Journal of Integer Sequences, Vol. 18 (2015),
Article 15.1.5

Hankel Determinant for a Sequence that Satisfies a Three-Term Recurrence Relation

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Abstract

In this paper, we give an explicit expression for the Hankel determinants for a sequence satisfying a three-term recurrence relation. Our method of evaluation is based on the theory of orthogonal polynomials. In particular, we analyze the linear functional associated with such a sequence.

1 Introduction

The problem of evaluation of Hankel determinants has emerged in the 19th century. These classes of determinants are important in many mathematical areas, such as number theory, integrable systems, approximation theory, and linear system theory, among others. Several authors have focused on the evaluation of the Hankel determinants using different methods. Here, we can cite Radoux [11] who evaluated Hankel determinants by using methods based primarily on matrix decomposition techniques. Krattenthaler [6] used combinatorial techniques and graph theory to evaluate a large class of these determinants. Egecioglu et al. [4, 5] evaluated Hankel determinants with binomial coefficient entries by using a technique based on the solution of differential equations. Chammam et al. [2] dealt with Hankel determinants with polynomial coefficient entries by using a method based mainly on the theory of orthogonal polynomials.

For instance, let $(a_n)_{n\geq 0}$ be a sequence satisfying the three-term recurrence relation

$$a_{n+1} = -\frac{2n+1}{n+1}\, a_n - a_{n-1},\ n \geq 0,$$

with initial values $a_0 = 1$, $a_{-1} = 0$.

In this work, we are interested in the evaluation of the Hankel determinants

$$H_n := \det\left[a_{i+j}\right]_{i,j=0}^{n} = \begin{vmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} \end{vmatrix},\ n \geq 0.$$

Our method of evaluation of such determinants is based on the study of the linear functional $\mathcal{L}$ associated with the sequence $(a_n)_{n\geq 0}$, given by $(\mathcal{L})_n = a_n$, $n \geq 0$. Mainly, we show that $\mathcal{L}$ is a solution of a functional first-order linear differential equation. Under certain conditions, $\mathcal{L}$ is related to the classical linear functional of Bessel (with parameter $\alpha = 1$) by the relation $\mathcal{L} = -x\big(h_{-\frac{1}{2}} \circ \tau_2 \mathcal{B}^{(1)}\big)'$. Sfaxi and Alaya [12] gave special kinds of linear functionals that allow to us to obtain a necessary and sufficient condition for the quasi-definiteness of the linear functional $\mathcal{L}$. By using the theory of orthogonal polynomials, this allows us to give an explicit expression of our Hankel determinants.

The paper is organized as follows. In Section 2, we introduce the basic background and notations to be used throughout the paper. In Section 3, first we deduce a necessary and sufficient condition for the quasi-definiteness of the linear functional $\mathcal{L}$. Finally, we give an explicit expression of the Hankel determinants $H_n := \det\left[a_{i+j}\right]_{i,j=0}^{n}$, $n \geq 0$.

2 Notation and preliminary results

2.1 Orthogonality and semi-classical character

Let $\mathbb{P}$ be the linear space of polynomials in one variable with complex coefficients and $\mathbb{P}'$ its algebraic dual space. We denote by $\langle \mathscr{U}, p\rangle$ the action of $\mathscr{U} \in \mathbb{P}'$ on $p \in \mathbb{P}$ and by $(\mathscr{U})_n := \langle \mathscr{U}, x^n\rangle$, $n \geq 0$, the sequence of moments of $\mathscr{U}$ with respect to the polynomial sequence $\{x^n\}_{n\geq 0}$. Let us define the following operations in $\mathbb{P}'$: for linear functionals $\mathscr{U}$ and $\mathscr{V}$, any polynomial q, and any $(a, b, c) \in \mathbb{C}^* \times \mathbb{C}^2$, let $D\mathscr{U} = \mathscr{U}'$, $q\mathscr{U}$, $(x-c)^{-1}\mathscr{U}$, $\tau_{-b}\mathscr{U}$ and $h_a\mathscr{U}$ be the linear functionals defined by duality [10]:

