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Some Explicit Estimates for the Möbius Function

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Abstract

In this work, we provide some explicit upper bounds for certain sums involving the Möbius function. Thanks to recent results proved by Balazard, some of these bounds improve on earlier estimates given by El Marraki.

1 Introduction and results

It is often a hard task to give completely explicit results in analytic number theory. For instance, it is well-known that if χ is a quadratic Dirichlet character to the modulus q, then, for any $\varepsilon \in \left(0, \frac{1}{2}\right)$, there exists a non-effectively computable constant $c_\varepsilon > 0$ such that

$$L(1,\chi) > \frac{c_\varepsilon}{q^\varepsilon}$$

and all attempts at providing a value to c_ε for sufficiently small ε have been unsuccessful.

On the other hand, a wide class of functions of prime numbers have been successfully explicitly estimated during the last fifty years, starting with the benchmarking paper of Rosser and Schœnfeld [11]. For instance, refining an earlier estimate of Dusart [4], Trudgian [14] proved that, for $x \geq 229$

$$|\pi(x) - \mathrm{Li}(x)| < 0.279\,5\,\frac{x}{(\log x)^{3/4}} \exp\left(-\sqrt{\frac{\log x}{6.455}}\right)$$

where Li is the usual logarithmic integral function. As for the Möbius function, the prime number theorem is known to be equivalent to the estimate

$$\sum_{n \le x} \mu(n) = o(x) \quad (x \to \infty)$$

and the method of contour integration may lead to bounds of the form

$$\sum_{n \le x} \mu(n) \ll x \exp\left(-c\sqrt{\log x}\right) \quad (x \ge 2,\ c > 0)$$

but no explicit result of this form is known. Refining a method of von Sterneck, based itself upon the work of Chebyshev, MacLeod [8] showed that

$$\left|\sum_{n \le x} \mu(n)\right| \le \frac{x+1}{80} + \frac{11}{2} \quad (x \ge 1)\,. \tag{1}$$

Bounds of the form $x(\log x)^{-\alpha}$ with $\alpha > 0$ were then obtained by Schœnfeld [12] and El Marraki [5] who, among others, proved that [5, Théorème 2]

$$\left|\sum_{n \le x} \mu(n)\right| < \frac{0.109\,17x}{\log x} \quad (x \ge 685) \tag{2}$$

and [5, Théorème 3]

$$\left|\sum_{n \le x} \mu(n)\right| < \frac{362.7x}{(\log x)^2} \quad (x > 1)\,. \tag{3}$$

Using an inequality coming from a convolution relation and partial summation, El Marraki [6] deduced from (2) and (3) that

$$\left|\sum_{n \le x} \frac{\mu(n)}{n}\right| \le \frac{0.218\,5}{\log x} \ (x \ge 33) \quad \text{and} \quad \left|\sum_{n \le x} \frac{\mu(n)}{n}\right| \le \frac{726}{(\log x)^2} \ (x > 1)\,. \tag{4}$$

Ramaré [10] refined the first estimate by showing

$$\left|\sum_{n \le x} \frac{\mu(n)}{n}\right| \le \frac{1}{69 \log x} \quad (x \ge 96\,955)\,.$$

Our first result improves on (4) in the following way.

Theorem 1.

(a) *For all* $x \geq 33$

$$\left|\sum_{n\leq x}\frac{\mu(n)}{n}\right| < \frac{0.19}{\log x}.$$

(b) *For all* $x > 1$

$$\left|\sum_{n\leq x}\frac{\mu(n)}{n}\right| < \frac{546}{(\log x)^2}.$$

It may be interesting to estimate similar sums twisted by additional conditions. For instance, Ramaré [9, 10] studied sums of the type

$$\sum_{\substack{n\leq x\\(n,k)=1}}\frac{\mu(n)}{n}$$

where $k \in \mathbb{Z}_{\geq 1}$, and showed among others that

$$\left|\sum_{\substack{n\leq x\\(n,k)=1}}\frac{\mu(n)}{n}\right| \leq \frac{k}{\varphi(k)}\,\frac{0.78}{\log(x/k)} \qquad (1 \leq k < x)\,.$$

Our second result is a complement to Ramaré's bound.

