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A Simpler Normal Number Construction for Simple Lüroth Series

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Abstract

Champernowne famously proved that the number

$$0.(1)(2)(3)(4)(5)(6)(7)(8)(9)(10)(11)(12)\cdots$$

formed by concatenating all the integers one after another is normal to base 10. We give a generalization of Champernowne's construction to various other digit systems, including generalized Lüroth series with a finite number of digits. For these systems, our construction simplifies a recent construction given by Madritsch and Mance. Along the way we give an estimation of the sum of multinomial coefficients above a tilted hyperplane in Pascal's simplex, which may be of general interest.

1 Introduction

A number $x \in [0,1)$ with base b expansion $x = 0.d_1d_2d_3\cdots$ is said to be normal to base b if for any string $s = a_1a_2\cdots a_k$ of base b digits, we have

$$\lim_{N\to\infty} \frac{\#\{0 \le n \le N \mid d_{n+i} = a_i, \quad 1 \le i \le k\}}{N} = b^{-k}.$$

This may be interpreted as saying that for a normal number x, each digit string appears with the same relative frequency as every other digit string having the same length.

While many methods (most notably the Birkhoff ergodic theorem) can be used to show that almost all real numbers $x \in [0, 1)$ are normal to any fixed base b, we know of very few examples of normal numbers. None of the well-known irrational constants, such as e or π, are known to be normal to any base, and the only examples we have of normal numbers are those explicitly constructed to be normal. The first and still most famous of these constructions is Champernowne's constant [7], which in base 10 looks like

$$0.(1)(2)(3)(4)(5)(6)(7)(8)(9)(10)(11)(12)(13)\cdots,$$

formed by concatenating all the integers in succession. He derived this construction after proving the base 10 normality of the following number

$$0.(0)(1)(2)(3)(4)(5)(6)(7)(8)(9)(00)(01)(02)(03)\cdots, \tag{1}$$

formed by concatenating all base 10 digit strings of length 1 in lexicographical order, then all the digit strings of length 2 in lexicographical order, and so on.

Constructions for base b normal numbers usually fall into one of three methods: the combinatorial method first introduced by Copeland and Erdős [8], that is perhaps the most natural generalization of Champernowne's techniques; the exponential sum method first introduced by Davenport and Erdős [10]; and the method of pseudo-random number generators used most powerfully by Bailey and Crandall [2, 3].

Recently, mathematical interest has turned to providing constructions of normal numbers in other systems. In many cases, these proofs draw from techniques used by Champernowne, Copeland, and Erdős. We shall be concerned here with ergodic fibred systems [5, 17]. Common examples of fibred systems include base b expansions, continued fraction expansions, generalized Lüroth series, and β-expansions.

Definition 1. Ergodic fibred systems consist with a transformation T that maps a set Ω to itself, a measure μ on Ω that is finite and T-invariant, a digit set $\mathcal{D} \subset \mathbb{N}$, and a countable collection of disjoint subsets $\{I_d\}_{d\in\mathcal{D}}$ such that every point in Ω is in some I_d. The map T is injective on each subset I_d and T is ergodic with respect to μ, *i.e.*, for every $A \subset \Omega$ with $T^{-1}A = A$ (up to some set of μ-measure 0), either $\mu(A) = 0$ or $\mu(A) = 1$.

The T-expansion of a point $x \in \Omega$ is then given by $x = [d_1, d_2, d_3, \ldots]$ where d_n is defined by $T^{n-1}x \in I_{d_n}$.[1] For a given fibred system, we say a point $x \in \Omega$ with expansion $x = [d_1, d_2, d_3, \ldots]$ is T-normal if for any string $s = [a_1, a_2, \ldots, a_k]$ of digits from $\mathcal{D}$ we have

$$\lim_{N\to\infty} \frac{\#\{0 \le n \le N-k \mid d_{n+i} = a_i, \quad 1 \le i \le k\}}{N} = \mu(C[s]). \tag{2}$$

where $C[s]$ is the cylinder set for the string s, i.e.,

$$C[s] = \{x = [d_1, d_2, \ldots] \mid d_i = a_i, \quad 1 \le i \le k\}.$$

We will often denote the measure of a cylinder set s by λ_s, and if s consists of a single digit d, then we will often shorthand the measure of the set $C[d]$ by λ_d.

[1] Our definitions here, in particular the part where $\{I_d\}_{d\in\mathcal{D}}$ form a partition of Ω, are somewhat nonstandard, but they guarantee that these expansions always exist and are unique for a given x.

We will denote the number of digits in the digit set $\mathcal{D}$ by D, and often assume that $\mathcal{D} = \{1, 2, \ldots, D\}$.

Madritsch and Mance [12] recently provided a normal number construction that works for many ergodic fibred systems, including those listed above. Their construction works roughly as follows:

1. Let ϵ_k be some small positive number shrinking to 0 very quickly as k increases, and let $S_k = \{s_1, s_2, s_3, \ldots, s_n\}$ be a set enumerating all strings of length k whose corresponding cylinder sets have measure at least ϵ_k.

2. Let M_k be at least $1/\epsilon_k$, and construct a string X_k formed by concatenating first $\lfloor M_k \lambda_{s_1} \rfloor$ copies of s_1, then $\lfloor M_k \lambda_{s_2} \rfloor$ copies of s_2 and so on until ending with $\lfloor M_k \lambda_{s_n} \rfloor$ copies of s_n. By construction, we expect that for strings s with length much smaller than k that s should appear in X_k with close to the correct frequency.

3. We chose a quickly growing sequence l_k and construct a digit x by first concatenating l_1 copes of X_1, then l_2 copies of X_2, and so forth. The l_k's are chosen so that l_k copies of X_k are vastly longer than the concatenated copies of X_1 up to X_{k-1} that precede it.

The strings X_k are constructed to have better and better small-scale normality properties and then are repeated so many times in the construction of x that their behavior swamps the behavior of what came before them. This construction was based on earlier work of Altomare and Mance [1], and Mance [13, 14] independently. The construction also bears resemblence to an earlier, but less general construction of Martinelli [15], although their results appear to be independent.