$$\begin{aligned}
&\langle \mathscr{U}', p\rangle := -\langle \mathscr{U}, p'\rangle, \quad \langle q\mathscr{U}, p\rangle := \langle \mathscr{U}, qp\rangle, \\
&\big\langle (x-c)^{-1}\mathscr{U},\ p\big\rangle := \langle \mathscr{U},\ \theta_c p\rangle = \Big\langle \mathscr{U},\ \frac{p(x)-p(c)}{x-c}\Big\rangle, \\
&\langle \tau_{-b}\mathscr{U},\ p\rangle := \langle \mathscr{U},\ \tau_b p\rangle = \langle \mathscr{U},\ p(x-b)\rangle, \\
&\langle h_a\mathscr{U},\ p\rangle := \langle \mathscr{U},\ h_a p\rangle = \langle \mathscr{U},\ p(ax)\rangle, \qquad p \in \mathbb{P}.
\end{aligned}$$

A linear functional $\mathscr{U}$ is called *normalized* if it satisfies $(\mathscr{U})_0 = 1$.

With any sequence of complex numbers $(a_n)_{n\geq 0}$, we can associate a unique linear functional $\mathscr{U} \in \mathbb{P}'$ given by $(\mathscr{U})_n = a_n,\ n \geq 0$.

The linear functional $\mathscr{U}$ is said to be *quasi-definite* (*regular*) if the Hankel determinant $H_n(\mathscr{U}) = \det[a_{i+j}]_{i,j=0}^{n} \neq 0$, for every integer $n \geq 0$. In such a case, there exists a unique *sequence of monic polynomials* (SMP) $\{P_n\}_{n\geq 0}$, i.e., $\deg P_n = n$ and their leading coefficients are equal to 1, such that

$$\langle \mathscr{U}, x^{\nu} P_n \rangle = 0,\ 0 \leq \nu \leq n-1 \text{ and } \langle \mathscr{U}, x^n P_n \rangle \neq 0.$$

The sequence $\{P_n\}_{n\geq 0}$ is said to be the *sequence of monic orthogonal polynomials* (SMOP) with respect to $\mathscr{U}$.

Notice that $P_n(x)$ can be represented by the determinantal formula

$$P_0(x) = 1,\ P_n(x) = \frac{1}{H_{n-1}(\mathscr{U})} \begin{vmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_n & \cdots & a_{2n-1} \\ 1 & x & \cdots & x^n \end{vmatrix},\ n \geq 1.$$

The orthogonality of $\{P_n\}_{n\geq 0}$ can be characterized by a three-term recurrence relation (TTRR, in short) [3]

$$P_{n+1}(x) = (x - \beta_n) P_n(x) - \gamma_n P_{n-1}(x),\ n \geq 0,$$

with initial values $P_0(x) = 1,\ P_{-1}(x) = 0$, where $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_n\}_{n\geq 0}$ are sequences of complex numbers such that $\gamma_n \neq 0,\ n \geq 1$, and the convention $\gamma_0 = (\mathscr{U})_0$.

Furthermore,

$$\beta_n = \frac{\langle \mathscr{U}, xP_n^2 \rangle}{\langle \mathscr{U}, P_n^2 \rangle}, \quad \gamma_{n+1} = \frac{\langle \mathscr{U}, P_{n+1}^2 \rangle}{\langle \mathscr{U}, P_n^2 \rangle},\ n \geq 0, \tag{1}$$

$$\langle \mathscr{U}, P_n^2 \rangle = \prod_{\nu=0}^{n} \gamma_{\nu} = \frac{H_n(\mathscr{U})}{H_{n-1}(\mathscr{U})},\ n \geq 0, \quad \big(H_{-1}(\mathscr{U}) = 1\big), \tag{2}$$

$$H_n(\mathscr{U}) = \prod_{k=0}^{n} \prod_{\nu=0}^{k} \gamma_{\nu},\ n \geq 0. \tag{3}$$

