(F_{z(p)}) \\ &= \nu_p(m) + \nu_p(n) + \nu_p(F_{z(p)}) = \nu_p(m) + \nu_p(L_n).\end{aligned}$$

From the above inequalities, we obtain that $\nu_p(m) \ge \nu_p(L_n^{k+1})$ for every prime p dividing L_n. Therefore $L_n^{k+1} \mid m$ which contradicts the fact that $L_n^k \parallel m$. Hence $L_n^{k+3} \nmid 4F_{nm}$. This proves (vi) and the proof is complete. $\square$

4 Examples

In this section, we give some examples to clarify the results in Theorems 3 and 4. Then we give some open questions at the end of this article. First we will show that the assumption that m is odd in Theorem 3 and that m is even in Theorem 4 cannot be omitted.

Example 7. Let $n \equiv 0 \pmod 3$, $k \ge 1$, and $m = L_n^k$. Then m is even, $L_n^k \mid m$, but $\nu_2(L_n^{k+1}) \ge k + 1 > 1 = \nu_2(L_{nm})$. Therefore $L_n^{k+1} \nmid L_{nm}$. This shows that Theorem 3 does not hold if m is not odd.

Example 8. Let $n \ge 2$, $n \equiv 1, 2 \pmod 3$, $k \ge 1$, and $m = L_n^k$. Then m is odd and $L_n^k \mid m$. As in the argument used in beginning of the proof of Theorem 4(i), we see that $L_n^{k+1} \mid \binom{m}{m-2j-1} L_n^{m-(2j+1)}$ for every $0 \le j \le \frac{m-3}{2}$ and $F_{nm} \cdot 2^{m-1} \equiv F_n^m 5^{\frac{m-1}{2}} \pmod{L_n^{k+1}}$. Since $n \equiv 1, 2 \pmod 3$, we obtain by (7) that $\gcd(L_n, F_n) = 1$. As $L_n \ge 3$ and $\gcd(L_n, 5) = 1$ with respect to Lemma 5, we obtain $L_n^{k+1} \nmid F_n^m 5^{\frac{m-1}{2}}$. Hence $L_n^{k+1} \nmid F_{nm}$. This shows that Theorem 4 does not hold if m is not even.

The conclusion $L_n^{k+2} \mid F_{nm}$ in Theorem 4(iv) may or may not be $L_n^{k+2} \parallel F_{nm}$ as shown in the next example.

Example 9. Let $k \ge 1$, $\ell \ge 2k + 2$, $n = 6$, and $m = 2^k 3^\ell$. Then m is even, $n \equiv 0 \pmod 6$, $L_n = 18$, $L_n^k \parallel m$, and $\frac{L_n^{k+1}}{2} \mid m$. Then we obtain

$$\begin{aligned}\nu_2(F_{nm}) &= \nu_2(nm) + 2 = k + 3 = (k+3)\nu_2(L_n) = \nu_2(L_n^{k+3}), \\ \nu_3(F_{nm}) &= \nu_3(nm) + \nu_3(F_4) = \ell + 2, \quad \text{and} \\ \nu_3(L_n^{k+3}) &= \nu_3(18^{k+3}) = 2k + 6.\end{aligned}$$

From this we see that if $\ell \ge 2k + 4$, then $L_n^{k+3} \mid F_{nm}$ and if $2k + 2 \le \ell < 2k + 4$, then $L_n^{k+3} \nmid F_{nm}$. This shows that the conclusion in Theorem 4(iv) may or may not be replaced by an exact divisibility.

In view of Theorems 2, 3, and 4, the reader may expect to see the result concerning the divisibility of Lucas numbers by powers of the Fibonacci numbers. But this is not possible in general. For example, let $n \ge 3$, $n \equiv 0 \pmod 3$, $k \ge 1$, $m = F_n^k$, then $\nu_2(F_n^{k+1}) \ge k + 1 \ge 2 > \nu_2(L_{nm})$ and therefore $F_n^k \parallel m$ but $F_n^{k+1} \nmid L_{nm}$. Even though we assume $n \not\equiv 0 \pmod 3$, the desired result is still false as shown in the next example.

Example 10. Let $n \geq 3$ and $n \not\equiv 0 \pmod 3$. Let $k, m \geq 1$ and $F_n^k \mid m$. Then by Lemma 6, $F_n^{k+1} \mid \binom{m}{2j} F_n^{2j}$ for every $1 \leq j \leq \left\lfloor \frac{m}{2} \right\rfloor$. By (8), we obtain

$$L_{nm} \cdot 2^{m-1} = \sum_{j=0}^{\left\lfloor \frac{m}{2} \right\rfloor} \binom{m}{2j} F_n^{2j} L_n^{m-2j} 5^j \equiv L_n^m \pmod{F_n^{k+1}}.$$

As $\gcd(F_n, L_n) = 1$ by (7), we have $F_n^{k+1} \nmid L_n^m$. Therefore $F_n^{k+1} \nmid L_{nm}$.