The advantage of the Madritsch-Mance construction is that it is extremely general, working even for the notoriously difficult β-expansions. The disadvantage of the Madritsch-Mance construction is its inefficency. For example, if we apply the Madritsch-Mance construction to create a normal number to base 10, it, like Champernowne's secondary construction (1), concatenates every digit string at some point; however, while Champernowne's second construction uses each digit string exactly 1 time, the Madritsch-Mance construction concatenates a string of length k at least $k^{2k} \log k$ times.

A different construction, by Bertrand-Mathis and Volkmann [6], is more efficient than the Madritsch-Mance construction; however it is more restricted in application. Bertrand-Mathis and Volkmann treat their dynamical systems symbolically, and their results only apply to symbolic dynamical systems with maximal entropy (so they would apply to base 10 expansions, but not to any other dynamical systems which are symbolically equivalent to base 10 expansions, such as generalized Lüroth series with 10 digits, see citation below).

Our goal in this paper is to construct and prove a normal number construction that is simpler than the Madritsch-Mance construction in that, like Champernowne's construction, uses each digit string one time, and that also works for certain systems where the Bertrand-Mathis and Volkmann construction does not.

Definition 2. Given an ergodic fibred system, let $S = \{s_n\}_{n\in\mathbb{N}}$ be an enumeration of all possible finite length strings ordered according to the following rule: If $\lambda_{s_i} > \lambda_{s_j}$, then $i < j$. (We do not care how strings whose cylinder sets have the same measure are ordered compared to one another. Although, if we want a rigorous definition of S, we may impose a lexicographical order on these strings.)

Let x_S be the point constructed by concatenating the strings s_i in order.

Note that if we consider a base 10 fibred system and impose a lexicographical ordering on those strings in S whose cylinder sets have the same measure, then we in fact get Champernowne's second construction (1) precisely. Therefore the construction of x_S given in Definition 2 is a true generalization of Champernowne's construction to other ergodic fibred systems.

Our goal in this paper will be to prove the following statement.

Theorem 3. *Consider an ergodic fibred system generated by a transformation T such that $\mathcal{D}$ is finite and such that for each string $s = [a_1, a_2, \ldots, a_k]$, we have $\lambda_s \asymp \lambda_{a_1}\lambda_{a_2}\cdots\lambda_{a_k}$.*

For such a system, the number x_S constructed in Definition 2 is T-normal.

The simplest example of such a fibred system are the generalized Lüroth series with finitely many digits, where we have, in fact, $\lambda_s = \lambda_{a_1}\lambda_{a_2}\cdots\lambda_{a_k}$. A good introduction to generalized Lüroth series is given in [9, Section 2.3].

We note that for some fibred systems, there may not be a point $x \in \Omega$ with T-expansion given by x_S. This is due to the possibility of inadmissable strings, strings s such that $\lambda_s = 0$. β-expansions, in particular, have many inadmissable strings, and in the Madritsch-Mance construction, they get around this obstruction by including padding, a long, but finite string of 0's inserted before each concatenated string s_i.

However, the condition in Theorem 3 that $\lambda_s \asymp \lambda_{a_1}\lambda_{a_2}\cdots\lambda_{a_k}$ guarantees that no inadmissable strings exist.

We leave as an open question—since we do not yet have enough information to be willing to state it as a conjecture—whether this construction works for other fibred systems, including generalized Lüroth series with an infinite number of digits, continued fraction expansions, and (with an appropriate padding, à la Madritsch-Mance) β-expansions.

In the proof we shall make use of the following theorem, known alternately as Pjatetskii-Shapiro normality criterion or the hot spot theorem [4, 16].

Theorem 4 (Pjatetskii-Shapiro)**.** *A point x with expansion $x = [d_1, d_2, d_3, \ldots]$ is T-normal if for any string $s = [a_1, a_2, \ldots, a_k]$ we have*

$$\limsup_{N\to\infty} \frac{\#\{0 \le n \le N-k \mid d_{n+i} = a_i, \quad 1 \le i \le k\}}{N} \le c\cdot\lambda_s.$$

for some constant c that is uniform over all strings.

This normality criterion is quite useful because it means that instead of having to prove a precise asymptotic for the counting function on the left-hand side of (2), we need only know its value up to a constant multiple.

We will need another result on a sum of multinomial coefficients, which we present here. Define the set T_ϵ by

$$T_\epsilon = \left\{\mathbf{x} = (x_1, \ldots, x_D) \in \mathbb{R}^D \;\middle|\; \lambda_1^{x_1}\lambda_2^{x_2}\cdots\lambda_D^{x_D} \geq \epsilon,\ x_i \geq 0,\ 1 \leq i \leq D\right\}.$$

We will use $\mathbf{m} = (m_1, m_2, \ldots, m_D) \in \mathbb{Z}^D$ to denote an integer lattice point. Then define

$$S(\epsilon) = \sum_{\mathbf{m}\in T_\epsilon} (m_1 + m_2 + \cdots + m_D)\frac{(m_1 + m_2 + \cdots + m_D)!}{m_1!m_2!\cdots m_D!} \tag{3}$$

and

$$S^{\#}(\epsilon) = \sum_{\mathbf{m}\in T_\epsilon} \frac{(m_1 + m_2 + \cdots + m_D)!}{m_1!m_2!\cdots m_D!}. \tag{4}$$

Theorem 5. *We have*

$$S(\epsilon) \asymp \frac{|\log\epsilon|}{\epsilon} \qquad S^{\#}(\epsilon) \asymp \frac{1}{\epsilon}$$

as ϵ *tends to* 0*.*

The proof of Theorem 3 will be broken down into the following steps.

1. In Section 3, we shall apply a counting argument to express

$$\#\{0 \leq n \leq N - k \mid d_{n+i} = a_i, \quad 1 \leq i \leq k\}/N$$

   in terms of the sums $S(\epsilon)$ and $S^{\#}(\epsilon)$, so that Theorem 3 is a simple consequence of Theorem 5.