When $\{P_n\}_{n\geq 0}$ is a SMOP with respect to a linear functional $\mathscr{U}$, then the sequence of monic polynomials $\{\tilde{P}_n\}_{n\geq 0}$, where $\tilde{P}_n(x) = a^{-n} P_n(ax + b)$, is also orthogonal with respect to $\tilde{\mathscr{U}} = (h_{a^{-1}} \circ \tau_{-b})\mathscr{U}$, and satisfies the following TTRR [8, 9, 10]

$$\tilde{P}_{n+1}(x) = (x - \tilde{\beta}_n) \tilde{P}_n(x) - \tilde{\gamma}_n \tilde{P}_{n-1}(x),\ n \geq 0,$$

with initial values $\tilde{P}_0(x) = 1$, $\tilde{P}_{-1}(x) = 0$, and where

$$\tilde{\beta}_n = a^{-1}(\beta_n - b),\ n \geq 0,$$
$$\tilde{\gamma}_{n+1} = a^{-2}\gamma_{n+1},\ n \geq 0,\ \big(\tilde{\gamma}_0 = (\mathscr{U})_0\big).$$

As a consequence, we get the following result.

Lemma 1. *Let $\{a_n\}_{n\geq 0}$ be a sequence of complex numbers and $\mathscr{U} \in \mathbb{P}'$ such that $(\mathscr{U})_n = a_n$, $n \geq 0$. Then, for any pair $(a, b) \in \mathbb{C}^* \times \mathbb{C}$, we have*

$$H_n\big((h_{a^{-1}} \circ \tau_{-b})\mathscr{U}\big) = a^{-n(n+1)} H_n(\mathscr{U}),\ n \geq 0,$$

where the moments of the shifted linear functional $(h_{a^{-1}} \circ \tau_{-b})\mathscr{U}$ are given by

$$\big((h_{a^{-1}} \circ \tau_{-b})\mathscr{U}\big)_n = a^{-n} \sum_{k=0}^{n} \binom{n}{k} (-b)^{n-k} a_k,\ n \geq 0.$$

A quasi-definite linear functional $\mathscr{U}$ is said to be *semi-classical* if it satisfies a functional equation (Pearson equation)

$$(\Phi\mathscr{U})' + \Psi\mathscr{U} = 0, \tag{4}$$

where Φ and Ψ are polynomials such that Φ is monic and $\deg(\Psi) \geq 1$.

The corresponding SMOP $\{P_n\}_{n\geq 0}$ is said to be *semi-classical* (for more details, see [1, 8, 10] and the literature therein).

If for each zero c of Φ we have

$$\mid \Phi'(c) + \Psi(c) \mid + \mid \langle \mathscr{U}, \theta_c^2\Phi + \theta_c\Psi\rangle \mid > 0, \tag{5}$$

then the nonnegative integer $s := \max\{\deg(\Phi) - 2, \deg(\Psi) - 1\}$ is said to be either the *class* of $\mathscr{U}$ or the class of $\{P_n\}_{n\geq 0}$.

The semi-classical character of a linear functional is invariant by shifting. Indeed, if $\mathscr{U}$ is a semi-classical linear functional of class s satisfying (4) and (5), then for any pair $(a, b) \in \mathbb{C}^2$ with $a \neq 0$, the shifted linear functional $\tilde{\mathscr{U}} = (h_{a^{-1}} \circ \tau_{-b})\mathscr{U}$ is also semi-classical of class s and satisfies $(\tilde{\Phi}\tilde{\mathscr{U}})' + \tilde{\Psi}\tilde{\mathscr{U}} = 0$ with $\tilde{\Phi}(x) = a^{-t}\Phi(ax + b)$, $\tilde{\Psi}(x) = a^{1-t}\Psi(ax + b)$ and where $t = \deg\Phi$.