Theorem 2. *Let* $k, m \in \mathbb{Z}_{\geq 1}$. *For all* $x \geq k^m$

$$\left|\sum_{\substack{n\leq x\\(n,k)=1}}\frac{\mu(n)}{n}\right| < \frac{k}{\varphi(k)}\,\frac{C_m}{\left(\log\left(exk^{-m}\right)\right)^2}$$

where

$$C_m = 1\,100\left(1 + 4e^{-1}\sqrt{\zeta\left(m+\tfrac{1}{2}\right)}\right)^2.$$

The first ten values of the ceiling of C_m are given below.

m	1	2	3	4	5	6	7	8	9	10
$\lceil C_m \rceil$	12 555	8 045	7 221	6 937	6 820	6 768	6 743	6 731	6 725	6 723

Next, we estimate the logarithmic mean of the Möbius function twisted by a Dirichlet character.

Proposition 3. *Let χ be a non-principal Dirichlet character to the modulus $q \geq 37$ and let $k \in \mathbb{Z}_{\geq 1}$. Then for all $x \geq 1$*

$$\left| \sum_{\substack{n \leq x \\ (n,k)=1}} \frac{\mu(n)\chi(n)}{n} \right| \leq \frac{k}{\varphi(k)} \frac{2\sqrt{q}\log q}{|L(1,\chi)|}.$$

Our last result deals with the following rather curious sum which does not seem to have been studied in the literature before.

Theorem 4. *For every $x \geq 1$, define*

$$S(x) = \sum_{n \leq x} \mu(n) \sum_{k=1}^{n} \mu(k).$$

Then for all $x \geq 1\,664$

$$\frac{x}{2\zeta(2)} - 0.067\sqrt{x} \leq S(x) \leq 0.000\,6 \left(\frac{x}{\log x}\right)^2 + \frac{x}{2\zeta(2)} + 0.067\sqrt{x}.$$

Furthermore, the prime number theorem implies that, for x sufficiently large

$$S(x) \ll x^2 e^{-0.419\,6\,(\log x)^{3/5}(\log\log x)^{-1/5}}.$$

Finally, the Riemann hypothesis is true if and only if, for all $\varepsilon > 0$ and x sufficiently large

$$S(x) \ll x^{1+\varepsilon}.$$

In what follows, we define the functions $M(x)$ and $m(x)$ by

$$M(x) = \sum_{n \leq x} \mu(n) \quad \text{and} \quad m(x) = \sum_{n \leq x} \frac{\mu(n)}{n}.$$

2 Tools

Our first lemma follows easily from well-known convolution techniques.

Lemma 5. *Let f be a completely multiplicative function and $a \in \mathbb{Z}_{\geq 1}$. Then uniformly for any real number $x \geq 1$*

$$\sum_{\substack{n \leq x \\ (n,a)=1}} \frac{\mu(n)f(n)}{n} = \sum_{\substack{k \leq x \\ k | a^\infty}} \frac{f(k)}{k} \sum_{m \leq x/k} \frac{\mu(m)f(m)}{m}.$$

Proof. If $\mathbf{1}_a^\infty$ is the characteristic function of the set of integers $k \geq 1$ such that $k \mid a^\infty$, then one can easily see that, for any $n \in \mathbb{Z}_{\geq 1}$

$$\left(\mathbf{1}_a^\infty \star \mu\right)(n) = \begin{cases} \mu(n), & \text{if } (n,a) = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Now inserting this in the left-hand side, interchanging the summations and taking the complete multiplicativity of f into account achieve the proof. □

The first result is an inequality coming from the work of Balazard [1], depending on Möbius' inversion formula and some special properties of the Bernoulli functions, and improving on the inequality used in [6, 9] by a factor log.