2. In Sections 4 and 5, we will show that the bounds in Theorem 5 follow from bounds on similar sums, where T_ϵ is replaced by a hyperplane segment

$$\mathcal{H}_\epsilon := \left\{\mathbf{x} = (x_1, x_2, \ldots, x_D) \;\middle|\; \sum_{i=1}^{D} x_i \log\lambda_i = \log\epsilon, \quad x_i \geq 0, 1 \leq i \leq D\right\}.$$

3. In Section 6, we analyze the size of the resulting sum over $\mathcal{H}_\epsilon$ by applying the Laplace method (see [11] for more details).

In this paper we will frequently use the Landau and Vinogradov asymptotic notation, such as $\ll$, $\gg$, $\asymp$, big-O, and little-o, all with the usual meanings.

2 Some additional results

We need a few general lemmas, which we will present here.

Lemma 6. *Let* $1 < x < y$ *and suppose that* $0 < \delta < \min\{1, x-1\}$*, then we have, uniformly in all variables*

$$\frac{\Gamma(y-\delta)}{\Gamma(x-\delta)} \ll \frac{\Gamma(y)}{\Gamma(x)} \ll \frac{\Gamma(y+\delta)}{\Gamma(x+\delta)} \quad \text{and} \quad x \pm \delta \asymp x.$$

Proof. The first relation follows immediately from the fact that $\Gamma(x+\alpha) \asymp \Gamma(x)x^{\alpha}$ provided x and $x+\alpha$ are on subset of the positive reals bounded away from 0. The second relation is trivial. □

Lemma 7. *Let* n *be a positive integer,* $\{p_i\}_{i=1}^{n}$ *be a set of real numbers, and* $\{q_i\}_{i=1}^{n}$ *be a set of positive numbers. Then we have that*

$$\frac{\left(\sum_{i=1}^{n} p_i\right)^2}{\sum_{i=1}^{n} q_i} \le \sum_{i=1}^{n} \frac{p_i^2}{q_i},$$

with equality if and only if all the fractions $\{p_i/q_i\}_{i=1}^{n}$ *have the same value.*

Proof. This follows immediately from the Cauchy–Schwarz inequality:

$$\left(\sum_{i=1}^{n} \sqrt{q_i} \cdot \frac{p_i}{\sqrt{q_i}}\right)^2 \le \left(\sum_{i=1}^{n} \sqrt{q_i}^2\right)\left(\sum_{i=1}^{n} \left(\frac{p_i}{\sqrt{q_i}}\right)^2\right)$$

with equality if and only if there exists a constant C such that $C\sqrt{q_i} = p_i/\sqrt{q_i}$. □

Lemma 8. *For a fixed constant* C*, we have*

$$\sum_{-Z^{2/3} \le k \le Z^{2/3}} \exp\left(-\frac{C}{Z}k^2\right) = \sqrt{\frac{\pi Z}{C}}(1+o(1))$$

as Z *tends to* ∞*.*

Proof. We apply Euler-Maclaurin summation:

$$\begin{aligned}
\sum_{-Z^{2/3} \le k \le Z^{2/3}} \exp\left(-\frac{C}{Z}k^2\right) &= \int_{-Z^{2/3}}^{Z^{2/3}} \exp\left(-\frac{C}{Z}x^2\right)\, dx \\
&\quad + O\left(\int_{-Z^{2/3}}^{Z^{2/3}} \frac{C|x|}{Z} \exp\left(-\frac{C}{Z}x^2\right)\, dx\right) \\
&\quad + O\left(\exp\left(-\frac{C}{Z}Z^{4/3}\right)\right) \\
&= \sqrt{\frac{\pi Z}{C}} - 2\sqrt{\frac{Z}{C}} \cdot \int_{Z^{1/6}C^{1/2}}^{\infty} \exp\left(-x^2\right)\, dx \\
&\quad + O\left(\int_0^{Z^{1/6}C^{1/2}} |x| \exp\left(-x^2\right)\, dx\right) + O(1) \\
&= \sqrt{\frac{\pi Z}{C}}(1 + o(1)).
\end{aligned}$$

□

3 Proving Theorem 3 from Theorem 5

By Theorem 4, it suffices to show that for any string $s = [a_1, a_2, \ldots, a_k]$, we have

$$\frac{\#\{0 \le n \le N - k \mid d_{n+i} = a_i, \quad 1 \le i \le k\}}{N} \ll \lambda_s,$$

with implicit constant uniform over all strings.

The counting function

$$\#\{0 \le n \le N - k \mid d_{n+i} = a_i, \quad 1 \le i \le k\}$$

is very difficult to compute directly, so we will instead estimate its size in terms of other, simpler functions. The Nth digit of x, d_N, must appear in the concatenation of some string s_n, for which we have $\mu(C[s_n]) = \epsilon = \epsilon(N)$.

Let $A(\epsilon; s)$ denote the number of time the string s occurs *within* the strings s_i where $\lambda_{s_i} \ge \epsilon$. Let $A(\epsilon)$ just denote the total number of digits in all the strings s_i where $\lambda_{s_i} \ge \epsilon$. We will also use $A^{\#}(\epsilon)$ to denote the total number of strings s_i where $\lambda_{s_i} \ge \epsilon$.