We find in [1, 9] a full description of the class $s = 0$. This corresponds to the *classical linear functionals* (*Hermite*, *Laguerre*, *Bessel* and *Jacobi*).

Further, we need the following properties of the monic Bessel polynomials $\{B_n^{(\alpha)}\}_{n\geq 0}$, (orthogonal with respect to the linear functional $\mathcal{B}^{(\alpha)}$) [3, 7].

$$
\boxed{\begin{array}{l}
B_n^{(\alpha)}(x) = \sum_{\nu=0}^{n} \binom{n}{\nu} \dfrac{2^{n-\nu}\Gamma(n+2\alpha+\nu-1)}{\Gamma(2n+2\alpha-1)}\, x^{\nu},\ n \geq 0, \quad (\alpha \neq -\dfrac{n}{2},\ n \geq 0).\\
\beta_0 = -\dfrac{1}{\alpha}, \quad \beta_n = \dfrac{1-\alpha}{(n+\alpha-1)(n+\alpha)},\ n \geq 0,\\
\gamma_n = -\dfrac{n(n+2\alpha-2)}{(2n+2\alpha-3)(n+\alpha-1)^2(2n+2\alpha-1)},\ n \geq 1.\\
\Phi(x) = x^2, \quad \Psi(x) = -2(\alpha x+1).\\
(\mathcal{B}^{(\alpha)})_n = \dfrac{(-2)^n\Gamma(2\alpha)}{\Gamma(n+2\alpha)},\ n \geq 0.\\
H_n(\mathcal{B}^{(\alpha)}) = (-4)^{\frac{n(n+1)}{2}} \prod_{k=0}^{n} k! \dfrac{\Gamma(k+2\alpha-1)\Gamma(2\alpha)}{\Gamma(2k+2\alpha-1)\Gamma(2k+2\alpha)},\ n \geq 0.
\end{array}}
$$

Table 1: Some basic characteristics of Bessel polynomials.

2.2 Quasi-definiteness condition of the linear functional $-(x-c)\mathscr{C}'$

Let $\{S_n\}_{n\geq 0}$ be a classical sequence of monic polynomials, orthogonal with respect to a normalized linear functional $\mathscr{C}$ that satisfies

$$
(\Phi\mathscr{C})' - (\lambda + \Phi')\mathscr{C} = 0, \tag{6}
$$

where Φ is monic, $\deg \Phi = 2$, and $\lambda \neq 0$. In other words, $\mathscr{C}$ is either a shifted Jacobi or a shifted Bessel linear functional.

We have the following results.

Lemma 2. [12] *For any $c \in \mathbb{C}$ where $\Phi(c) \neq 0$, the following statements are equivalent.*

(i) *The linear functional $-(x-c)\mathscr{C}'$ is quasi-definite.*

(ii) $A_{n+1}(c) \neq 0,\ n \geq 0,$ *where* $A_{n+1}(x) = \Phi(x)S_n'(x) + \lambda S_n(x),\ n \geq 0.$

Lemma 3. [12] *Let $\mathscr{C}$ be a classical linear functional satisfying (6). When it is quasi-definite, the linear functional $w_0(c) = -(x-c)\mathscr{C}'$ is semi-classical of class one and satisfies $(\varphi w_0(c))' + \psi w_0(c) = 0$, where $\varphi(x) = (x-c)\Phi(x)$ and $\psi(x) = -\big(2\Phi(x) + \lambda(x-c)\big)$.*

Under the hypothesis of the previous lemmas, let $\{Q_n\}_{n\geq 0}$ be the SMOP with respect to $w_0(c)$. Suppose that the SMOP $\{S_n\}_{n\geq 0}$ satisfies

$$
S_{n+1}(x) = (x-\xi_n)S_n(x) - \rho_n S_{n-1}(x),\ n \geq 0,
$$

with initial values $S_0(x) = 1$, $S_{-1}(x) = 0$, where $\rho_n \neq 0,\ n \geq 1$.