Lemma 6. *For all $x \geq 1$*

$$x\,|m(x)| \leq |M(x)| + \frac{1}{x}\int_1^x |M(t)|\,\mathrm{d}t + \frac{8}{3}.$$

In fact, this inequality is a special case of a more general result stating that, for every $k \in \mathbb{Z}_{\geq 1}$, there exist constants $C_k > 0$ and $D_k > 0$ such that

$$|xm(x) - M(x)| \leq C_k x^{2-k} \int_1^x |M(t)|\, t^{k-3} \mathrm{d}t + D_k$$

which implies Lemma 6 by taking $k = 3$. The case $k = 2$ provides the bound

$$x\,|m(x)| \leq |M(x)| + \int_1^x \frac{|M(t)|}{t}\,\mathrm{d}t + 2 - \frac{2}{x}$$

which proves to be slightly weaker than Lemma 6.

The next tool is an explicit bound for a certain class of integrals.

Lemma 7. *Let $a > 1$, $\alpha > 0$ be real numbers. For all $x \geq a$*

$$\int_a^x \frac{t\,\mathrm{d}t}{(\log t)^\alpha} \leq \frac{C_\alpha x^2}{(\log x)^\alpha}$$

where

$$C_\alpha = \frac{\alpha^{-1}}{2}\left(\frac{\alpha}{(2e\log a)^{\alpha/(\alpha+1)}} + \alpha^{1/(\alpha+1)}\right)^{\alpha+1}.$$

Proof. For any $b > 1$, we get

$$\int_a^x \frac{t\,\mathrm{d}t}{(\log t)^\alpha} = \left(\int_a^{x^{1/b}} + \int_{x^{1/b}}^x\right)\frac{t\,\mathrm{d}t}{(\log x)^\alpha} \leq \frac{1}{2}\left(\frac{x^{2/b}}{(\log a)^\alpha} + \frac{b^\alpha x^2}{(\log x)^\alpha}\right).$$

The inequality $\log x \leq Ce^{-1}x^{1/C}$, used with $C = \dfrac{b\alpha}{2(b-1)}$, yields

$$(\log x)^\alpha \leq \left(\frac{b\alpha}{2e(b-1)}\right)^\alpha x^{2-2/b} \Longleftrightarrow x^{2/b} \leq \left(\frac{b\alpha}{2e(b-1)}\right)^\alpha \frac{x^2}{(\log x)^\alpha}$$

and hence

$$\int_a^x \frac{t\,\mathrm{d}t}{(\log t)^\alpha} \leq \frac{x^2}{2}\left(\frac{b}{\log x}\right)^\alpha \left(\left(\frac{\alpha}{2e(b-1)\log a}\right)^\alpha + 1\right)$$

and choosing

$$b = 1 + \left(\frac{\alpha}{2e\log a}\right)^{\alpha/(\alpha+1)}$$

concludes the proof. ◻

The next lemma will be proved to be useful in the proof of Theorem 2.

Lemma 8. *Let $k, m \in \mathbb{Z}_{\geq 1}$. For all $x \geq 1$*

$$\sum_{\substack{n\leq x\\(n,k)=1}} \frac{\mu(n)}{n} = \sum_{d_1|k} \frac{\mu(d_1)^2}{d_1} \sum_{d_2|d_1} \frac{\mu(d_2)^2}{d_2} \cdots \sum_{d_m|d_{m-1}} \frac{\mu(d_m)^2}{d_m} \sum_{\substack{h\leq \frac{x}{d_1\cdots d_m}\\ h|d_m^\infty}} \frac{1}{h} \sum_{j\leq \frac{x}{hd_1\cdots d_m}} \frac{\mu(j)}{j}.$$