With $\epsilon = \epsilon(N)$, we clearly have

$$\#\{0 \le n \le N - k \mid d_{n+i} = a_i, \quad 1 \le i \le k\} \le A(\epsilon; s) + kA^{\#}(\epsilon),$$

where the latter term comes from a trivial estimate on how many times the string s could occur starting in one string s_i and ending another string s_j. Moreover, the number N itself is at least $A(2\epsilon)$, and thus

$$\frac{\#\{0 \le n \le N-k \mid d_{n+i} = a_i, \quad 1 \le i \le k\}}{N} \le \frac{A(\epsilon; s) + kA^{\#}(\epsilon)}{A(2\epsilon)}.$$

Now we wish to bound the A functions, in terms of the S functions (3) and (4). Following the assumption from Theorem 3, let us assume that for a string $s = [a_1, a_2, \ldots, a_k]$ we have

$$c_1 \lambda_{a_1} \lambda_{a_2} \cdots \lambda_{a_k} \le \lambda_s \le c_2 \lambda_{a_1} \lambda_{a_2} \cdots \lambda_{a_k}.$$

Suppose we want to count the total number of ways one can concatenate the string s together with m_d copies of the digit d. If counted with multiplicity, this will correctly count the total number of times s occurs in strings that have $m_d + e_d$ copies of the digit d, where e_d is the number of times d occurs in s. There are precisely

$$(1 + m_1 + m_2 + \cdots + m_D) \cdot \frac{(m_1 + m_2 + \cdots + m_D)!}{m_1! m_2! \cdots m_D!}$$

such strings (counted with multiplicity), each of which will have a cylinder set of measure in the interval

$$\left[\frac{c_1}{c_2} \lambda_s \cdot \prod_{d \le D} \lambda_d^{m_d}, \frac{c_2}{c_1} \lambda_s \cdot \prod_{d \le D} \lambda_d^{m_d} \right].$$

Thus if we let

$$\begin{aligned} S(\epsilon; s) &= \sum_{\substack{m_1, m_2, \ldots, m_D \\ \lambda_1^{m_1} \lambda_2^{m_2} \cdots \lambda_D^{m_D} \ge \epsilon / \lambda_s}} (1 + m_1 + m_2 + \cdots + m_D) \frac{(m_1 + m_2 + \cdots + m_D)!}{m_1! m_2! \cdots m_D!} \\ &= S(\epsilon/\lambda_s) + S^{\#}(\epsilon/\lambda_s), \end{aligned}$$

then we clearly have

$$S\left(\frac{c_2}{c_1}\epsilon; s\right) \le A(\epsilon; s) \le S\left(\frac{c_1}{c_2}\epsilon; s\right).$$

By a similar argument we can show

$$S\left(\frac{c_2}{c_1}\epsilon\right) \le A(\epsilon) \le S\left(\frac{c_1}{c_2}\epsilon\right) \qquad \text{and} \qquad S^{\#}\left(\frac{c_2}{c_1}\epsilon\right) \le A^{\#}(\epsilon) \le S^{\#}\left(\frac{c_1}{c_2}\epsilon\right).$$

Thus,

$$\frac{\#\{0 \le n \le N-k \mid d_{n+i} = a_i, \quad 1 \le i \le k\}}{N} \le \frac{S\left(\frac{c_1}{c_2 \lambda_s}\epsilon\right) + (k+1) S^{\#}\left(\frac{c_1}{c_2 \lambda_s}\epsilon\right)}{S\left(2\frac{c_2}{c_1}\epsilon\right)}.$$

Now applying Theorem 5 we obtain

$$\frac{\#\{0\le n\le N-k \mid d_{n+i}=a_i,\quad 1\le i\le k\}}{N} \ll \frac{\left(\frac{c_1}{c_2\lambda_s}\epsilon\right)^{-1}\left|\log\left(\frac{c_1}{c_2\lambda_s}\epsilon\right)\right| + (k+1)\left(\frac{c_1}{c_2\lambda_s}\epsilon\right)^{-1}}{\left(2\frac{c_2}{c_1}\epsilon\right)^{-1}\left|\log\left(2\frac{c_2}{c_1}\epsilon\right)\right|}$$
$$\ll \lambda_s,$$

and these bounds are uniform in s, which completes the proof of Theorem 3.

4 Proof of Theorem 5

We will consider two new functions $H(\epsilon)$ and $H^{\#}(\epsilon)$ given by the following.

Let $\mathcal{H}_\epsilon$ denote the hyperplane segment

$$\mathcal{H}_\epsilon := \left\{ \mathbf{x}=(x_1,x_2,\ldots,x_D) \;\middle|\; \sum_{i=1}^{D} x_i\log\lambda_i = \log\epsilon,\quad x_i\ge 0, 1\le i\le D \right\}.$$

Note that

$$x_1\log\lambda_1^{-1}+\cdots+x_D\log\lambda_D^{-1}\le\log\epsilon^{-1}$$

is equivalent to $\lambda_1^{x_1}\cdots\lambda_D^{x_D}\ge\epsilon$. We will consider "lattice" points $\mathbf{m}\in\mathcal{H}_\epsilon$ to be given by $(m_1,m_2,\ldots,m_D)$ where $m_2,\ldots,m_D\in\mathbb{Z}$, and $m_1=M$ is a real number determined by the other coordinates via the formula

$$M=\frac{\log\left(\epsilon/\left(\lambda_2^{m_2}\lambda_3^{m_3}\cdots\lambda_D^{m_D}\right)\right)}{\log\lambda_1}.$$

We then define $H(\epsilon)$ and $H^{\#}(\epsilon)$ by

$$H(\epsilon) := \sum_{\mathbf{m}\in\mathcal{H}_\epsilon}(M+m_2+m_3+\cdots+m_D)\frac{(M+m_2+m_3+\cdots+m_D)!}{M!m_2!m_3!\cdots m_D!}$$
$$H^{\#}(\epsilon) := \sum_{\mathbf{m}\in\mathcal{H}_\epsilon}\frac{(M+m_2+m_3+\cdots+m_D)!}{M!m_2!m_3!\cdots m_D!}$$

We extend the factorial to real values in the natural way by $x!=\Gamma(x+1)$.

While the functions $S(\epsilon)$ and $S^{\#}(\epsilon)$ look at all values lying *above* the hyperplane $\mathcal{H}_\epsilon$, the functions $H(\epsilon)$ and $H^{\#}(\epsilon)$ instead look at values *on* the hyperplane $\mathcal{H}_\epsilon$.

Theorem 5 (and therefore Theorem 3) will follow from the following two lemmas, which we prove in subsequent sections.