Then the SMOP $\{Q_n\}_{n\geq 0}$ satisfies the TTRR [12]

$$
Q_{n+1}(x) = (x-\alpha_n)Q_n(x) - \eta_n Q_{n-1}(x),\ n \geq 0, \tag{7}
$$

with initial values $Q_0(x) = 1$, $Q_{-1}(x) = 0$, and where

$$\alpha_0 = 2\zeta_0 - c,\ \alpha_1 = c - \frac{2\lambda\rho_1}{A_2(c)},$$
$$\alpha_n = c - \frac{(n+1)}{n}\frac{A_n(c)}{A_{n+1}(c)}\rho_n - \frac{n-1}{n}\frac{A_{n+1}(c)}{A_n(c)},\ n \geq 2,$$
$$\eta_1 = -A_2(c),\ \eta_2 = \lambda\rho_1\frac{A_3(c)}{A_2^2(c)},$$
$$\eta_{n+1} = \frac{A_{n+2}(c)A_n(c)}{A_{n+1}^2(c)}\rho_n,\ n \geq 2.$$

The following result is a straightforward consequence of (3) and (7).

Theorem 4. *Let $\{S_n\}_{n\geq 0}$ be a classical polynomial sequence, orthogonal with respect to a normalized linear functional $\mathscr{C}$ satisfying $(\Phi\mathscr{C})' - (\lambda + \Phi')\mathscr{C} = 0$, where Φ is monic, $\deg \Phi = 2$, and $\lambda \neq 0$. Then, for any $c \in \mathbb{C}$ such that $\Phi(c) \neq 0$ and $A_{n+1}(c) := \Phi(c)S_n'(c) + \lambda S_n(c) \neq 0,\ n \geq 0$, we get*

$$H_n\big(-(x-c)\mathscr{C}'\big) = (-1)^n\lambda^{n-1}A_{n+1}(c)H_{n-1}(\mathscr{C}) \neq 0,\ n \geq 0,\ \textit{with } H_{-1}(\mathscr{C}) = 1.$$

3 Expression of the Hankel determinants

Let $(a_n)_{n\geq 0}$ be the sequence given by

$$a_{n+1} = -\frac{2n+1}{n+1}\ a_n - a_{n-1},\ n \geq 0, \tag{8}$$

with initial values $a_0 = 1$ and $a_{-1} = 0$.

We can associate with the sequence $(a_n)_{n\geq 0}$ a unique linear functional $\mathcal{L}$ given by its moments with respect to the monomial sequence $\{x^n\}_{n\geq 0}$,

$$(\mathcal{L})_n = a_n,\ n \geq 0. \tag{9}$$

The aim of this section is to evaluate the Hankel determinants of the linear functional $\mathcal{L}$, i.e., $H_n(\mathcal{L}) = \det\big[a_{i+j}\big]_{i,j=0}^n$, $n \geq 0$. To do it, we need, first, to establish some useful properties for the linear functional $\mathcal{L}$.

Using (8) and (9), we get

$$(\mathcal{L})_{n+1} + \frac{2n+1}{n+1}(\mathcal{L})_n + (\mathcal{L})_{n-1} = 0,\ n \geq 0, \tag{10}$$

with $(\mathcal{L})_0 = 1$ and $(\mathcal{L})_{-1} = 0$.

From the above relation for the moments, the linear functional $\mathcal{L}$ satisfies

$$(\varphi\mathcal{L})' + \psi\mathcal{L} = 0,$$

where $\varphi(x) = x(x+1)^2$ and $\psi(x) = -(2x^2 + 3x + 2)$.

We have the following fondamental result.