Proof. We proceed by induction on m. For $m = 1$, we have

$$\begin{aligned}
\sum_{\substack{n\leq x\\(n,k)=1}} \frac{\mu(n)}{n} &= \sum_{d_1|k} \frac{\mu(d_1)}{d_1} \sum_{h\leq x/d_1} \frac{\mu(hd_1)}{h} \\
&= \sum_{d_1|k} \frac{\mu(d_1)^2}{d_1} \sum_{\substack{h\leq x/d_1\\(h,d_1)=1}} \frac{\mu(h)}{h} \\
&= \sum_{d_1|k} \frac{\mu(d_1)^2}{d_1} \sum_{\substack{h\leq x/d_1\\ h|d_1^\infty}} \frac{1}{h} \sum_{j\leq x/(hd_1)} \frac{\mu(j)}{j}
\end{aligned}$$

where we used Lemma 5 in the last equality. Now assume that the statement is true for some $m \geq 1$. From Lemma 5 again

$$\sum_{\substack{h\leq \frac{x}{d_1\cdots d_m}\\ h|d_m^\infty}} \frac{1}{h} \sum_{j\leq \frac{x}{hd_1\cdots d_m}} \frac{\mu(j)}{j} = \sum_{\substack{h\leq \frac{x}{d_1\cdots d_m}\\ (h,d_m)=1}} \frac{\mu(h)}{h}$$

so that using the induction hypothesis

$$\begin{aligned}
\sum_{\substack{n\le x\\(n,k)=1}}\frac{\mu(n)}{n} &= \sum_{d_1|k}\frac{\mu(d_1)^2}{d_1}\sum_{d_2|d_1}\frac{\mu(d_2)^2}{d_2}\cdots\sum_{d_m|d_{m-1}}\frac{\mu(d_m)^2}{d_m}\sum_{\substack{h\le\frac{x}{d_1\cdots d_m}\\(h,d_m)=1}}\frac{\mu(h)}{h}\\
&= \sum_{d_1|k}\frac{\mu(d_1)^2}{d_1}\cdots\sum_{d_m|d_{m-1}}\frac{\mu(d_m)^2}{d_m}\sum_{d_{m+1}|d_m}\mu\left(d_{m+1}\right)\sum_{h\le\frac{x}{d_1\cdots d_m d_{m+1}}}\frac{\mu\left(hd_{m+1}\right)}{hd_{m+1}}\\
&= \sum_{d_1|k}\frac{\mu(d_1)^2}{d_1}\cdots\sum_{d_m|d_{m-1}}\frac{\mu(d_m)^2}{d_m}\sum_{d_{m+1}|d_m}\frac{\mu\left(d_{m+1}\right)^2}{d_{m+1}}\sum_{\substack{h\le\frac{x}{d_1\cdots d_m d_{m+1}}\\(h,d_{m+1})=1}}\frac{\mu(h)}{h}\\
&= \sum_{d_1|k}\frac{\mu(d_1)^2}{d_1}\cdots\sum_{d_{m+1}|d_m}\frac{\mu\left(d_{m+1}\right)^2}{d_{m+1}}\sum_{\substack{h\le\frac{x}{d_1\cdots d_{m+1}}\\h|d_{m+1}^\infty}}\frac{1}{h}\sum_{j\le\frac{x}{hd_1\cdots d_{m+1}}}\frac{\mu(j)}{j}
\end{aligned}$$

achieving the proof. □

The identity below may be proved by induction. We leave the details to the reader.

Lemma 9. *Let $k,m\in\mathbb{Z}_{\ge1}$. Then*

$$\sum_{d_1|k}\frac{\mu(d_1)^2}{d_1}\sum_{d_2|d_1}\frac{\mu(d_2)^2}{d_2}\cdots\sum_{d_m|d_{m-1}}\frac{\mu(d_m)^2}{d_m}\prod_{p|d_m}\left(1-\frac{1}{p^{1/2}}\right)^{-1}=\frac{k}{\varphi(k)}\prod_{p|k}\left(1+\frac{1}{p^{m+1/2}}\right).$$

3 Proofs of the Theorems

3.1 Theorem 1

Proof.