Lemma 9. *We have*

$$H(\epsilon/\lambda_1)\ll S(\epsilon)\ll H(\epsilon\cdot\lambda_2)\qquad\text{and}\qquad H^{\#}(\epsilon/\lambda_1)\ll S^{\#}(\epsilon)\ll H^{\#}(\epsilon\cdot\lambda_2).$$

Lemma 10. *We have*

$$H(\epsilon)\asymp\frac{|\log\epsilon|}{\epsilon}\qquad H^{\#}(\epsilon)\asymp\frac{1}{\epsilon}$$

as ϵ tends to 0.

5 Proof of Lemma 9

We shall provide bounds for $S(\epsilon)$. The method for $S^{\#}(\epsilon)$ is similar.

First, we place a lower bound on $S(\epsilon)$. We have

$$\begin{aligned} S(\epsilon) &= \sum_{\substack{m_2,\ldots,m_D \\ \lambda_2^{m_2}\cdots\lambda_D^{m_D}\geq\epsilon}} \left(\sum_{\substack{m_1 \\ \lambda_1^{m_1}\lambda_2^{m_2}\cdots\lambda_D^{m_D}\geq\epsilon}} (m_1+m_2+\cdots+m_D)\frac{(m_1+m_2+\cdots+m_D)!}{m_1!m_2!\cdots m_D!} \right) \\ &\gg \sum_{\substack{m_2,\ldots,m_D \\ \lambda_2^{m_2}\cdots\lambda_D^{m_D}\geq\epsilon}} (M'+m_2+\cdots+m_D)\frac{(M'+m_2+m_3+\cdots+m_D)!}{M'!m_2!m_3!\cdots m_D!}, \end{aligned}$$

where in each summand M' is the largest integer such that

$$\lambda_1^{M'}\lambda_2^{m_2}\lambda_2^{m_3}\cdots\lambda_D^{m_D}\geq\epsilon. \tag{5}$$

Increasing the size of ϵ in the index of summation but not in the definition of M' will only result in removing terms, therefore,

$$S(\epsilon) \gg \sum_{\substack{m_2,\ldots,m_D \\ \lambda_2^{m_2}\cdots\lambda_D^{m_D}\geq\epsilon/\lambda_1}} (M'+m_2+\cdots+m_D)\frac{(M'+m_2+m_3+\cdots+m_D)!}{M'!m_2!m_3!\cdots m_D!}.$$

Comparing this series term by term with $H(\epsilon/\lambda_1)$ and noting that M' for this sum is greater than and within 1 of the corresponding M in the terms of $H(\epsilon/\lambda_1)$, we get that $S(\epsilon)\gg H(\epsilon/\lambda_1)$ by Lemma 6.

For the reverse inequality, we have, for fixed $m_2, m_3, \ldots, m_D$ and with M' defined as in (5), that

$$\begin{aligned} &\sum_{\substack{m_1 \\ \lambda_1^{m_1}\lambda_2^{m_2}\cdots\lambda_D^{m_D}\geq\epsilon}} (m_1+m_2+\cdots+m_D)\frac{(m_1+m_2+\cdots+m_D)!}{m_1!m_2!\cdots m_D!} \\ &\quad\leq (M'+m_2+m_3+\cdots+m_D)\cdot \sum_{\substack{m_1 \\ \lambda_1^{m_1}\lambda_2^{m_2}\cdots\lambda_D^{m_D}\geq\epsilon}} \frac{(m_1+m_2+\cdots+m_D)!}{m_1!m_2!\cdots m_D!} \\ &\quad= (M'+m_2+m_3+\cdots+m_D)\cdot\frac{(m_2+m_3+\cdots+m_D)!}{m_2!m_3!\cdots m_D!}\times \\ &\qquad \sum_{\substack{m_1 \\ \lambda_1^{m_1}\lambda_2^{m_2}\cdots\lambda_D^{m_D}\geq\epsilon}} \binom{m_1+m_2+m_3+\cdots+m_D}{m_2+m_3+\cdots+m_D} \end{aligned}$$

$$= (M' + m_2 + m_3 + \cdots + m_D) \cdot \frac{(m_2 + m_3 + \cdots + m_D)!}{m_2! m_3! \cdots m_D!} \times$$
$$\times \binom{M' + 1 + m_2 + m_3 + \cdots + m_D}{1 + m_2 + m_3 + \cdots + m_D}$$
$$= \frac{m_2 + 1}{1 + m_2 + m_3 + \cdots + m_D} (M' + m_2 + m_3 + \cdots + m_D) \times$$
$$\binom{M' + (m_2 + 1) + m_3 + m_4 + \cdots + m_D}{M, m_2 + 1, m_3, m_4, \ldots, m_D}$$
$$\ll (M' + (m_2 + 1) + m_3 + \cdots + m_D) \cdot \frac{(M' + (m_2 + 1) + m_3 + m_4 + \cdots + m_D)!}{M!(m_2 + 1)! m_3! m_4! \cdots m_D!}.$$

By summing over all possible $m_2, m_3, \ldots, m_D$ for which the sum is non-empty, we obtain most of the terms from $H(\epsilon \cdot \lambda_2)$, namely all the terms where $m_2 \geq 1$. So therefore we have $S(\epsilon) \ll H(\epsilon \cdot \lambda_2)$.

6 Proof of Lemma 10

We shall provide the proof for $H(\epsilon)$, as the proof for $H^{\#}(\epsilon)$ is similar.

We want to begin by examining the terms of $H(\epsilon)$, using Stirling's formula. We will use a somewhat non-standard form as follows:

$$x! \asymp \sqrt{2\pi(x+1)} \left(\frac{x}{e}\right)^x. \tag{6}$$

This clearly follows from the usual Stirling's formula for large x, since replacing x by $x + 1$ inside the square root introduces an error of at most $1 + O(x^{-1})$; however this function has the added advantage of being true and uniform for all non-negative x, because the function on the right is bounded away from 0.