Proposition 5. *The linear functional $\mathcal{L}$ is related to a classical linear functional $\mathcal{B}^{(1)}$ (the Bessel form with parameter $\alpha = 1$), by the following relation*

$$\mathcal{L} = -x\Big((h_{-\frac{1}{2}} \circ \tau_2)\mathcal{B}^{(1)}\Big)'.$$

Proof. Let $\mathcal{V} = -x\Big((h_{-\frac{1}{2}} \circ \tau_2)\mathcal{B}^{(1)}\Big)'$. We show that $(\mathcal{V})_n,\ n \geq 0$, also satisfies (10). Indeed, we have

$$(\mathcal{V})_n = (-1)^n 2^{-n}(n+1)\Big\langle \mathcal{B}^{(1)}, (x+2)^n \Big\rangle,\ n \geq 0. \tag{11}$$

By Table 1, the linear functional $\mathcal{B}^{(1)}$ satisfies $\big(x^2\mathcal{B}^{(1)}\big)' - 2(x+1)\mathcal{B}^{(1)} = 0$. Then, we obtain

$$\Big\langle \big(x^2\mathcal{B}^{(1)}\big)' - 2(x+1)\mathcal{B}^{(1)}, (x+2)^n \Big\rangle = 0,\ n \geq 0.$$

Equivalently

$$(n+2)\Big\langle \mathcal{B}^{(1)}, \big(x+2\big)^{n+1} \Big\rangle - 2(2n+1)\Big\langle \mathcal{B}^{(1)}, (x+2)^n \Big\rangle + 4n\Big\langle \mathcal{B}^{(1)}, \big(x+2\big)^{n-1} \Big\rangle = 0,\ n \geq 0.$$

By using (11), it follows that

$$(\mathcal{V})_{n+1} + \frac{2n+1}{n+1}\,(\mathcal{V})_n + (\mathcal{V})_{n-1} = 0,\ n \geq 0,\ \text{with } (\mathcal{V})_0 = 1,\ (\mathcal{V})_{-1} = 0.$$

We, thus, obtain $\mathcal{V} = \mathcal{L}$. □

Under the hypothesis of the previous result and from Lemma 2, we are going to establish a necessary and sufficient condition for the quasi-definiteness of the linear functional $\mathcal{L}$. Next, according to Theorem 4, we will give an explicit expression of our Hankel determinants.

Let $\{S_n\}_{n\geq 0}$ be the SMOP with respect to $\mathcal{L}$, and let $\{A_{n+1}\}_{n\geq 0}$ be the sequence of polynomials given by

$$A_{n+1}(x) := \Phi(x)S_n'(x) + \lambda S_n(x).$$

By Proposition 5, we have $\mathcal{L} = -x\Big((h_{-\frac{1}{2}} \circ \tau_2)\mathcal{B}^{(1)}\Big)'$, that satisfies (6) with $\Phi(x) = (x+1)^2$ and $\lambda = -1 \neq 0$. In this case, $S_n(x) = \big(\frac{-1}{2}\big)^n B_n^{(1)}\big(-2(x+1)\big),\ n \geq 0$, where $\{B_n^{(1)}(x)\}_{n\geq 0}$ is the Bessel SMOP with parameter $\alpha = 1$. By using to Table 1, we get

$$S_n(x) = \sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} \frac{(2n-\nu)!}{(2n)!}(x+1)^{n-\nu},\ n \geq 0.$$

According to Lemma 2, the linear functional $\mathcal{L}$ is quasi-definite if and only if $A_{n+1}(0) := S_n'(0) - S_n(0) \neq 0,\ n \geq 0$, i.e., $\sum_{\nu=0}^{n+1} \lambda_{n+1,\nu} \neq 0$, where

$$\begin{aligned}
&\lambda_{n+1,0} = n,\\
&\lambda_{n+1,\nu} = \frac{(-1)^\nu(2n-\nu+1)!\binom{n}{\nu-1}}{(2n)!}\Big[\frac{(n-\nu)(n-\nu+1)}{(2n-\nu+1)\nu} + 1\Big],\ 1 \leq \nu \leq n+1.
\end{aligned}$$

On the other hand, we can deduce the following result.