(a) We check numerically the inequality for $x\in[33,6\,000]$, and we assume $x>6\,000$. Let $T\in[685,x]$ be a parameter at our disposal. From Lemma 6 and the bounds (1) and (2), we infer

$$\begin{aligned}
x|m(x)| &\le |M(x)|+\frac{1}{x}\left(\int_1^T+\int_T^x\right)|M(t)|\,\mathrm{d}t+\frac{8}{3}\\
&< \frac{0.109\,17x}{\log x}+\frac{T(T+882)}{160x}-\frac{883}{160x}+\frac{0.109\,17}{x}\int_T^x\frac{t\,\mathrm{d}t}{\log t}+\frac{8}{3}\\
&\le \frac{0.109\,17x}{\log x}+\frac{T^2(1+882/685)}{160x}+\frac{0.109\,17}{x}\int_T^x\frac{t\,\mathrm{d}t}{\log t}+\frac{8}{3}
\end{aligned}$$

and Lemma 7 implies that

$$x|m(x)| < \frac{0.109\,17x}{\log x} + \frac{1\,567}{685}\,\frac{T^2}{160x} + \frac{0.109\,17}{2}\left(1+\frac{1}{\sqrt{2e\log T}}\right)^2\frac{x}{\log x} + \frac{8}{3}.$$

We choose $T = 0.337\,x(\log x)^{-1/2}$. Since $x > 6\,000$, we have $T > 685$ and thus

$$\begin{aligned} x|m(x)| &< \frac{x}{\log x}\Bigg(0.109\,17 + 0.001\,63 + \frac{0.109\,17}{2}\left(1+\frac{1}{\sqrt{2e\log 685}}\right)^2 \\ &\qquad + \frac{8}{3}\,\frac{\log 6\,000}{6\,000}\Bigg) \\ &< \frac{0.19x}{\log x} \end{aligned}$$

achieving the proof of the inequality.

(b) The inequality is first checked on $]1,2[$ via

$$\frac{546}{(\log x)^2} > \frac{546}{(\log 2)^2} > 1 = |m(x)|$$

and then numerically for $x \in [2, 33]$. If $x \in [33, e^{2\,873}]$, then

$$|m(x)| < \frac{0.19}{\log x} < \frac{546}{(\log x)^2}$$

so that we may suppose $x > e^{2\,873}$. Using Lemma 6 as above, Lemma 7 with $\alpha = 2$, and (3), we get for any $T > 1$

$$\begin{aligned} x|m(x)| &\le \frac{362.7x}{(\log x)^2} + \frac{T^2(1+882/T)}{160x} - \frac{883}{160x} + \frac{362.7}{x}\int_T^x \frac{t\,\mathrm{d}t}{(\log t)^2} + \frac{8}{3} \\ &< \frac{362.7x}{(\log x)^2} + \frac{T^2(1+882/T)}{160x} \\ &\qquad + \frac{362.7}{4}\left(\frac{2}{(2e\log T)^{2/3}} + 2^{1/3}\right)^3\frac{x}{(\log x)^2} + \frac{8}{3}. \end{aligned}$$

Choosing $T = x(\log x)^{-1} > \dfrac{e^{2\,873}}{2\,873}$, we obtain

$$\begin{aligned} x|m(x)| &< \frac{x}{(\log x)^2}\Bigg(362.7 + \frac{1}{160}\left(1+\frac{882\times 2\,873}{e^{2\,873}}\right) \\ &\qquad + \frac{362.7}{4}\left(2^{1/3} + 2\left(e\log\left(\frac{e^{2\,873}}{2\,873}\right)\right)^{-2/3}\right)^3 + \frac{8}{3}\,\frac{2\,873}{e^{2\,873}}\Bigg) \\ &< \frac{x}{(\log x)^2}\left(362.7 + 0.006\,25 + 182.738\,23 + \frac{8}{3}\,\frac{2\,873}{e^{2\,873}}\right) \\ &< \frac{546x}{(\log x)^2}. \end{aligned}$$

The proof is completed. $\square$

3.2 Theorem 2

We first state the following result, which is an easy consequence of Theorem 1.