Now consider a given term of $H(\epsilon)$,

$$(M + m_2 + m_3 + \cdots + m_D) \frac{(M + m_2 + m_3 + \cdots + m_D)!}{M! m_2! m_3! \cdots m_D!}, \tag{7}$$

where, as before,

$$M = \frac{\log\left(\epsilon / \left(\lambda_2^{m_2} \lambda_3^{m_3} \cdots \lambda_D^{m_D}\right)\right)}{\log \lambda_1}.$$

Applying Stirling's formula (6) gives that (7) is of the order of $G(\mathbf{m}) \cdot \exp(F(\mathbf{m}))$, where

$$G(\mathbf{m}) := \frac{(M + m_2 + \cdots + m_D + 1)^{3/2}}{\sqrt{(M+1)(m_2+1)(m_3+1)\cdots(m_D+1)}}$$

and

$$F(\mathbf{m}) := (M + m_2 + m_3 + \cdots + m_D)\log(M + m_2 + m_3 + \cdots + m_D) \\ - M\log M - \sum_{i=2}^{D} m_i \log m_i.$$

The function G is fairly smooth and, compared to the exponential of F, quite small. Therefore we shall focus our studies primarily on understanding the properties of F.

6.1 Understanding F

In order to understand the properties of F better, it is helpful to work with an auxiliary function. Let

$$\tilde{F}(\mathbf{x}) := (x_1 + \cdots + x_D)\log(x_1 + \cdots + x_D) - \sum_{i=1}^{D} x_i \log x_i$$

be a function on $\mathcal{H}_\epsilon$.

We think of F as being a function of $D-1$ variables. (The value of $m_1 = M$ is determined by the others.) However, we will think of $\tilde{F}$ as a function on D free variables, and then restrict our attention to the $D-1$-dimensional hyperplane $\mathcal{H}_\epsilon$.

Proposition 11. *Let $l = (a_1t+b_1, a_2t+b_2, \ldots, a_Dt+b_D)$ be a line parallel to and intersecting the hyperplane segment $\mathcal{H}_\epsilon$. Then the second directional derivative of $\tilde{F}$ along this line is negative.*

Proof. Since l is parallel to and intersecting $\mathcal{H}_\epsilon$, we have that

$$\sum_{i=1}^{D}(a_i t + b_i)\log\lambda_i = \log\epsilon.$$

By isolating the coefficient of t, we obtain

$$\sum_{i=1}^{D} a_i \log\lambda_i = 0.$$

In particular, since all the $\log\lambda_i$ are negative, there must exist at least one positive and one negative a_i.

The second derivative of $\tilde{F}$ along this line is given by

$$\frac{d^2}{dt^2}\tilde{F}(a_1t+b_1, \ldots, a_Dt+b_D) = \frac{\left(\sum_{i=1}^{D} a_i\right)^2}{\sum_{i=1}^{D}(a_it+b_i)} - \sum_{i=1}^{D}\frac{a_i^2}{a_it+b_i}.$$

By Lemma 7, this is never positive, and is zero if and only if $a_i/(a_it+b_i)$ has the same value for all i; however, in order to be in the domain of $\tilde{F}$, all the a_it+b_i must be positive, and as we noted earlier, at least one m_i must be positive and at least one m_i must be negative, therefore the $a_i/(a_it+b_i)$ cannot all have the same value. The second derivative is therefore strictly negative. $\square$

This proposition produces two immediate consequences. First, $\tilde{F}$ must have a unique local maximum on $\mathcal{H}_\epsilon$: it must have a maximum on $\mathcal{H}_\epsilon$ since it is a continuous function on a compact set, and there cannot be two local maximums since on the line between them $\tilde{F}$ would have strictly negative second derivative. Second, on any line passing through this maximum, the function $\tilde{F}$ is strictly decreasing away from the maximum.

Lemma 12. *The function $\tilde{F}(\mathbf{x})$ has its unique maximum on $\mathcal{H}_\epsilon$ at the point $\mathbf{p} = (\lambda_1 L, \lambda_2 L, \lambda_3 L, \ldots, \lambda_D L)$, where*

$$L = \frac{\log(\epsilon)}{\lambda_1 \log(\lambda_1) + \lambda_2 \log(\lambda_2) + \cdots + \lambda_D \log(\lambda_D)}.$$

Moreover, $\tilde{F}(\mathbf{p}) = -\log\epsilon$.

Proof. It is easy to see that $\mathbf{p}$ is on the hyperplane segment $\mathcal{H}_\epsilon$. Since all the directional second derivatives parallel to $\mathcal{H}_\epsilon$ are negative, it suffices to show that, at the point $\mathbf{p}$, all the directional first derivatives parallel to $\mathcal{H}_\epsilon$ are 0.

As before, consider a line $l(t) = (a_1t + \lambda_1 L, \ldots, a_nt + \lambda_D L)$ passing through the point $\mathbf{p}$. We again have

$$\sum_{i=1}^{D} a_i \log\lambda_i = 0.$$

The directional derivative of F at p along this line (in the positive t direction) is given by

$$\begin{aligned}
&\left(\sum_{i=1}^{D} a_i\right)\log\left(\sum_{i=1}^{D}\lambda_i L\right) - \sum_{i=1}^{D}\left(a_i\log(\lambda_i L)\right)\\
&\quad= \left(\sum_{i=1}^{D} a_i\right)\log\left(\frac{\log(\epsilon)\sum_{i=1}^{D}\lambda_i}{\sum_{i=1}^{D}\lambda_i\log\lambda_i}\right) - \sum_{i=1}^{D}\left(a_i\log\left(\frac{\log(\epsilon)\lambda_i}{\sum_{j=1}^{D}\lambda_j\log\lambda_j}\right)\right)\\
&\quad= \left(\sum_{i=1}^{D} a_i\right)\log\left(\frac{\log(\epsilon)}{\sum_{i=1}^{D}\lambda_i\log\lambda_i}\right) - \sum_{i=1}^{D}\left(a_i\log\left(\frac{\log(\epsilon)}{\sum_{j=1}^{D}\lambda_j\log\lambda_j}\right)\right)\\
&\qquad - \sum_{i=1}^{D} a_i\log\lambda_i\\
&\quad= 0.
\end{aligned}$$