Theorem 6. *The Hankel determinants with coefficient* $(a_n)_{n\geq 0}$*, are given by*

$$H_n := \det\left[a_{i+j}\right]_{i,j=0}^{n} = (-1)^{\frac{(n-2)(n+1)}{2}} A_{n+1}(0) \prod_{k=0}^{n-1} \frac{(k!)^2}{(2k)!(2k+1)!},$$

for any non-negative integer number n *and the convention* $\prod_{k=0}^{-1} = 1$*.*

Proof. From Table 1 and Lemma 1, we get for $n \geq 0$,

$$H_n\big((h_{-\frac{1}{2}} \circ \tau_2)\mathcal{B}^{(1)}\big) = \left(\frac{-1}{2}\right)^{n(n+1)} H_n(\mathcal{B}^{(1)}) = (-1)^{\frac{n(n+1)}{2}} \prod_{k=0}^{n} \frac{(k!)^2}{(2k)!(2k+1)!}.$$

According to Theorem 4, such that $A_{n+1}(0) \neq 0,\ n \geq 0$, we obtain the desired result. □

4 Acknowledgements

The author would like to thank the anonymous referee for their valuable comments and suggestions for improving the original version of this manuscript.

References

[1] A. Branquinho, J. Petronilho, and F. Marcellán, Classical orthogonal polynomials, a functional approach. *Acta Appl. Math.* **34** (1994), 283–303.

[2] W. Chammam, F. Marcellán, and R. Sfaxi, Orthogonal polynomials, Catalan numbers, and a general Hankel determinant evaluation. *Linear Algebra Appl.* **436** (2012), 2105–2116.

[3] T. S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, 1978.

[4] Ö. Egecioglu, A prime sensitive Hankel determinant of Jacobi symbol enumerators. *Ann. Comb.* **14** (2010), 443–456.

[5] Ö. Egecioglu, T. Redmond, and C. Ryavec, A multilinear operator for almost product evaluation of Hankel determinants. *J. Combin. Theory, Ser. A.* **117** (2010), 77–103.

[6] C. Krattenthaler, Advanced determinant calculus: a complement. *Linear Algebra Appl.* **411** (2005), 68–166.

[7] N. N. Lebedev, *Special Functions and their Applications.* Revised English Edition, Dover Publications, 1972.

[8] F. Marcellán and R. Sfaxi, A characterization of weakly regular linear functionals. *Rev. Acad. Colombiana Cienc. Exact. Fis. Natur.* **31** (119) (2007), 285–295.

[9] P. Maroni, Fonctions Eulériennes, polynômes orthogonaux classiques. In *Techniques de l'Ingénieur*, Traité Généralités (Sciences Fondamentales) **A 154** Paris, (1994), 1–30.

[10] P. Maroni, Une théorie algébrique des polynômes orthogonaux Applications aux polynômes orthogonaux semi-classiques. In C. Brezinski, L. Gori, and A. Ronveaux, eds., *Orthogonal Polynomials and their Applications*, *IMACS Ann. Comput. Appl. Math.* **9** (1991), 95–130.

[11] C. Radoux, Déterminant de Hankel construit sur des polynômes liés aux nombres de dérangements. *European J. Combin.* **12** (1991), 327–329.

[12] R. Sfaxi and J. Alaya, On orthogonal polynomials with respect to the form $-(x-c)\mathbf{S}'$, *Period. Math. Hungar.* **52** (2006), 67–99.

2010 *Mathematics Subject Classification*: Primary 33C45; Secondary 42C05.
Keywords: Orthogonal polynomial, three-term recurrence relation, semi-classical linear functional, Hankel determinant, Bessel polynomial.

Received October 22 2014; revised versions received December 15 2014; December 18 2014.
Published in *Journal of Integer Sequences*, January 7 2015.

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