Lemma 10. *For all* $N \in \mathbb{Z}_{\geq 1}$

$$\left|\sum_{n=1}^{N} \frac{\mu(n)}{n}\right| \leq \frac{1}{\log\left(\frac{e}{2}(N+1)\right)} \quad \textit{and} \quad \left|\sum_{n=1}^{N} \frac{\mu(n)}{n}\right| < \frac{550}{(\log(e(N+1)))^2}.$$

Proof. We check numerically the first inequality for $N \in \{1, \dots, 32\}$ and, if $N \geq 33$, then by Theorem 1

$$\left|\sum_{n=1}^{N} \frac{\mu(n)}{n}\right| < \frac{0.19}{\log N} \leq \frac{1}{\log\left(\frac{e}{2}(N+1)\right)}.$$

Now let us have a look at the second inequality. If $N \in \{1, \dots, 10^{119}\}$, then

$$\left|\sum_{n=1}^{N} \frac{\mu(n)}{n}\right| \leq \frac{1}{\log\left(\frac{e}{2}(N+1)\right)} < \frac{550}{(\log(e(N+1)))^2}$$

and for $N > 10^{119}$

$$\left|\sum_{n=1}^{N} \frac{\mu(n)}{n}\right| < \frac{546}{\log^2 N} \leq \frac{550}{(\log(e(N+1)))^2}$$

concluding the proof. $\square$

We now are in a position to show Theorem 2.

Proof of Theorem 2. Setting

$$T = \left(exk^{-m}\right)^{\frac{4\sqrt{\zeta(m+1/2)}}{4\sqrt{\zeta(m+1/2)}+e}}$$

and using Lemmas 8 and 10, the sum at the left-hand side does not exceed

$$
\begin{aligned}
&< \quad 550 \sum_{d_1|k} \frac{\mu(d_1)^2}{d_1} \sum_{d_2|d_1} \frac{\mu(d_2)^2}{d_2} \cdots \sum_{d_m|d_{m-1}} \frac{\mu(d_m)^2}{d_m} \sum_{\substack{h \le \frac{x}{d_1\cdots d_m} \\ h|d_m^\infty}} \frac{1}{h\left(\log\left(\frac{ex}{hd_1\cdots d_m}\right)\right)^2} \\
&= \quad 550 \sum_{d_1|k} \frac{\mu(d_1)^2}{d_1} \cdots \sum_{d_m|d_{m-1}} \frac{\mu(d_m)^2}{d_m} \left(\sum_{\substack{h\le T \\ h|d_m^\infty}} + \sum_{\substack{T<h\le \frac{x}{d_1\cdots d_m} \\ h|d_m^\infty}} \right) \frac{1}{h\left(\log\left(\frac{ex}{hd_1\cdots d_m}\right)\right)^2} \\
&\le \quad 550 \sum_{d_1|k} \frac{\mu(d_1)^2}{d_1} \sum_{d_2|d_1} \frac{\mu(d_2)^2}{d_2} \cdots \sum_{d_m|d_{m-1}} \frac{\mu(d_m)^2}{d_m} \frac{1}{\left(\log\left(\frac{ex}{Td_1\cdots d_m}\right)\right)^2} \frac{d_m}{\varphi(d_m)} \\
&\qquad + 550\, T^{-1/2} \sum_{d_1|k} \frac{\mu(d_1)^2}{d_1} \sum_{d_2|d_1} \frac{\mu(d_2)^2}{d_2} \cdots \sum_{d_m|d_{m-1}} \frac{\mu(d_m)^2}{d_m} \sum_{h|d_m^\infty} \frac{1}{h^{1/2}}
\end{aligned}
$$

where in the second sum we used the fact that, if $h > T$, then $h^{-1} < (hT)^{-1/2}$. Now from Lemma 9 we get