This shows that $\mathbf{p}$ is the maximum. The value $\tilde{F}$ takes at this point is given by

$$\begin{aligned}
&\left(\sum_{i=1}^{D} \lambda_i L\right) \log\left(\sum_{i=1}^{D} \lambda_i L\right) - \sum_{i=1}^{D} \lambda_i L \log(\lambda_i L) \\
&\quad = \frac{\log \epsilon}{\sum_{j=1}^{D} \lambda_j \log \lambda_j} \log\left(\frac{\log \epsilon}{\sum_{j=1}^{D} \lambda_j \log \lambda_j}\right) \\
&\qquad - \sum_{i=1}^{D} \frac{\lambda_i \log \epsilon}{\sum_{j=1}^{D} \lambda_j \log \lambda_j} \log\left(\frac{\lambda_i \log \epsilon}{\sum_{j=1}^{D} \lambda_j \log \lambda_j}\right) \\
&\quad = -\sum_{i=1}^{D} \frac{\lambda_i \log \epsilon}{\sum_{j=1}^{D} \lambda_j \log \lambda_j} \log \lambda_i \\
&\quad = -\log \epsilon,
\end{aligned}$$

which completes the proof.

□

We will abuse notation and consider $\mathbf{x} \in \mathcal{H}_\epsilon$ as being both the vector $(x_1, x_2, \ldots, x_D)$ and the vector $(x_2, \ldots, x_D)$ with implied extra variable

$$x_1 = \frac{1}{\log \lambda_1}\left(\log \epsilon - \sum_{i=2}^{n} x_i \log \lambda_i\right).$$

And likewise we will consider $\mathbf{p} \in \mathcal{H}_\epsilon$ as being both the vector $(\lambda_1 L, \lambda_2 L, \ldots, \lambda_D L)$ and the vector $(\lambda_2 L, \lambda_3 L, \ldots, \lambda_D L)$.

Therefore F can be given by

$$\begin{aligned}
F(\mathbf{x}) = &\left(\frac{\log \epsilon}{\log \lambda_1} + \sum_{i=2}^{D} x_i \left(1 - \frac{\log \lambda_i}{\log \lambda_1}\right)\right) \log\left(\frac{\log \epsilon}{\log \lambda_1} + \sum_{i=2}^{D} x_i \left(1 - \frac{\log \lambda_i}{\log \lambda_1}\right)\right) \\
&- \left(\frac{\log \epsilon}{\log \lambda_1} - \sum_{i=2}^{D} x_i \frac{\log \lambda_i}{\log \lambda_1}\right) \log\left(\frac{\log \epsilon}{\log \lambda_1} - \sum_{i=2}^{D} x_i \frac{\log \lambda_i}{\log \lambda_1}\right) \\
&- \sum_{i=2}^{D} x_i \log x_i.
\end{aligned}$$

Given $2 \leq i, j \leq D$, we have

$$\begin{aligned}
\frac{\partial^2}{\partial x_i \partial x_j} F(\mathbf{x}) = &\frac{\left(1 - \frac{\log \lambda_i}{\log \lambda_1}\right)\left(1 - \frac{\log \lambda_j}{\log \lambda_1}\right)}{\frac{\log \epsilon}{\log \lambda_1} + \sum_{i=2}^{D} x_i \left(1 - \frac{\log \lambda_i}{\log \lambda_1}\right)} \\
&- \frac{\frac{\log \lambda_i \log \lambda_j}{(\log \lambda_1)^2}}{\frac{\log \epsilon}{\log \lambda_1} - \sum_{i=2}^{D} x_i \frac{\log \lambda_i}{\log \lambda_1}} - \frac{\delta_{i,j}}{x_i}.
\end{aligned}$$

So, if we consider the second partial derivatives at p arranged in a matrix, then we see that there exists a fixed real symmetric matrix A (independent of ϵ), such that

$$\frac{\partial^2}{\partial x_i \partial x_j} F(\mathbf{p}) = \frac{1}{\log \epsilon} A_{i-1,j-1}.$$

Since $(\log \epsilon)^{-1} A$ is a real symmetric matrix, it can be diagonalized by orthogonal matrices. In particular, this implies that there exist unit vectors $\mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_D \in \mathbb{R}^{D-1}$ and fixed eigenvalues $l_2, l_3, \ldots, l_D$ (again not dependent on ϵ) such that

$$\frac{\partial^2}{\partial \mathbf{u}_i \partial \mathbf{u}_j} F(\mathbf{p}) = \begin{cases} \dfrac{l_j}{\log \epsilon}, & \text{if } i = j; \\ 0, & \text{otherwise.} \end{cases}$$

By Proposition 11 the second directional derivatives must always be negative, so l_j must be positive.

Consider a ball $\mathcal{B}_\epsilon$ around the point $\mathbf{p}$, given by

$$\mathcal{B}_\epsilon = \left\{ \mathbf{p} + t_2 \mathbf{u}_2 + \cdots + t_D \mathbf{u}_D \;\middle|\; \sum_{i=2}^{D} t_i^2 \leq |\log \epsilon|^{2/3} \right\}.$$

Note that for sufficiently small ϵ, we have $\mathcal{B}_\epsilon \subset \mathcal{H}_\epsilon$. We also consider a box B_ϵ given by

$$B_\epsilon = \left\{ \mathbf{p} + t_2 \mathbf{e}_2 + \cdots + t_D \mathbf{e}_D \;\middle|\; |t_i| \leq \frac{1}{\sqrt{D-1}} |\log \epsilon|^{2/3} \right\},$$

where $\mathbf{e}_i$ are the elementary basis vectors. We have that $B_\epsilon \subset \mathcal{B}_\epsilon$.