5 Open questions

It is natural to ask if the converse of each theorem holds. Marques [9, Corollary 3] shows that a partial converse of Theorem 1 holds. More precisely, he shows that if $F_n^{k+1} \mid F_{nm}$ and m is odd, then $F_n^k \mid m$. Whether or not the converse of Theorem 1 holds when m is even is still open. Similar questions are the following.

Q1 Assume that $F_n^{k+1} \parallel F_{nm}$ and m is odd. Can we conclude that $F_n^k \parallel m$?

Q2 Assume that $L_n^{k+1} \parallel L_{nm}$ and m is odd (or any other assumption). Can we conclude that $L_n^k \parallel m$?

Q3 Assume that $L_n^{k+1} \parallel F_{nm}$ and m is even (or any other assumption). Can we conclude that $L_n^k \parallel m$?

Q4 Suppose that Q1, Q2, or Q3 has a negative answer. Can we say something about the divisibility of m by powers of the Fibonacci and Lucas numbers?

Q5 Benjamin and Rouse [2] show that Theorem 1 holds if the Fibonacci numbers are replaced by the generalized Lucas numbers of the first kind (given by $u_0 = 0$, $u_1 = 1$, and $u_n = au_{n-1} + bu_{n-2}$ for $n \geq 2$). Does Theorem 1 hold for other sequences? Can we extend the results to include exact divisibility by powers of the generalized Lucas numbers of the first kind?

6 Acknowledgment

The author receives financial support from Faculty of Science, Silpakorn University, contract number SRF-PRG-2557-07. The author would like to thank the referee for his/her careful reading and for his/her suggestions which improve the presentation of this article.

References

[1] A. Benjamin and J. Quinn, The Fibonacci numbers-exposed more discretely, *Math. Mag.* **76** (2003), 182–192.

[2] A. Benjamin and J. Rouse, When does F_m^L divide F_n? A combinatorial solution, in *Proceedings of the Eleventh International Conference on Fibonacci Numbers and Their Applications*, Vol. 194, *Congressus Numerantium*, 2009, pp. 53–58.

[3] R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics*, 2nd edition, Addison-Wesley, 1994.

[4] J. H. Halton, On the divisibility properties of Fibonacci numbers, *Fibonacci Quart.* **4** (1966), 217–240.

[5] V. E. Hoggatt, Jr. and M. Bicknell-Johnson, Divisibility by Fibonacci and Lucas squares, *Fibonacci Quart.* **15** (1977), 3–8.

[6] D. Kalman and R. Mena, The Fibonacci numbers-exposed, *Math. Mag.* **76** (2003), 167–181.

[7] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Wiley, 2001.

[8] T. Lengyel, The order of the Fibonacci and Lucas numbers, *Fibonacci Quart.* **33** (1995), 234–239.

[9] D. Marques, The order of appearance of powers of Fibonacci and Lucas numbers, *Fibonacci Quart.* **50** (2012), 239–245.

[10] Y. Matijasevich, Enumerable sets are Diophantine, *Soviet Math.* **11** (1970), 354–358.

[11] Y. Matijasevich, My collaboration with Julia Robison, *Math. Intelligencer* **14** (1992), 38–45.

[12] Y. Matijasevich, *Hilbert's Tenth Problem*, MIT Press, 1996.

[13] K. Onphaeng and P. Pongsriiam, Subsequences and divisibility by powers of the Fibonacci numbers, *Fibonacci Quart.* **52** (2014), 163–171.

[14] C. Panraksa, A. Tangboonduangjit, and K. Wiboonton, Exact divisibility properties of some subsequences of Fibonacci numbers, *Fibonacci Quart.* **51** (2013), 307–318.

[15] P. Ribenboim, *My Numbers, My Friends: Popular Lectures on Number Theory*, Springer, 2000.

[16] J. Seibert and P. Trojovský, On divisibility of a relation of the Fibonacci numbers, *Int. J. Pure Appl. Math.* **46** (2008), 443–448.

[17] A. Tangboonduangjit and K. Wiboonton, Divisibility properties of some subsequences of Fibonacci numbers, *East-West J. Math.* Spec. Vol. (2012), 331–336.

[18] S. Vajda, *Fibonacci and Lucas Numbers and the Golden Section: Theory and Applications*, Dover Publications, 2008.

2010 *Mathematics Subject Classification*: Primary 11B39; Secondary 11A07.
Keywords: Fibonacci number, Lucas number, exact divisibility.

(Concerned with sequences A000032, and A000045.)

Received July 24 2014; revised versions received November 5 2014; November 6 2014. Published in *Journal of Integer Sequences*, November 7 2014.

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