$$
\begin{aligned}
\left| \sum_{\substack{n\le x \\ (n,k)=1}} \frac{\mu(n)}{n} \right| \quad &\le \quad \frac{550}{\left(\log\left(\frac{ex}{Tk^m}\right)\right)^2} \sum_{d_1|k} \frac{\mu(d_1)^2}{d_1} \sum_{d_2|d_1} \frac{\mu(d_2)^2}{d_2} \cdots \sum_{d_m|d_{m-1}} \frac{\mu(d_m)^2}{\varphi(d_m)} \\
&\qquad + 550\, T^{-1/2} \sum_{d_1|k} \frac{\mu(d_1)^2}{d_1} \cdots \sum_{d_m|d_{m-1}} \frac{\mu(d_m)^2}{d_m} \prod_{p|d_m} \left(1 - \frac{1}{p^{1/2}}\right)^{-1} \\
&= \quad \frac{550k}{\varphi(k)} \left(\frac{1}{\left(\log\left(\frac{ex}{Tk^m}\right)\right)^2} + T^{-1/2} \prod_{p|k} \left(1 + \frac{1}{p^{m+1/2}}\right) \right) \\
&\le \quad \frac{550k}{\varphi(k)} \left(\frac{1}{\left(\log\left(\frac{ex}{Tk^m}\right)\right)^2} + \frac{16\,\zeta\left(m+\frac{1}{2}\right)}{(e\log T)^2} \right)
\end{aligned}
$$

giving the asserted result if we replace T by its value given above. □

3.3 Proposition 3

Proof. Since a Dirichlet character is a completely multiplicative function, we get from Lemma 5

$$
\sum_{\substack{n\le x \\ (n,k)=1}} \frac{\mu(n)\chi(n)}{n} = \sum_{\substack{n\le x \\ n|k^\infty}} \frac{\chi(n)}{n} \sum_{m\le x/n} \frac{\mu(m)\chi(m)}{m}
$$

and we conclude using the inequality [2, page 4]

$$\left|\sum_{n\leq x}\frac{\mu(n)\chi(n)}{n}\right|\leq\frac{2\sqrt{q}\log q}{|L(1,\chi)|}$$

valid whenever $q\geq 37$. $\square$

3.4 Theorem 4

Proof. The proof will follow from the identity

$$S(x)=\frac{1}{2}\left(M(x)^2+\sum_{k\leq x}\mu(k)^2\right). \tag{5}$$

Indeed

$$\begin{aligned}
S(x) &= \sum_{n\leq x}\mu(n)\sum_{k=1}^{n}\mu(k)=\sum_{k\leq x}\mu(k)\sum_{k\leq n\leq x}\mu(n)\\
&= M(x)\sum_{n\leq x}\mu(n)-\sum_{k\leq x}\mu(k)\sum_{n=1}^{k-1}\mu(n)\\
&= M(x)^2-\sum_{k\leq x}\mu(k)\left(\sum_{n=1}^{k}\mu(n)-\mu(k)\right)\\
&= M(x)^2-S(x)+\sum_{k\leq x}\mu(k)^2
\end{aligned}$$

giving (5). Now from [3] we know that

$$\left|\sum_{k\leq x}\mu(k)^2-\frac{x}{\zeta(2)}\right|\leq 0.133\,3\sqrt{x}$$

as soon as $x\geq 1\,664$. This along with (2) leads to the explicit inequality of the theorem. The second inequality follows from the fully explicit bound for the Riemann zeta-function given in [7] providing

$$M(x)\ll xe^{-0.209\,8(\log x)^{3/5}(\log\log x)^{-1/5}}.$$

Now assume RH. Using Soundararajan's result [13, Theorem 1] we infer

$$S(x)\ll xe^{2(\log x)^{1/2}(\log\log x)^{14}}\ll x^{1+\varepsilon}.$$

Conversely, if $S(x)\ll x^{1+\varepsilon}$, then

$$M(x)^2=2S(x)-\sum_{n\leq x}\mu(n)^2\ll x^{1+\varepsilon}$$

so that $M(x)\ll x^{1/2+\varepsilon}$, which is known to be equivalent to the Riemann hypothesis. The proof of Theorem 4 is complete. $\square$

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