If $\mathbf{x} \in \mathcal{B}_\epsilon$, then each coordinate x_i of $\mathbf{x}$ must be on the order of $|\log x|$. Therefore for all points $\mathbf{x} \in \mathcal{B}_\epsilon$, the third partial derivative of F satisfies the following bound:

$$\frac{\partial^3}{\partial \mathbf{u}_j \partial \mathbf{u}_D \partial \mathbf{u}_l} F(\mathbf{x}) \ll |\log \epsilon|^{-2}.$$

By Taylor's Theorem, for any point $\mathbf{x} = \mathbf{p} + t_2 \mathbf{u}_2 + \cdots + t_D \mathbf{u}_D \in \mathcal{B}_\epsilon$, we have

$$F(\mathbf{x}) = -\log \epsilon + \sum_{i=2}^{D} \frac{l_i}{\log \epsilon} t_i^2 + O(1). \tag{8}$$

Let F_+ and F_- be given by

$$F_+(\mathbf{x}) = -\log \epsilon + \left(\max_{2 \leq i \leq D} \frac{l_i}{\log \epsilon} \right) \sum_{i=2}^{D} t_i^2$$

and

$$F_-(\mathbf{x}) = -\log \epsilon + \left(\min_{2 \leq i \leq D} \frac{l_i}{\log \epsilon} \right) \sum_{i=2}^{D} t_i^2,$$

so that

$$F_-(\mathbf{x}) \le F(\mathbf{x}) + O(1) \le F_+(\mathbf{x}).$$

These functions are advantageous because $\sum_{j=2}^{D} t_j^2$ is invariant under rotation around $\mathbf{p}$. If $\mathbf{x} = \mathbf{p} + y_2\mathbf{e}_2 + \cdots + y_D\mathbf{e}_D$ is in the box B_ϵ, then

$$F_+(\mathbf{x}) = -\log\epsilon + \left(\max_{2\le i\le n} \frac{l_i}{\log\epsilon}\right)\sum_{i=2}^{n} y_i^2,$$

and likewise for F_-.

Moreover, for each point $\mathbf{x}$ outside of the box B_ϵ, we can draw a line between $\mathbf{x}$ and $\mathbf{p}$ and note that by Lemma 12, F increases along the line as we move towards $\mathbf{p}$. Therefore, the value of F at $\mathbf{x} \notin B_\epsilon$ is at most the maximum of F on the boundary of B_ϵ, and by (8), this is at most $-\log\epsilon - C|\log\epsilon|^{1/3}$ for some fixed *positive* constant C.

6.2 Returning to the full sum

For points $\mathbf{x} \in B_\epsilon$, it is easy to see that $G(\mathbf{x})$ is on the order of $|\log\epsilon|^{(3-D)/2}$ and for $\mathbf{x} \notin B_\epsilon$, the value $G(\mathbf{x})$ could be as large as $|\log\epsilon|$. Therefore,

$$\begin{aligned}\sum_{\mathbf{m}\in\mathcal{H}_\epsilon\setminus B_\epsilon} G(\mathbf{m})\exp(F(\mathbf{m})) &\ll \sum_{\mathbf{m}\in\mathcal{H}_\epsilon\setminus B_\epsilon} |\log\epsilon|\exp(-\log\epsilon - C|\log\epsilon|^{1/3}) \\ &\ll |\log\epsilon|^D \exp(-\log\epsilon - C|\log\epsilon|^{1/3}) = o(\epsilon^{-1}).\end{aligned}$$

Here we used the fact that $m_i \ll |\log\epsilon|$.

Therefore

$$H(\epsilon) \asymp \sum_{\mathbf{m}\in B_\epsilon} G(\mathbf{m})\exp(F(\mathbf{m})) + o(\epsilon^{-1}).$$

Since, as noted above $G(\mathbf{m})$ is on the order of $|\log\epsilon|^{(3-D)/2}$ for $\mathbf{m} \in B_\epsilon$, to complete the proof it suffices to prove that

$$\sum_{\mathbf{m}\in B_\epsilon} \exp(F(\mathbf{m})) \asymp \frac{|\log\epsilon|^{D-1}}{\epsilon}.$$

First we note that

$$\sum_{\mathbf{m}\in B_\epsilon} \exp(F_-(\mathbf{m})) \ll \sum_{\mathbf{m}\in B_\epsilon} \exp(F(\mathbf{m})) \ll \sum_{\mathbf{m}\in B_\epsilon} \exp(F_+(\mathbf{m})).$$

There exists a point $\mathbf{p}'$ within distance $\sqrt{D-1}/2$ from $\mathbf{p}$, such that $\mathbf{p}'$ is an integer lattice point. For $\mathbf{m} \in B_\epsilon$, let $\mathbf{m}' = \mathbf{m} + \mathbf{p} - \mathbf{p}'$. Then we have $F_\pm(\mathbf{m}) - F_\pm(\mathbf{m}') = O(|\log\epsilon|^{-1/3}) = O(1)$. Moreover, each vector $\mathbf{m}'$ can be written as $\mathbf{p} + k_2\mathbf{e}_2 + \cdots + k_D\mathbf{e}_D \in B_\epsilon$ with each k_i in the interval

$$I = \left[-c|\log\epsilon|^{2/3} - \frac{\sqrt{D-1}}{2}, c|\log\epsilon|^{2/3} + \frac{\sqrt{D-1}}{2}\right].$$

Therefore

$$\sum_{\mathbf{m}\in B_\epsilon} \exp(F_+(\mathbf{m})) \asymp \sum_{\mathbf{m}\in B_\epsilon} \exp(F_+(\mathbf{m}+\mathbf{p}-\mathbf{p}')) \tag{9}$$

$$\leq \frac{1}{\epsilon}\prod_{i=2}^{D}\left(\sum_{k_i\in I}\exp\left(\left(\max_{2\leq i\leq D}\frac{l_i}{\log\epsilon}\right)k_i^2\right)\right), \tag{10}$$

and likewise

$$\sum_{\mathbf{m}\in B_\epsilon} \exp(F_-(\mathbf{m})) \gg \frac{1}{\epsilon}\prod_{i=2}^{D}\left(\sum_{k_i\in J}\exp\left(\left(\max_{2\leq i\leq D}\frac{l_i}{\log\epsilon}\right)k_i^2\right)\right), \tag{11}$$

where

$$J = \left[-c|\log\epsilon|^{2/3} + \frac{\sqrt{D-1}}{2}, c|\log\epsilon|^{2/3} - \frac{\sqrt{D-1}}{2}\right].$$

Applying Lemma 8 to (10) and (11) completes the proof.

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